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# Decoherence in time evolution of bound entanglement

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**Abstract.** We study a dynamic process of disentanglement by considering the time evolution of bound entanglement for a quantum open system, two qutrits coupling to a common environment. Here, the initial quantum correlations of the two qutrits are characterized by the bound entanglement. Both bosonic and spin environments are considered. We found that the bound entanglement displays collapses and revivals, and it can be stable against small temperature and time change. The thermal fluctuation effects on bound entanglement are also considered.

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## 1 Introduction

Entanglement [1], as an essential feature of quantum mechanics, helps us to distinguish the classical and quantum nature of matter world. It has become a key ingredient in quantum information processing, such as quantum computing, quantum teleportation and quantum cryptography [2–4]. On the other hand, generally a realistic system is surrounded by an environment. Thus the effects of the quantum decoherence such as quantum dephasing on quantum entanglement should be considered for quantum open systems. It is reasonable that when we study quantum effects induced by entanglement, the two-particle system should hold phase relations between the components of the entangled states. Thus perceivably, due to the interactions with environment, we can expect that the dephasing of two-particle system can demonstrate some exotic properties.

Recently, Yu and Eberly [5] showed that two entangled qubits became completely disentangled in a finite time under the influence of pure vacuum noise. Surprisingly, they found that the behaviors of local decoherence is different from the spontaneous disentanglement. The decoherence effects take an infinite time evolution under the influence of vacuum noise while the entanglement displays a "sudden death" in a finite time. In their investigations and other studies on disentanglement in open quantum systems [6,7], only qubit systems are considered. Here, the disentanglement process is characterized by time evolution of the concurrence [8]. It is well-known that concurrence cannot be calculated analytically in higher dimensions.

For the systems with spins larger than 1/2, one can use the positive partial transpose (PPT) criterion [9] to study disentanglement. For the mixed states of two spin halves and (1/2, 1) mixed spins, the PPT method can fully characterize entanglement. However, in the case of two qutrits and even larger spins, one only know that if a state does not have PPT, the state must be entangled. In other words, one may use the method to witness entanglement. Actually, in the case of higher dimension, there are two qualitatively different types of entanglement [10], free entanglement (FE) which corresponds to the states without a PPT, and bound entanglement (BE) corresponding to the entangled states, however, with a PPT. The BE is an intrinsic property and cannot be distilled to a singlet form, thus it cannot be used alone for quantum communication. Nevertheless, the BE can be activated and then contribute to quantum communication [11]. A formal entanglement-energy analogy [12] implies that the bound entanglement is like the energy of a system confined in a shallow potential well. If we add a small amount of extra energy, behaving as a perturbation, to the system, its energy can be deliberated. The existence of bound entangled states reveals a transparent form of irreversibility in entanglement processing [13].

In this paper, we consider an open composite system, a two-qutrit system commonly coupled to an environment, and study a type of dynamical process of disentanglement, where the two qutrits are initially prepared in a

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bound entangled state. We would like to reveal that different environment gives different dynamics of entanglement. Firstly, a bosonic heat bath is considered. We remark that this modeling of environment is universal [14,15] in the sense that any environment weakly coupled to a system can be approximated by a collection of harmonic oscillators. Secondly, we consider a spin environment consisting of spin halves which can be considered as a fermionic environment. We let two types of bound entangled states being initial state of the two qutrits in order to find the different properties of bound entangled state during the quantum dephasing. And initially the environments are assumed to be at thermal equilibrium states, which helps us to find effects of the thermal fluctuation on dynamics of quantum entanglement.

This paper is organized as follows. In Section 2, we consider the bosonic environment and give the analytical results of FE and BE. we numerically study the BE to illustrate the details of the dynamics of entanglement. In Section 3, the two qutrits are coupled to a spin environment. Also the analytical and numerical results are given to show the effects of coupling strength, temperature and energy spectrum structure on the dynamical behaviors of entanglement. The conclusion is given in Section 4.

