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Franck-Condon effect in central spin system

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Abstract. We study the quantum transitions of a central spin surrounded by a collective-spin environment. It is found that the influence of the environmental spins on the absorption spectrum of the central spin can be explained with the analog of the Franck-Condon (FC) effect in conventional electron-phonon interaction system. Here, the collective spins of the environment behave as the vibrational mode, which makes the electron transitioned mainly with the so-called "vertical transitions" in the conventional FC effect. The "vertical transition" for the central spin in the spin environment manifests as, the certain collective spin states of the environment is favored, which corresponds to the minimal change in the average of the total spin angular momentum.

The Franck-Condon (FC) principle, which determines the relative intensity of the vibration-assisted electron transition spectrum, is of much significance in molecular physics [1,2]. In these excitation and de-excitation processes, the transition probabilities are proportional to the square of the overlap integrals between the initial and final vibrational states (the FC factors). Compared with the fast electronic transition, the vibrational motion is extraordinary slow. As a result, during the electronic transition, vibrational coordinates nearly keep stationary. This corresponds to a "vertical transition" picture on the effective vibrational potential energy surface, and is called FC effect. The FC principle was originally proposed by Franck to study the mechanism of photon-induced chemical reactions [3] and later expanded to the semi-classical formulation by Condon [1]. And then Lax applied this principle to solid-state physics [4]. However, all the previous works focused on electron-phonon coupling system [5–9].

In this paper, we will study the FC effect induced by spin-spin interaction for a model of a central spin in collective-spin environment, like a central spin in quantum dot (QD) [10–14] or in nitrogen-vacancy (NV) center [15–22]. In these systems, the unavoidable hyperfine interaction between the central spin and the collective environmental spins is the chief culprit of decoherence of the interested central spin. Thus, it is important to investigate the effect of the spin-spin FC principle on the dynamics of the central spin. On the other hand, in an ideal environment with specific inter-spin coupling (such as Ising type), the central spin can be used as a probe to explore the supersensitivity of a quantum critical multi-spin system [23–25]. This theoretical prediction has been tested in several experiments [26–28] and its robustness has been numerically shown as the longitudinal field (equivalent to the tranverse hyperfine coupling in our model) does not effect on the decoherence behaviour around the critical point [27,29].

We consider the model of a central spin immersed in an environment of nuclear spins. In general, the central spin can be a nuclear spin or an electron spin. The central spin is initially polarized by the crystal field in the z direction. The collective environmental spins behave as the vibrational mode in conventional electron-phonon interaction model of FC effect. And the longitudinal hyperfine coupling between the central spin and its spin environment is analogous to the diagonal electron-phonon coupling, resulting in the effective Hamiltonian of the environment spin being central-spin-dependent. Due to this hyperfine coupling, when the central spin is excited by the external field, the spin bath will be excited simultaneously and this co-excitation generates the collective-spin-based FC effect. And the FC factors, which were originally the overlap integrals between the initial and final displaced vibrational Fock states, are defined as the overlaps of the rotated collective spin states in our system. An earlier paper investigated the spin FC effect, but it was only devoted to demonstrating the Stokes shift in a spin-spin interaction system [30]. In contrast to that work [30], we study detailedly and systematically the collective-spin-based FC effect and reveal its underlying physical mechanism.

We find that there exists the similar FC effect in our spin-spin interacting system. In zero temperature case, the original Lorentz absorption spectrum of a naked spin is shifted and split into few small peaks by the weak

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hyperfine coupling, just as same as the vibronic transition spectrum [31]. The distribution of the relative transition intensity is determined by the FC factors. And the most probable transitions, which have largest FC factors, are ruled by the "vertical transition" mechanism. On the other hand, if the collective-spin environment is at finite temperature, the peaks of the absorption spectrum of the central spin are depressed and broadened significantly. Especially, when the hyperfine coupling is strong enough, the excitation of the central spin is suppressed intensively. This behavior is called FC blockade [32,33].

In the next section, we present our central spin model and its implementation in N-V center in detail. In Section 2, we discuss the low excitation limit of our central spin model and interpret the conventional FC effect schematically. The collective-spin-based FC effect in our central spin system are addressed in Section 3. In Section 4, we study the collective-spin-based FC effect with vertical transition in schematic perspective. Finally, the summery of our main results is given in Section 5. Some details about the rotated Dicke state are displayed in Appendix.

1 Model setup: central spin in environment

In the central spin system (e.g., QDs or NV center systems), the coupling to the environmental nuclear spins destroys the coherence of the central spin (electronic spin or large nuclear spin) primarily [34]. While we find that these hyperfine couplings can arouse another interesting effect – the collective-spin-based FC effect.

We consider a general model of a central spin immersed in an N-spin environment, with the Hamiltonian described by $(\hbar = 1)$

$$H = \mathcal{H}_s\left(S_z\right) + A\sum_{j=1}^{N} S_z \cdot I_z^{(j)} + \omega_{\mathrm{nu}}\sum_{j=1}^{N} I_x^{(j)} + \omega_{\mathrm{el}} S_x, \quad (1)$$

where S_α and $I_\alpha^{(j)}$ $(\alpha = x, y, z)$ are the angular momentum operators of the central spin and the jth nuclear spin, respectively. By virtue of the existence of the crystal field, the central spin is usually polarized in a particular direction (z direction). As a result, $\mathcal{H}_s\left(S_z\right)$ generally depends on S_z, e.g., for NV center system, $\mathcal{H}_s\left(S_z\right) = DS_z^2$. When a transverse external field along the x-axis is applied, the central and environmental spins get additional Zeeman splittings ω_{el} and ω_{nu}, respectively. As we know, the last term in equation (1), $\omega_{\mathrm{el}} S_x$ will induce transitions between the ground and excited states of the central spin. Because of the existence of the spin environment, these transitions will be modulated by the longitudinal hyperfine coupling $A(>0)$. For simplicity, the hyperfine coupling between the central spin and the environment-spin ensemble has been assumed to be homogeneous.

It is convenient to define the collective polarization operators for the nuclear spin ensemble as

$$J_\alpha = \sum_j I_\alpha^{(j)}. \quad (2)$$

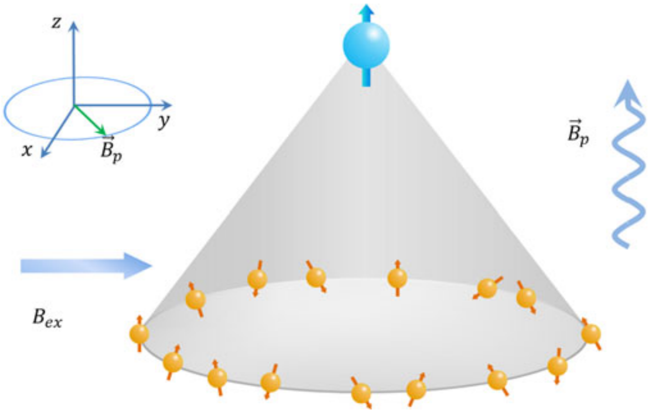

Fig. 1. (Color online) Schematic of central spin model. The central spin (the top blue one) is polarized in the z direction. The flipping of the central spin induced by the polarized field $\boldsymbol{B}_p$ is modulated by the longitudinal hyperfine coupling between the central spin and its nuclear-spin environment.

It is ready to find that these collective operators satisfy the following commutation relations:

$$[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma} J_\gamma, \;\; [J^2, J_\alpha] = 0, \quad (3)$$

with $\epsilon_{\alpha\beta\gamma}$ the totally antisymmetric Levi-Civita tensor. As a result, the collective environmental spins have collective eigenstates (i.e., Dicke state [35,36]) and the dynamic symmetry is described by the algebra SO (3).

Now let us focus on the model of a central electronic spin implemented in the negatively charged NV center in diamond (Fig. 1). The electronic ground state of the center, which we concern, is a spin triplet $(S = 1)$ and there is zero-field splitting (ZFS) $D = 2.87$ GHz between states $|s = 0\rangle$ and $|s = \pm 1\rangle$ ($|s\rangle$ is the eigenstate of S_z). It should be noted that the $|s = \pm 1\rangle$ levels are degenerate. In the high purity dimond, since the nitrogen concentration is negligible, we just consider the effect of ^{13}C nuclear spins (spin-1/2) to the dynamics of the central spin. We rewrite our model Hamiltonian as $H = H_0 + H_1$, with

$$H_0 = DS_z^2 + \omega_{\mathrm{nu}} J_x + AS_z J_z, \quad (4)$$

and

$$H_1 = \omega_{\mathrm{el}} S_x + \Omega\left(S_x \cos\omega t + S_y \sin\omega t\right).$$

The applied static magnetic field is weak $B_{\mathrm{ex}} = 12$ Gauss and the corresponding Zeeman splittings of the electronic and nuclear spins are $\omega_{\mathrm{el}} = g_e\mu_B B_{\mathrm{ex}} \approx 211.35$ MHz and $\omega_{\mathrm{nu}} = g_n\mu_n B_{\mathrm{ex}} \approx 0.15$ MHz, respectively [18]. We have taken the isotropic Landé g-factor of the electron and nuclear as $g_e \approx 2.0$ and $g_n = 1.4$, respectively. $\mu_B(\mu_n)$ is the Bohr (nuclear) magneton and $\mu_n \approx 10^{-3}\mu_B$. The hyperfine coupling constant A is of the order of (kHz–MHz). And in order to probe the absorption spectrum of the central spin, an external circularly polarized magnetic field $\boldsymbol{B}_p = (B_p\cos\omega t, B_p\sin\omega t, 0)$ is added with $\Omega = g_e\mu_e B_p = D/20$. Since $\omega_{\mathrm{el}}, \Omega \ll D$, we take these terms as perturbations. The longitudinal hyperfine coupling has a much larger influence on the dynamics of the

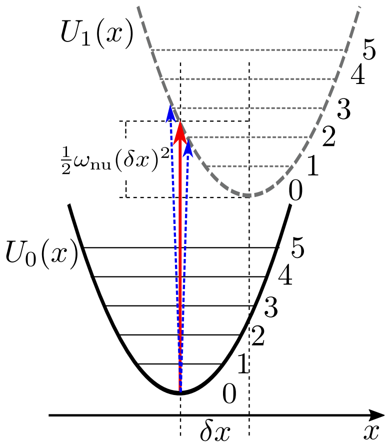

Fig. 2. (Color online) Schematic of the conventional vertical transition.

central spin than the transverse one. And the strength of the transverse hyperfine coupling is much smaller than the driving field Ω. Thus we neglected the corresponding terms in the Hamiltonian (4).