## 2 Bound entanglement in a bosonic environment

We start with a well-known model of the pure dephasing [15,16], where two qutrits interact with the environment, which is modeled as a heat bath with many harmonic oscillators of frequency $\omega_j$. The model Hamiltonian reads

$$H = \sum_j^L H_j = \sum_j^L \left[\hbar\omega_j b_j^\dagger b_j + g(b_j^\dagger + b_j)(S_{1z} + S_{2z})\right], \tag{1}$$

where $b_j^\dagger$ and $b_j$ are creation and annihilation operators, respectively, $S_{1z}$ and $S_{2z}$ are $z$ components of two spin-1 operators, and $g$ denotes the coupling strength between the spins and the heat bath.

In order to study the dynamical process of entanglement in our system, it is convenient for us to study the time evolution in the interaction picture. Here,

$$H_0 = \sum_j \hbar\omega_j b_j^\dagger b_j \tag{2}$$

is the free Hamiltonian, and the interaction Hamiltonian

$$H_I = \sum_j g(b_j^\dagger + b_j)(S_{1z} + S_{2z}). \tag{3}$$

Then, through the Wei-Norman method [17], the time evolution operator in the interaction picture is factorized as,

$$U(t) = \prod_j e^{i\Phi_j(t)S_z^2} D\left[z_j(t)S_z\right], \tag{4}$$

where $S_z = S_{1z} + S_{2z}$,

$$\Phi_j(t) = \frac{g^2}{\hbar^2\omega_j^2}\left(\omega_j t - \sin\omega_j t\right), \tag{5}$$

$$z_j(t) = \frac{g}{\hbar\omega_j}\left(1 - e^{i\omega_j t}\right) \tag{6}$$

and $D(z_j S_z) = \exp\left[\left(z_j b_j^\dagger - z_j^* b_j\right) S_Z\right]$ is the displacement operator.

Before discussing the dynamical process of entanglement, we introduce two quantities to quantitatively study entanglement. One is the negativity [18], which can be used to study FE. For a state $\rho$, negativity is defined in terms of the trace norm of the partial transposed matrix

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_1}\| - 1}{2}, \tag{7}$$

where $T_1$ denotes the partial transpose with respect to the first subsystem. If $\mathcal{N} > 0$, then the two-spin state is free entangled. As an entanglement measure, the negativity is operational and easy to compute, and it has been used to study entanglement behavior in large spin systems [19–22].

In order to characterize BE, one can use the so-called realignment criterion (cross-norm criterion) which proved to be very efficient [23]. The operation of realignment on the density matrix is just as $(\rho^R)_{ij,kl} = \rho_{ik,jl}$. A separable state $\rho$ always satisfies $||\rho^R|| \leq 1$. Thus, a quantity for the BE can be defined as

$$\mathcal{R}(\rho) = \max\left\{0, ||\rho^R|| - 1\right\}. \tag{8}$$

We call this the entanglement witness-like quantity for BE. When the negativity vanishes the positive value of $\mathcal{R}$ can quantify the nontrivial BE. However, since the realignment criterion is a necessary criterion for separability. Here it should be noted that, in this paper we only consider the special kind BE which can be detected by $\mathcal{R}$. The realignment criterion is simple and computable, and it has shown powerful ability to identify most bound entangled state discussed in the literature. Thus, the quantity $\mathcal{R}$ can give us some useful information about the properties of bound entangled states. Furthermore, as a byproduct of the criterion, one can estimate the degree of entanglement of the quantum states by use of the quantity $\mathcal{R}$ [23].

### 2.1 Horodecki's bound entangled state

In the following discussions, we consider the dynamical evolution process of the two-spin 1 system, derived by the Hamiltonian (1) with the initial state being in the Horodecki's bound entangled state [11].

#### 2.1.1 Analytical results

The bound entangled state reads [11]:

$$\rho_a(0) = \frac{2}{7}\mathbf{P}_+ + \frac{a}{7}\varrho_+ + \frac{5-a}{7}\varrho_-, \tag{9}$$
$$2 \leq a \leq 5,$$

where

$$\mathbf{P}_+ = |\Psi_+\rangle\langle\Psi_+|, \tag{10}$$
$$|\Psi_+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle),$$
$$\varrho_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|),$$
$$\varrho_- = \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|) \tag{11}$$

where $|m_1 m_2\rangle, (m_1, m_2 = 0, 1, 2)$ are the eigenvectors of $S_z = S_{1z} + S_{2z}$, with the corresponding eigenvalues $m_1 + m_2 - 2$, respectively.