Now we consider the specific case, where the frequency ω of the circularly polarized field is nearly resonant with the ZFS of the central electronic spin, i.e., $\omega \approx D$. Under the rotating wave approximation, the H_1 part reduces to

$$H_1 \approx \frac{\Omega}{\sqrt{2}}\left[|1\rangle\langle 0| e^{-i\omega t} + |0\rangle\langle 1| e^{i\omega t}\right], \tag{5}$$

where we have used the condition $\omega_{\rm el}, \Omega \ll D$ as well as $\omega \approx D$.

2 Low excitation approximation

To see why the above system enjoys the FC effect, we first consider the semi-classical interpretation for the conventional FC principle in an electron-phonon interacting system: in the approximation of the linearization of the vibronic coupling, the phonons will get electron-dependent displaced effective potentials (see Fig. 2). The electronic transition is so fast that the geometry structure of the vibrational freedom of degrees remain unchanged during this process, i.e., vertical transition takes place most possibly. This conventional FC phenomenon could be described by a simple model of a two-level system coupling to a single vibrational mode, with the Hamiltonian,

$$H = \frac{\varepsilon}{2}\sigma_z + \omega_p b^\dagger b + g\sigma_z(b^\dagger + b), \tag{6}$$

where σ_z is the Pauli operator, ε is the energy difference between the ground and excited states of the eletron, $b^\dagger(b)$ is the bosonic creation (annihilation) operator of the vibration mode with frequency ω_p, and g is the elctron-phonon coupling constant.

Next we will show that our central electronic spin model can be reduced into the above model in low excitation limit. In the subspace spanned by the states $\{|N/2, m\rangle\}$ $(m = 0, 1, 2 \ldots, N)$, where $|N/2, m\rangle$ is the eigenstate of $\{J^2,\ J_x\}$, one can use the Holstein-Primakoff (HP) transformation

$$\tilde{J}_+ \equiv J_z - iJ_y = b^\dagger\sqrt{N - b^\dagger b}, \tag{7}$$

$$\tilde{J}_- \equiv J_z + iJ_y = \sqrt{N - b^\dagger b}\, b, \tag{8}$$

$$J_x = b^\dagger b - \frac{N}{2}, \tag{9}$$

with defining the bosonic creation and annihilation operators $b^\dagger$ and b. In the low excitation limit $\langle b^\dagger b\rangle \ll N$, HP transformation (9) can be given, expanded to the lowest order, as

$$\tilde{J}_+ \approx \sqrt{N} b^\dagger,\ \tilde{J}_- \approx \sqrt{N} b,\ J_x = b^\dagger b - \frac{N}{2}. \tag{10}$$

Then we obtain the Hamiltonian $\tilde{H} = \tilde{H}_0 + \tilde{H}_1$ of a electron-phonon-like interaction model with

$$\tilde{H}_0 = DS_z^2 + \omega_{\rm nu} b^\dagger b + \frac{1}{2}\sqrt{N} A S_z\left(b^\dagger + b\right) - \frac{N}{2}\omega_{\rm nu}, \tag{11}$$

and the perturbation part $\tilde{H}_1 = H_1$ describing the transition between the electronic spin states $|s = 0\rangle$ and $|s = 1\rangle$. The $\tilde{H}_0$ part can be diagonalized by the displaced Fock state $|\xi_s, m; s\rangle = \exp\left(\xi_s b^\dagger - \xi_s^* b\right)|m\rangle \otimes |s\rangle$ [37,38], where s denotes the eigenstate of S_z, $|m\rangle$ $(m = 0, 1, \ldots, N)$ is the mth Fock state of $b^\dagger b$, and the electron-dependent displacement

$$\xi_s = -\xi_s^* = -\sqrt{N}\frac{sA}{2\omega_{\rm nu}}. \tag{12}$$

The FC factor of the transition from $|\xi_0, m; 0\rangle$ to $|\xi_1, n; 1\rangle$ is defined as the overlap integral between the two relative displaced Fock states

$$\langle \xi_1, n \mid \xi_0, m\rangle = e^{-|\xi|^2/2}\sqrt{\frac{m!}{n!}} L_m^{n-m}\left(|\xi|^2\right)(\xi)^{n-m}, \tag{13}$$

with $\xi = \xi_0 - \xi_1$ and $L_m^{n-m}(x)$ being the generalized Laguerre polynomial.

We can formally define dimensionless canonical coordinates of the vibrational mode as $x \equiv \left(b^\dagger + b\right)/\sqrt{2}$ and $p \equiv i\left(b^\dagger - b\right)/\sqrt{2}$, with $[p, x] = -i$. The effective potentials of the nuclear spins are different correponding to different eigenstates $(|s\rangle)$ of the central spin:

$$U_s(x) = \frac{1}{2}\omega_{\rm nu} x^2 + \sqrt{\frac{N}{2}} sAx - \frac{N}{2}\omega_{\rm nu}. \tag{14}$$

As we know, if the electron spin and the nuclear spins are both in the ground state $|\xi_0, 0; 0\rangle$ initially, when the electron spin is exited by $\tilde{H}_1$ to its higher level $|1\rangle$, the nuclear spins could be co-excited to all possible eigenstates $|\xi_1, n; 1\rangle$ of $U_1(x)$. But the transition, which has the largest FC factor, is most favored. From equation (13), we find that the most favored final state $|n_{\rm mf}\rangle$ is determined by

$$n_{\rm mf} = \left[\frac{1}{2}(\delta x)^2\right]. \tag{15}$$

Here $\delta x = \sqrt{2}\xi$ and $[\ldots]$ means the rounding operation. As shown in Figure 2, the most favored transition coincides with the vertical transition: the sole transition (red solid arrow) or two adjacent transitions (blue dashed arrows). As a consequence, when the system is initially in the ground state, vertical transition takes place most possibly.