In reference [10], Horodecki demonstrated that

$$\rho_a \text{ is } \begin{cases} \text{separable for } 2 \le a \le 3, \\ \text{bound entangled for } 3 < a \le 4, \\ \text{free entangled for } 4 < a \le 5. \end{cases} \tag{12}$$

And the density matrix for the initial state of the total system is a simple direct product

$$\rho_{\text{tot}}(0) = \rho_a \otimes \rho_E, \tag{13}$$

where $\rho_E$ is the density matrix of environment.

Driven by the time evolution operator (4), the system will evolve from the bound entangled state $\rho_a$ into the state described by

$$\begin{aligned} \rho_{1,2}(t) &= \mathrm{Tr}_E\left[U(t)\,\rho_{\text{tot}}(0)\,U^\dagger(t)\right] \\ &= \frac{2}{21}\big[\,(|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22|) \\ &\quad + (F_1(t)\,|00\rangle\langle 11| + \text{H.c.}) \\ &\quad + (F_2(t)\,|22\rangle\langle 11| + \text{H.c.}) \\ &\quad + (F_3(t)\,|00\rangle\langle 22| + \text{H.c})\,\big] \\ &\quad + \frac{a}{7}\varrho_+ + \frac{5-a}{7}\varrho_- \end{aligned} \tag{14}$$

where

$$\begin{aligned} F_1(t) &= \mathrm{Tr}_E\left[\rho_E U_1^\dagger(t)\,U_0(t)\right] \\ F_2(t) &= \mathrm{Tr}_E\left[\rho_E U_1^\dagger(t)\,U_2(t)\right] \\ F_3(t) &= \mathrm{Tr}_E\left[\rho_E U_2^\dagger(t)\,U_0(t)\right] \end{aligned} \tag{15}$$

are *decoherence factors* [15]. The unitary operators $U_0(t)$, $U_1(t)$, and $U_2(t)$ are derived from equation (4) just by replacing operator $S_z = S_{1z} + S_{2z}$ with numbers $-2, 0$ and 2, respectively.

From the reduced density matrix (14), the realigned matrix becomes

$$(\rho_{12}(t))^R = \frac{1}{21}\left(A_{3\times 3} \oplus B_{2\times 2}^{(1)} \oplus B_{2\times 2}^{(2)} \oplus B_{2\times 2}^{(3)}\right),$$
$$A_{3\times 3} = \begin{pmatrix} 2 & a & 5-a \\ 5-a & 2 & a \\ a & 5-a & 2 \end{pmatrix},$$
$$B_{2\times 2}^{(k)} = \begin{pmatrix} 2F_k & 0 \\ 0 & 2F_k^* \end{pmatrix} (k = 1, 2, 3). \tag{16}$$

Then, the witness-like quantity $\mathcal{R}$ is obtained as

$$\begin{aligned} \mathcal{R}(\rho_{1,2}) &= \max\left\{||\rho_{1,2}^R(t)|| - 1, 0\right\} \\ &= \frac{2}{21}\max\Big\{\sqrt{3a^2 - 15a + 19} \\ &\qquad + 2(|F_1| + |F_2| + |F_3|) - 7, 0\Big\}. \end{aligned} \tag{17}$$

As mentioned above, the positive witness-like quantity can quantify the nontrivial BE only when the negativity vanishes. Thus, we need to calculate the time evolution of negativity.