Thus, in the present spin-spin interaction system, there exists conventional FC effect in low excitation limit.

3 Colletive-spin-based Franck-Condon effect

In the previous section, we have shown that our central spin model is approximately equivalent to an electron-phonon interaction model in low excitation limit. In this section, we investigate the influence of the nuclear spin ensemble on the transition spectrum of the central spin beyond this limit and find collective-spin-based FC effect exists in this central spin system.

The Hamiltonian H_0 is diagonalized in its direct product Hilbert space as: $H_0^{(s)} \otimes |s\rangle\langle s|$. Here $H_0^{(s)}$ is the effective Hamiltonian of the nuclear-spin environment with the central spin at state $|s\rangle$ and given by

$$H_0^{(s)} \equiv sAJ_z + \omega_{\text{nu}} J_x + s^2 D. \tag{16}$$

As shown in Appendix, the eigenstate of $H_0^{(s)}$ is the rotated Dicke state $|\theta_s, m\rangle = \exp(-i\theta_s J_y)|m\rangle$, where $|m\rangle$ is the eigenstate of J_z with eigenvalue $(m - N/2)$ and the mixing angle is determined by

$$\sin\theta_s = \frac{\omega_{\text{nu}}}{\tilde{\omega}_{\text{nu}}(s)}, \; \cos\theta_s = \frac{sA}{\tilde{\omega}_{\text{nu}}(s)}, \tag{17}$$

with

$$\tilde{\omega}_{\text{nu}}(s) = \sqrt{\omega_{\text{nu}}^2 + (sA)^2}. \tag{18}$$

As a result, we obtain the total eigenstate of H_0 as

$$|\theta_s, m; s\rangle = |\theta_s, m\rangle \otimes |s\rangle \tag{19}$$

and the corresponding eigenvalue

$$E_{s,m} = s^2 D + \left(m - \frac{N}{2}\right)\tilde{\omega}_{\text{nu}}(s) \tag{20}$$

with $s = 0, \pm 1$ and $m = 0, 1, 2, \ldots, N$.

3.1 Franck-Condon factor

As we know, the conventional FC effect describes the phenomenon that, in the electron-phonon interacting system, the electronic transition is modulated by the vibronic coupling. The transition probability is proportional to the square of the FC factor, which is specifically defined as the overlap integral between the two vibrational states involved in the transition. Here we can also obtain the counterpart of such factor in spin-spin coupling system as the overlap of two rotated Dicke states:

$$f_{m\to n} \equiv \langle\theta_1, n \mid \theta_0, m\rangle = d_{n,m}^{N/2}(\theta), \tag{21}$$

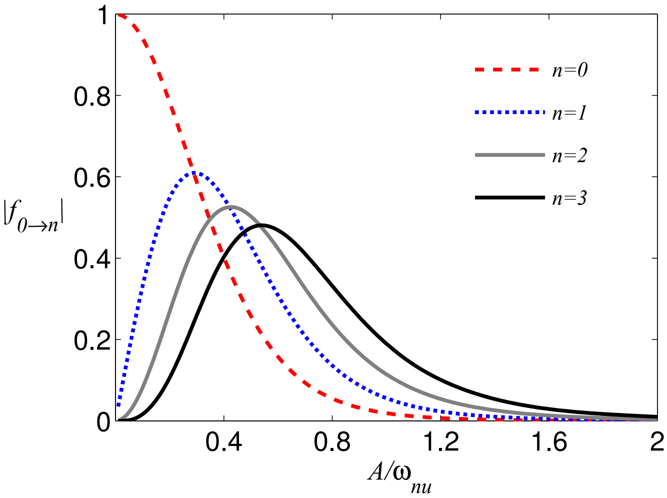

Fig. 3. (Color online) The amplitude of the Franck-Condon factor vs. the hyperfine coupling A for transition from $|\theta_0, 0; 0\rangle$ to $|\theta_1, n; 1\rangle$.

with the rotation-angle difference $\theta_0 - \theta_1 \equiv \theta \in (0, \pi/2)$ and the element of the Wigner's (small) d-matrix

$$d_{n,m}^{N/2} = [m!(N-m)!n!\,(N-n)!]^{\frac{1}{2}} \times \sum_k \frac{(-)^k \left(\cos\frac{\theta}{2}\right)^{N+m-n-2k} \left(-\sin\frac{\theta}{2}\right)^{n-m+2k}}{(N-n-k)!\,(m-k)!\,(k+n-m)!k!}. \tag{22}$$

For the special transition from $|\theta_0, 0; 0\rangle$ to $|\theta_1, n; 1\rangle$, the FC factor reduces to a simple form

$$f_{0\to n} = \sqrt{\frac{N!}{(N-n)!n!}} \left(\cos\frac{\theta}{2}\right)^{N-n} \left(-\sin\frac{\theta}{2}\right)^n. \tag{23}$$

As depicted in Figure 3, for a specific final state $|\theta_1, n; 1\rangle$, the amplitude of the FC factor $f_{0\to n}$ has its sole maximum and the corresponding hyperfine coupling strength to the maximum, $A_n^{\max}$, increases gradually with n. But the maximum of $|f_{0\to n}|$ decreases with n and the FC factor is drastically suppressed in the case of strong coupling A. It is worth noting that, for a given coupling strength A, there exists a sole or two adjacent most favored collective-spin states $|\theta_1, n\rangle$, which have largest FC factors. This is one of the essences lying at the core of the FC principle, since it will correspond to a very intuitive physical picture – vertical transition.

Different from $f_{0\to n}$, the monotonicity of $f_{m\to n}$ $(m \neq 0)$ is much complicated, sine there are multiple local maximums for a given A. We numerically calculate the FC factors of the all the possible transitions for coupling constant $A = 0.2\omega_{\text{nu}}$ or $A = 2\omega_{\text{nu}}$, as shown in Figure 4. It is found that neighboring transitions ($|m - n|$ is small) have larger FC factors when the hyperfine coupling is weak, but these transitions are depressed in strong coupling case.

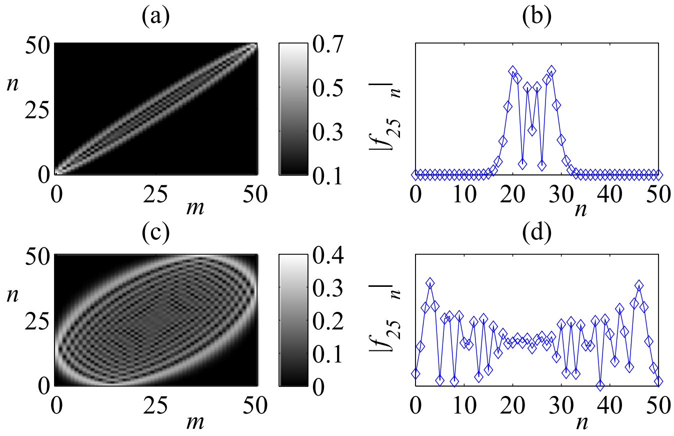

Fig. 4. (Color online) Franck-Condon factor for transition from $|\theta_0,m;0\rangle$ to $|\theta_1,n;1\rangle$ with different hyperfine couplings. (a) $A=0.2\omega_{\mathrm{nu}}$. (b) $A=0.2\omega_{\mathrm{nu}}$ and $m=25$. (c) $A=2\omega_{\mathrm{nu}}$. (d) $A=2\omega_{\mathrm{nu}}$ and $m=25$.

3.2 Transition probability and absorption spectrum

Starting from an initial state $|\theta_0,m;0\rangle$, the probability of the system in the state $|\theta_1,n;1\rangle$ at time t is given by

$$P_{1,n;0,m}(t)=\frac{\Omega^2}{2}F\left(\frac{\omega_{1n,0m}-\omega}{2},t\right)|f_{m\to n}|^2, \tag{24}$$

where we have defined $\omega_{s'n,sm}=E_{s',n}-E_{s,m}$ and the function $F(\omega,t)=\left(\sin^2\omega t\right)/\omega^2$. With the Fermi's golden rule, the transition rate of the central spin from $|\theta_0,m;0\rangle$ to $|\theta_1,n;1\rangle$ is obtained as

$$k_{m\to n}(\omega)=\lim_{t\to\infty}\frac{P_{1,n;0,m}(t)}{t}. \tag{25}$$

Now we first consider the zero temperature case. The system is initially in the ground state $|\theta_0,0;0\rangle$. It is found that the transition rates vary with the different final states. As shown in Figure 5a, for the weak coupling $A=0.2\omega_{\mathrm{nu}}$, there exist a favored transition, i.e., the one from the ground state to $|\theta_1,0;1\rangle$. From equations (24) and (25), we know that the transition rate is not only determined by the Fermi's golden rule, but also modified by the FC factor $f_{0\to n}$.

By summing up the transition rates over all the final states, we obtain the absorption spectrum as

$$I_s(\omega)=\sum_n k_{0\to n}(\omega). \tag{26}$$

It should be noted that when the time $t\to\infty$, we have this limit

$$\lim_{t\to\infty}\frac{F(\omega,t)}{t}=2\pi\delta(\omega). \tag{27}$$

Then the spectrum becames to many width-less lines. Actually, the observed absorption lines from experiments must have finite width resulting from the interaction of the radiation field on the central spin or any other interactions. However, the line-width correction is neglected, since the positions and the relative heights are mostly concerned and the linewidths of all the absorption lines are the same in our case. And we take a finite time $\omega_{\mathrm{nu}}t=10$ in equation (25) in our numerical calculations in Figures 5 and 6. As we know, the absorption spectrum of a naked electronic spin ($A=0$) given by the Fermi's golden rule is of the Lorentz form peaked at $\Delta\equiv\omega-D=0$ (the thin black line in Fig. 5b). The Lorentz spectrum is shifted and split into small peaks, as a result of the coupling between the electron spin and its nuclear-spin environment. When the coupling is weak $A=0.2\omega_{\mathrm{nu}}$, the spectrum is just split into few peaks neighboring the original one. However, in strong coupling case, the absorption spectrum is composed by many small discrete peaks centered at $\Delta=0$. Hence, the excitation of the central spin introduced by a polarization field is inhibited by the strong hyperfine coupling. And this phenomenon of transition suppression was called Franck-Condon blockade.