We first make the partial transpose of $\rho_{12}$ with respect to the second system and obtain

$$(\rho_{12}(t))^{T_2} = \frac{1}{21}\left(C_{3\times 3} \oplus D_{2\times 2}^{(1)} \oplus D_{2\times 2}^{(2)} \oplus D_{2\times 2}^{(3)}\right),$$
$$C_{3\times 3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
$$D_{2\times 2}^{(k)} = \begin{pmatrix} a & 2F_k \\ 2F_k^* & 5-a \end{pmatrix} (k = 1, 2, 3). \tag{18}$$

Then, from the above equation, we immediately obtain the negativity

$$\mathcal{N}(\rho_{1,2}) = \frac{1}{42}\sum_{k=1}^{3}\max\left\{0, \sqrt{(2a-5)^2 + 16|F_k|^2} - 5\right\}. \tag{19}$$

Thus, we have obtained analytical expressions of quantity $\mathcal{R}$ and negativity $\mathcal{N}$ in terms of the three decoherence factors. It is natural to see that if the decoherence factors are zero, namely, the completely decoherence occurs, from equations (17) and (19), we have $\mathcal{R} = \mathcal{N} = 0$. From equation (19), we can also see that in the region $3 < a \le 4$, negativity always gives zero at any time since $|F_k| \le 1$.

From the above discussions, once we know the decoherence factors, the two quantities $\mathcal{R}$ and $\mathcal{N}$ for detecting entanglement can be determined. So, we are left to obtain these decoherence factors. It is well-known that high temperature may enhance the decoherence, thus it is reasonable to choose a thermal equilibrium state as the initial state of the heat bath, which is described by the density matrix

$$\begin{aligned} \rho_E &= \prod_j \rho_{Ej} = \prod_j \frac{e^{-\beta\hbar\omega_j b_j^\dagger b_j}}{\mathrm{Tr}\left[e^{-\beta\hbar\omega_j b_j^\dagger b_j}\right]} \\ &= \prod_j \left(1 - e^{-\beta\hbar\omega_j}\right) e^{-\beta\hbar\omega_j b_j^\dagger b_j}, \end{aligned} \tag{20}$$

where $\beta = 1/k_B T$, $k_B$ is the Boltzmann's constant, and we choose $k_B = 1$ for simplicity in the following.

For the bosonic environment we calculate the decoherence factors in the coherent-state representation. The $P$-representation for the thermal state is given by

$$\rho_E(0) = \prod_j \rho_{Ej} = \prod_j \int \rho_j(\alpha) |\alpha\rangle \langle\alpha| \, d^2\alpha, \tag{21}$$

$$\rho_j(\alpha) = \frac{1}{\pi \langle n_j \rangle} \exp\left(-\frac{|\alpha|^2}{\langle n_j \rangle}\right), \tag{22}$$

where $\langle n_j \rangle = \left(e^{\beta\hbar\omega_j} - 1\right)^{-1}$ is the thermal excitation number of harmonic oscillators. From equation (15) and using the $P$-representation, one obtains the modulus of the decoherence factors [24]

$$\begin{aligned} |F_1(t)| = |F_2(t)| &= \prod_j |\mathrm{Tr}_{E_j}[\rho_{E_j} D(2z_j)] e^{i4\Phi_j}| \\ &= \prod_j e^{-2|z_j|^2 (2\langle n_j \rangle + 1)} \\ &= \exp\left[\sum_j \frac{-8g^2}{\hbar^2 \omega_j^2} (2\langle n_j \rangle + 1) \sin^2\left(\frac{\omega_j t}{2}\right)\right], \end{aligned} \tag{23}$$

$$|F_3(t)| = \prod_j \mathrm{Tr}_j(\rho_{Bj} D(-4z_j)) = |F_1(t)|^4. \tag{24}$$

As expected, the above three factors are smaller than or equal to unity. Now, we study the decoherence of BE, and choose parameter $a = 4$ in the bound entangled state in the following discussions. This choice of parameter maximize the quantity $\mathcal{R}$. Then, equation (17) simplifies to

$$\mathcal{R}(\rho_{1,2}) = \frac{2}{21} \max\left\{0, 2\left(2|F_1| + |F_1|^4\right) + \sqrt{7} - 7\right\}. \tag{25}$$

Then, we find that the dynamic properties of BE is thus directly related to the one single decoherence factors $|F_1(t)|$. By numerical calculation, one obtains the threshold point of

$$|F_1| \approx 0.839829, \tag{26}$$

before which the quantity $\mathcal{R}$ is larger than zero, and which shows a sign of BE. From equation (23), we note that the decoherence factors can be considered as quasi-periodic functions and the periodic properties lie on the distribution of the frequencies.

In the following we will choose some special spectrum distributions of the environment in order to study the dynamical behavior of the decoherence factor analytically. In some cases we will find that the decoherence factors decay as a Gaussian or a exponential form with time, and consequently we know the dynamical behaviors of the quantity $\mathcal{R}$.

$(i)$ Let us choose the frequency $\omega_j$ in the region $[0, \omega]$, where $\omega$ is an arbitrary value larger than zero. In the following part, we will do some approximate analysis in order to obtain a compact function of the decoherence factor, which will present the decay behavior of the factor clearly. Since the frequency starts from zero, it can be achieved for us to choose a sufficiently small cutoff frequency $\omega_{j_c}$ to make sure that at a finite time, the decoherence factor ($\hbar = 1$)

$$\begin{aligned} |F_1(t)| &\leq \exp \sum_j^{j_c} \left[\frac{-8g^2}{\omega_j^2} \sin^2\left(\frac{\omega_j t}{2}\right) (2\langle n_j \rangle + 1)\right] \\ &\approx \exp\left[-2g^2 \sum_j^{j_c} (2\langle n_j \rangle + 1) t^2\right] = e^{-\gamma t^2}, \end{aligned} \tag{27}$$

where

$$\gamma = 2g^2 \sum_j^{j_c} (2\langle n_j \rangle + 1). \tag{28}$$

It can be seen that the decoherence factor displays a Gaussian decay with time. Moreover one may observe that the decay parameter $\gamma$ increases at high temperature since $\langle n_j \rangle$ is a monotonically increasing function of temperature $T$, and enlarging the strength $g$ can also increase $\gamma$.

On the other hand, this Gaussian decay of the factors is a generic behavior for short times, and which is independent of the energy distribution of the environment. Hornberger has given a discussion about the short time case [25].

Substituting equation (27) to (25) leads to

$$\mathcal{R}(\rho_{1,2}) = \frac{2}{21} \max\left\{0, 2\left(2e^{-\gamma t^2} + e^{-4\gamma t^2}\right) + \sqrt{7} - 7\right\}, \tag{29}$$

and it will decay to zero in a fixed time $t_0$, which can be determined from equation (26)

$$t_0 = 0.4176/\sqrt{\gamma}. \tag{30}$$

When the evolution time is larger than the threshold value $t_0$, the BE which is detected by $\mathcal{R}$ suddenly vanishes.

$(ii)$ If we choose some continuous spectrum, the sum in the decoherence factors (we assume $\hbar = 1$) becomes

$$\ln|F_1(t)| = -\sum_j \left[\frac{8g_j^2}{\omega_j^2} \sin^2\left(\frac{\omega_j t}{2}\right)\right], \tag{31}$$

where $g_j = g\sqrt{(2\langle n_j \rangle + 1)}$. Assume a spectrum distribution $\rho(\omega_j)$, the above equation becomes

$$\ln|F_1(t)| = -\int_0^\infty \frac{8\rho(\omega_j) g_j^2}{\omega_j^2} \sin^2 \frac{\omega_j t}{2} d\omega_j. \tag{32}$$

For some concrete spectrum distributions, interesting circumstances may arise. For instance, when $\rho(\omega_j) = \gamma/(2\pi g_j^2)$ the integral converges to a negative number proportional to time $t$, precisely, $|F_1(t)| = \exp(-\gamma t)$, $|F_3(t)| = \exp(-4\gamma t)$. Thus, in this case, the reasonable assumption on the energy distribution brings us a exponential decay of decoherence factor with time. Thus $\mathcal{R}$ will decay and tend to zero in this case.