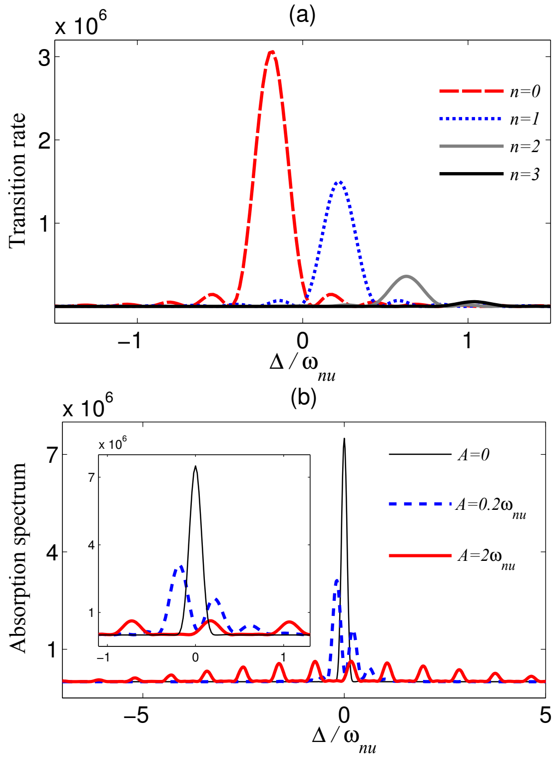

Fig. 5. (Color online) (a) Transition rates $k_{0\to n}$ for system from the state $|\theta_0,0;0\rangle$ to different final states $|\theta_1,n;0\rangle$ vs. Δ ($\Delta=\omega-D$), where the hyperfine coupling $A=0.2\omega_{\mathrm{nu}}$ and the number of the nuclear spins $N=50$. (b) The absorption spectra of the central spin with different hyperfine couplings.

In the case of finite ambient temperature, the collective nuclear spins are usually in the thermal equilibrium state

$$\rho_{\mathrm{nu}}=\frac{1}{Z}\sum_{m=0}^{N}\exp\left(-\frac{m\omega_{\mathrm{nu}}}{k_BT}\right)|\theta_0,m\rangle\langle\theta_0,m|, \tag{28}$$

where $Z=\sum_{m=0}^{N}\exp\left(-m\omega_{\mathrm{nu}}/k_BT\right)$ is the partition function, k_B is the Boltzmann constant, and T is the temperature of the environment. Since the energy splitting of the

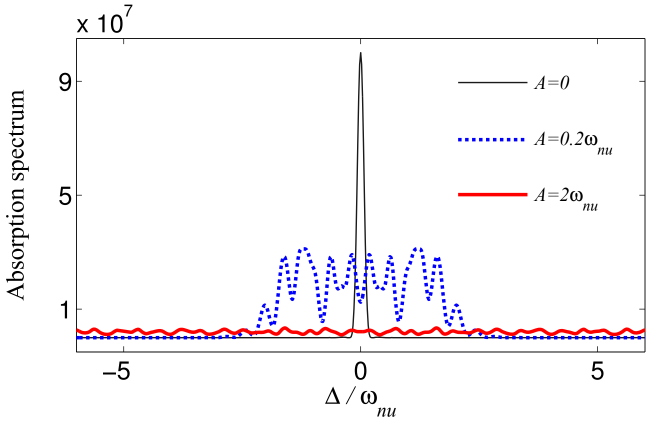

Fig. 6. (Color online) Spectrum of the central spin at room temperature with different hyperfine couplings A.

nuclear spins due to the weak external static field is small (i.e., $\omega_{\mathrm{nu}} \ll k_B T$), all the Dicke state $|\theta_0, m\rangle$ are nearly equally occupied in the case of high temperature.

The absorption spectrum of the central spin at room temperature $T = 300$ K is shown in Figure 6. For small A, only the neighboring transitions, which possess large FC factors, contribute significantly to the spectrum. Incorporated the Fermi's golden rule (i.e., the energy-conservation requirement $\omega_{1n,0m} - \omega = 0$), the absorption peaks are close to the resonant point $\Delta = 0$. Nevertheless, when the hyperfine coupling is large, the relative intensity of all the possible transition are almost the same. Consequently, the excitation of the central spin is drastically suppressed due to the Franck-Condon blockade.

4 Schematic perspective views for Franck-Condon effect

To reveal the physical essence of the FC effect in central spin system with vertical transitions, we will present a schematic perspective view for the collective-spin-based FC effect in comparison with the conventional picture of the electron-photon FC phenomenon.

As noted in the preceding section, when the system is at zero temperature, the transition probability is modulated by the FC factor $f_{0\to n}$ (Eq. (24)). The most favored transition must have the largest FC factor. And for a given hyperfine coupling strength A, there is only one maximum or two adjacent maxima ($|f_{0\to n}| = |f_{0\to n+1}|$). Hence, we require the most possible transition ($|\theta_0, 0; 0\rangle$ to $|\theta_1, n; 1\rangle$) to satisfy

$$|f_{0\to n}| - |f_{0\to n+1}| \geqslant 0, \tag{29}$$

$$|f_{0\to n}| - |f_{0\to n-1}| \geqslant 0. \tag{30}$$

From equation (23), we obtain the most favored final Dicke state $|n_{\mathrm{mf}}\rangle$ by

$$n_{\mathrm{mf}} \leqslant \frac{N+1}{2}(1 - \cos\theta) \leqslant n_{\mathrm{mf}} + 1. \tag{31}$$

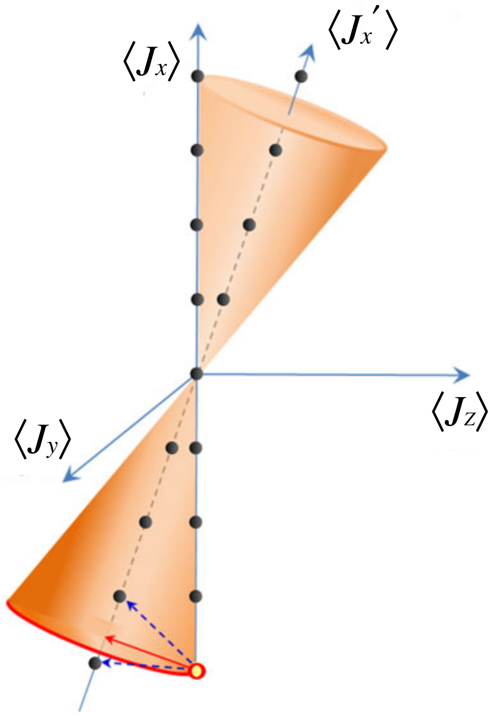

Fig. 7. (Color online) Time-evolution of the nuclear spins, which was assumed to be initialized on the state $|\theta_0, 0; 0\rangle$, with small hyperfine coupling $A = 0.2\omega_{\mathrm{nu}}$.

Here θ is dependent on the coupling constant A and $0 \leqslant n_{\mathrm{mf}} \leqslant (N+1)/2$. Thus the most favored transition is determined by hyperfine coupling A and number of the nuclear spins N.

In the space spanned by $\{\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle\}$, the eigenstates of $H_0^{(0)}$ (Eq. (16)) are designated by the discrete dark gray dots on the $\langle J_x\rangle$ axis in Figure 7. The initial state of the collective nuclear spins ($|\theta_0, 0\rangle$) is denoted by the bottommost red circle, with coordinate $\{\langle J_x\rangle = -N/2,\ \langle J_y\rangle = 0,\ \langle J_z\rangle = 0\}$. Let us consider the transition of the central spin from $|s = 0\rangle$ to $|s = 1\rangle$ induced by the polarization field. The corresponding effective Hamiltonian of the nuclear spins after this transition changes into

$$H_0^{(1)} = AJ_z + \omega_{\mathrm{nu}} J_x + D. \tag{32}$$

After making a rotation around y-axis with angle $(-\theta)$, we obtain

$$H_0^{\prime(1)} = \tilde{\omega}_{\mathrm{nu}} J_x' + D. \tag{33}$$

Its eigenstates distribute on the $\langle J_x'\rangle$ axis as shown in Figure 7.

From equation (31), the average of J_x' for the most favored state is given by

$$\langle \theta_1, n_{\mathrm{mf}}; 1 \mid J_x' \mid \theta_1, n_{\mathrm{mf}}; 1\rangle \approx -(N+1)\cos\theta/2. \tag{34}$$

It is nearly the same as the projection of the initial value of $\langle J_x\rangle$ on the J_x' axis $-N\cos\theta/2$. In other words, during the excitation of the central spin, the collective environmental spins like to jump vertically from the initial state $|\theta_0, 0; 0\rangle$ to the final state $|\theta_1, n_{\mathrm{mf}}; 1\rangle$, as depicted by the red arrow with a solid line in Figure 7. Usually, there is no such a eigenstates of the system corresponds to this vertical transition. As a consequence, the two adjacent transitions (characterized by the two blue arrows with dashed lines in Fig. 7) beside the vertical projection point are most likely to occur.

In the other hand, we investigate the dynamic evolution of the collective spins. In the Heisenberg picture, the operators satisfy the Heisenberg equations

$$\frac{\partial J_x'}{\partial t} = 0, \tag{35}$$

$$\frac{\partial J_y'}{\partial t} = -\tilde{\omega}_{\mathrm{nu}} J_z', \tag{36}$$

$$\frac{\partial J_z'}{\partial t} = \tilde{\omega}_{\mathrm{nu}} J_y'. \tag{37}$$

Since the transition of the central spin is very fast compared with the motion of the nuclear spins. During the excitation of the electronic spin from $|0\rangle$ to $|1\rangle$, the average of the nuclear-spin operators can be assumed to be the same before and after the transition, i.e., the geometry configuration of the nuclear spins keep invariant. This is the essence of vertical transition. Then we obtain corresponding initial mean values for the new angular operators $\langle J_x'(0)\rangle = -(N\cos\theta)/2$, $\langle J_y'(0)\rangle = 0$, and $\langle J_z'(0)\rangle = -(N\sin\theta)/2$. With these initial conditions, we obtain the solution of the former differential equations as

$$\langle J_x'(t)\rangle = -\frac{N}{2}\cos\theta, \tag{38}$$

$$\langle J_y'(t)\rangle = \frac{N}{2}\sin\theta\sin\tilde{\omega}_{\mathrm{nu}} t, \tag{39}$$

$$\langle J_z'(t)\rangle = -\frac{N}{2}\sin\theta\cos\tilde{\omega}_{\mathrm{nu}} t. \tag{40}$$

As shown in Figure 7, the trajectory of the state of collective nuclear spins is a circle (the red one) which crosses the initial point in the plane and is perpendicular to the $\langle J_x'\rangle$-axis. This intuitive understanding of vertical transition lead to the same result as equation (34). This confirms our former conjecture that the most favor final nuclear-spin state after the transition is ruled by the vertical transition.