$(iii)$ Now we will choose another more general distribution $\rho(\omega)$. Assume that all the coefficients $g_j$ are equal: $g_j = G$. If the frequencies lie within an interval $[\omega_1, \omega_2]$ and the distribution is homogeneous, we have $\rho(\omega_k) = L/(\omega_2 - \omega_1)$, thus [26]

$$\begin{aligned}
\ln |F_1(t)| &= -\sum_j \left[ \frac{8g_j^2}{\omega_j^2} \sin^2 \frac{\omega_j t}{2} \right] \\
&= -\int_{\frac{\omega_1}{2}}^{\frac{\omega_2}{2}} \sin^2(\omega_k t) \frac{2G^2 \rho(\omega_k)}{\omega_k^2} d\omega_k \\
&= \frac{-2G^2 L}{\omega_2 - \omega_1} \int_{\frac{\omega_1}{2}}^{\frac{\omega_2}{2}} \frac{\sin^2 \omega_k t}{\omega_k^2} d\omega_k \\
&\leq \frac{-2G^2 L}{\omega_2 - \omega_1} \frac{4}{\omega_2^2} \int_{\frac{\omega_1}{2}}^{\frac{\omega_2}{2}} \sin^2(\omega_k t) \, d\omega_k \\
&= \frac{-2G^2 L}{\omega_2^2} \left[ 1 - \frac{2\cos\left(\frac{\omega_2+\omega_1}{2} t\right) \sin\left(\frac{\omega_2-\omega_1}{2} t\right)}{(\omega_2 - \omega_1)\, t} \right].
\end{aligned} \tag{33}$$

By substituting the above equation into equation (25), we see that when the environment has sufficiently large size $L$, the decoherence factor and the quantity $\mathcal{R}$ will decay with time rapidly.

### 2.1.2 Numerical results

From the analytical results, we find that the quantity $\mathcal{R}$ can decay with time to zero in some cases, which means the BE detected by $\mathcal{R}$ will vanish in a finite time in these cases. Next, in this section we resort to numerical calculation.

Firstly we should start from the decoherence factor in equation (23). Now taking into account a finite interval of frequencies one should consider two time scales: the first one is $t_p = 2\pi/\bar{\omega}$, where $\bar{\omega}$ is the mean frequency of the interval, and this time scale will roughly give the period of the oscillation of the sum in the exponent. The second time scale is given as $t_r = 2\pi/\Delta\omega$, where $\Delta\omega$ is the width of the frequency interval. And all the terms in the exponent tend to completely randomize at the time $t_r$, which means that all oscillations are strongly suppressed around the time $t_r$. Therefore, revivals of the decoherence factors can be expected whenever $t_r/t_p > 1$. On the other hand, the revivals will be suppressed whenever $t_r/t_p < 1$. Easily, one can find that when the frequency interval starts from zero, the condition $t_r/t_p < 1$ can be satisfied.

In Figure 1, we choose a random distribution of the environment energy $\omega_j$ over a finite region $[0, \omega]$, and show the dynamical behaviors of the quantity $\mathcal{R}$. A Gaussian decay is exhibited, and which is consistent with the analytical discussion. In this case, the BE (detected by $\mathcal{R}$) occurs complete disentanglement in finite time. Secondly, we can see that increasing the coupling $g$ can accelerate the decay speed of the $\mathcal{R}$ just as expected in the analytical discussion. From the physical point of view, we can interpret

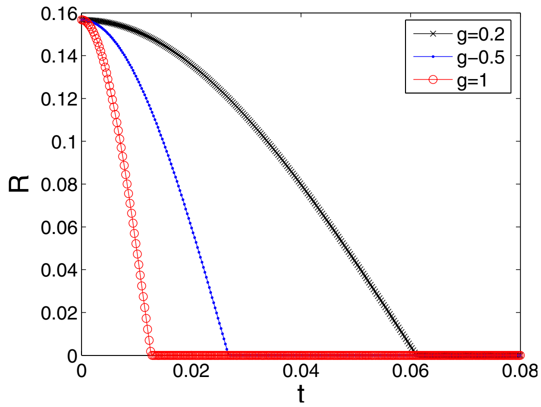


**Fig. 1.** (Color online) $\mathcal{R}$ versus time with different coupling parameter $g$. The frequencies of heat bath are chosen randomly in a low region $\omega_j \in [0, 5]$. The size of bath $L = 200$ and the system is at a finite temperature $T = 1$.