5 Conclusion

We studied the influence of the environmental spins on absorption spectrum of the central spin. It is found that there exists similar FC effect in the central spin model as that in conventional electron-phonon model. In the zero temperature case, the original Lorentz absorption spectrum of the bare central spin is shifted and split into few small peaks. And the most probable transitions, which make the largest contribution to the absorption spectrum, are governed by the "vertical transition" mechanism. If the system is at finite temperature, the peak of the absorption spectrum is markedly depressed and broadened. Especially, when the hyperfine coupling is strong enough, the excitation of the central spin is intensively suppressed, which leads to the spin F-C blockade.

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Appendix: Rotated Dicke state

We define the collective spin operators as

$$J_\pm = \sum_j I_\pm^{(j)},\ J_\alpha = \sum_j I_\alpha^{(j)},\ \text{for } \alpha = x, y, z, \tag{A.1}$$

where $I_\pm^{(j)} = I_x^{(j)} \pm i I_y^{(j)}$. It is easy to find that the defined operators satisfy

$$[J_i, J_j] = i\epsilon_{ijk} J_k,\ [J^2, J_i] = 0, \tag{A.2}$$

and

$$[J_z, J_\pm] = \pm J_\pm,\ [J_+, J_-] = 2J_z,\ [J^2, J_\pm] = 0, \tag{A.3}$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor, with $\epsilon_{xyz} = +1$. With the help of these operators, the collective spins may be characterized by the simultaneous eigenstates $|J, M\rangle$ of J^2 and J_z with [35,36]

$$J = \frac{N}{2},$$
$$M = -\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}-1, \frac{N}{2}.$$

It is proofed that all of these states $|J, M\rangle$ (Dicke states) are symmetric under permutations of the nucleus and all the symmetric states are in the subspace which is spanned by the states with maximal angular momentum $J = N/2$. Now we order the totally symmetric eigenstates as

$$\begin{aligned}
\left|\frac{N}{2}, \frac{N}{2}\right\rangle &= |1, 1, \ldots, 1\rangle \\
\left|\frac{N}{2}, \frac{N}{2}-1\right\rangle &= J_- |1, 1, \ldots, 1\rangle = \mathcal{S}_n |0, 1, \ldots, 1\rangle \\
\left|\frac{N}{2}, \frac{N}{2}-2\right\rangle &= J_-^2 |1, 1, \ldots, 1\rangle = \mathcal{S}_n |0, 0, 1, \ldots, 1\rangle \\
&\cdots \\
\left|\frac{N}{2}, -\frac{N}{2}\right\rangle &= J_-^N |1, 1, \ldots, 1\rangle = |0, 0, \ldots, 0\rangle,
\end{aligned}$$

where $\mathcal{S}_n$ is the symmetrization operator and $|N/2, -N/2\rangle$ the ground state of the atomic ensemble.

Since all the operations are proceeding in the subspace of $J = N/2$, we abbreviate the eigenfunction of the operators $\{J^2, J_z\}$ as

$$|m\rangle \equiv \left|\frac{N}{2}, -\frac{N}{2} + m\right\rangle, \; m = 0, 1, 2, \ldots, N \tag{A.4}$$

which satisfy

$$J_+ |m\rangle = \sqrt{(N-m)(m+1)}\, |m+1\rangle, \tag{A.5}$$

$$J_- |m\rangle = \sqrt{m(N-m+1)}\, |m+1\rangle, \tag{A.6}$$

$$J_z |m\rangle = \left(-\frac{N}{2} + m\right) |m\rangle. \tag{A.7}$$

It should be noted that we re-marked the eigenstate of the total angular momentum, with the excitation number m of the nuclear spins.

The rotating operator corresponding to a rotation about the y-axis reads as

$$R_y(\theta) = \exp[-i\theta J_y]. \tag{A.8}$$

Since J^2 commutes with the rotation operators (i.e. $[J^2,\ J_y] = 0$) and hence the subspace of $J = N/2$ is an invariant subspace of R_y. After a unitary transformation via equation (A.8), we get the rotated operators

$$J'^2 = R_y(\theta)\, J^2 R_y^\dagger(\theta) = J^2, \tag{A.9}$$

$$J'_z = R_y(\theta)\, J_z R_y^\dagger(\theta) = \cos\theta J_z + \sin\theta J_x, \tag{A.10}$$

and the rotated eigenstate

$$|\theta, m\rangle = R_y(\theta)\, |m\rangle = \sum_l d^{N/2}_{l,m}\, |l\rangle, \tag{A.11}$$

with the element of the Wigner's (small) d-matrix

$$d^{N/2}_{l,m}(\theta) = [m!(N-m)!l!\,(N-l)!]^{\frac{1}{2}} \times \sum_k \frac{(-)^k \left(\cos\frac{\theta}{2}\right)^{N+m-l-2k} \left(-\sin\frac{\theta}{2}\right)^{l-m+2k}}{(N-l-k)!\,(m-k)!\,(k+l-m)!k!}, \tag{A.12}$$

where $k = 0, 1, 2, \ldots$ and keep all the factorials non-negative.

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