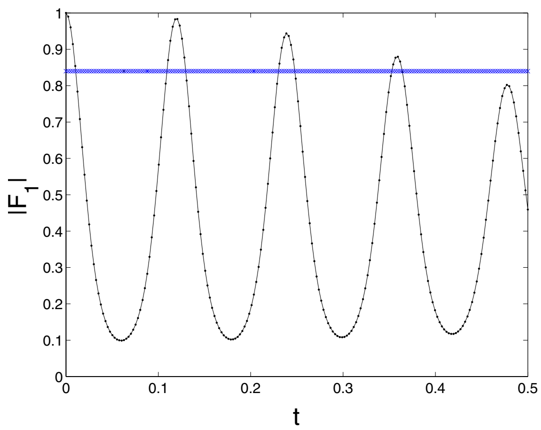


**Fig. 2.** The modulus of the decoherence factor $|F_1(t)|$ versus time with $g = 2$. The frequencies of heat bath are chosen randomly in a region $\omega_j \in [50, 55]$. The size of the bath is $L = 200$ and the system is at the temperature $T = 1$. The horizontal line in the figure corresponds to the threshold value $|F_1| = 0.8398$.

why coupling to the environment will destroy the entanglement: the interaction between the qutrit system and the environment drives the total system to be entangled, and entanglement has an exclusive quality [27], which means that the entanglement between the qutrits and the environment will destroy the entanglement between the two qutrits, thus stronger coupling is more effective to destroy the entanglement.

In Figure 2 we numerically show the modulus $|F_1(t)|$ versus time. The frequencies $\omega_j$ are chosen randomly in a region $\omega_j \in [50, 55]$, in which the condition $t_r/t_p > 1$ is satisfied. Then, the modulus $|F_1(t)|$ oscillates with time and periodically crosses the horizontal line corresponding to the threshold value $F_1 = 0.8398$. Obviously, the witness-like quantity $\mathcal{R}$ in equation (25) displays discontinuous behavior and below the line it becomes zero.

Consequently, in Figure 3, we numerical calculate $\mathcal{R}$ with the energy $\omega_j$ randomly in the region $[50, 50 + \delta]$, where $\delta$ is the width of the distribution. In this frequency region the time scale ratio $t_r/t_p > 1$ and which will in-

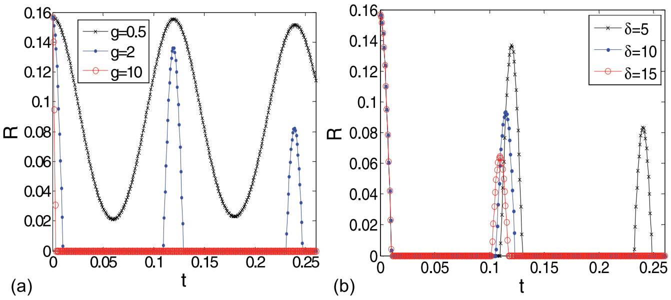


**Fig. 3.** (Color online) (a) $\mathcal{R}$ versus time with different coupling parameter $g$. The frequencies of heat bath are chosen randomly in a higher region $\omega_j \in [50, 55]$. The size of the bath $L = 200$ and the system is at the temperature $T = 1$. (b) We consider the frequency distribution randomly in $[50, 50+\delta]$, the coupling $g = 2$ and the system is at $T = 1$.

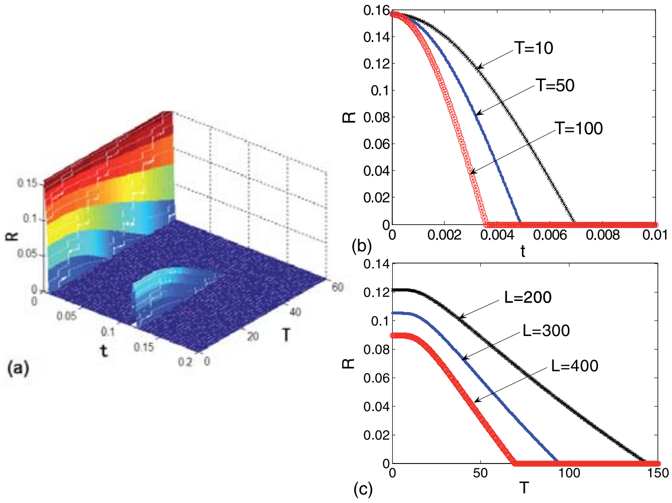


**Fig. 4.** (Color online) (a) Three dimensional (3D) diagram of $\mathcal{R}$ versus time and temperature, with $L = 200$ and $g = 3$. The region of frequencies is $\omega_j \in [50, 58]$. (b) Quantity $\mathcal{R}$ versus time at different temperatures. (c) $\mathcal{R}$ versus temperature at a fixed time $t = 0.003$ for different sizes.

troduce oscillations of the decoherence factors. For small values of the coupling constants, as shown in Figure 3a, $\mathcal{R}$ displays cyclic evolution and the maximum achievable level of $\mathcal{R}$ is reduced. When the coupling strength is strong enough, $\mathcal{R}$ decays rapidly to zero without revivals. Moreover, extended the time region, it remains zero. In this case the BE which can be detected by $\mathcal{R}$ is completely destroyed.

Figure 3b presents numerical results for different frequency widths. From the figure, we see that when the frequency width increases, the revival amplitude decreases. It can be understood as that increasing the width $\delta$ can decrease the time scale ratio $t_r/t_p$, and the randomized frequencies will suppress the periodic oscillation of the factors.

We can also consider the thermal effect on the BE. Figures 4a and 4b show that the thermal fluctuation can destroy entanglement and accelerate the decaying process of $\mathcal{R}$. With the joint effect of thermal fluctuation and strong coupling $g$, $\mathcal{R}$ vanishes in a finite time without reviving. In Figure 4c, we can see that with the temperature increasing $\mathcal{R}$ decreases to zero, and enlarging the size of heat bath can suppress $\mathcal{R}$ and accelerate the decay speed. Generally, thermal fluctuation is harmful in holding quantum entanglement [28], and here thermal fluctuation also suppress the BE which is detected by the quantity $\mathcal{R}$.

### 2.2 Second bound entangled state

We choose another $3 \times 3$ bound entangled state as the initial state of the two qutrits which was introduced by Bennett et al. [29] from the unextendible product bases:

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle),$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle,$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle),$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle) - |2\rangle)|0\rangle,$$

$$|\phi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle), \qquad (34)$$

from which the density matrix could be expressed as

$$\rho = \frac{1}{4}\left(I_{9\times 9} - \sum_{j=0}^{4}|\phi_j\rangle\langle\phi_j|\right). \qquad (35)$$

In this case, the dynamics of entanglement are determined by six decoherence factors, and analytical results are not available. We numerically calculate $\mathcal{R}$ and $\mathcal{N}$, and the results are shown in Figure 5.

In order to compare BE and FE, we have numerically given the time behaviors of both quantity $\mathcal{R}$ and negativity $\mathcal{N}$. We choose a frequency region $\omega_j \in [50, 50+\delta]$, which will induce periodic properties of $\mathcal{R}$ and $\mathcal{N}$. In Figure 5a, we see that under some condition the negativity is larger than zero, which is different from the case of the first bound entangled state (9). The nonzero negativity implies that the state is free entangled. This phenomena shows us that by choosing some appropriate energy distribution of the environment, the BE and FE can be realized alternately. When the coupling $g$ becomes stronger, as shown in Figure 5b, both $\mathcal{R}$ and $\mathcal{N}$ will tend to zero in finite time at last.

In Figure 5c, we extend the width of frequency region to $\delta = 9$. Expanding the energy region will prevent the revivals of $\mathcal{R}$ and $\mathcal{N}$. In Figure 5d, we show the behaviors of $\mathcal{R}$ and $\mathcal{N}$ against temperature for a fixed time. With the increasing temperature, quantity $\mathcal{R}$ and negativity $\mathcal{N}$ decrease gradually, and at last the thermal fluctuation completely destroys the BE (detected by $\mathcal{R}$) and FE of the two qutrits.

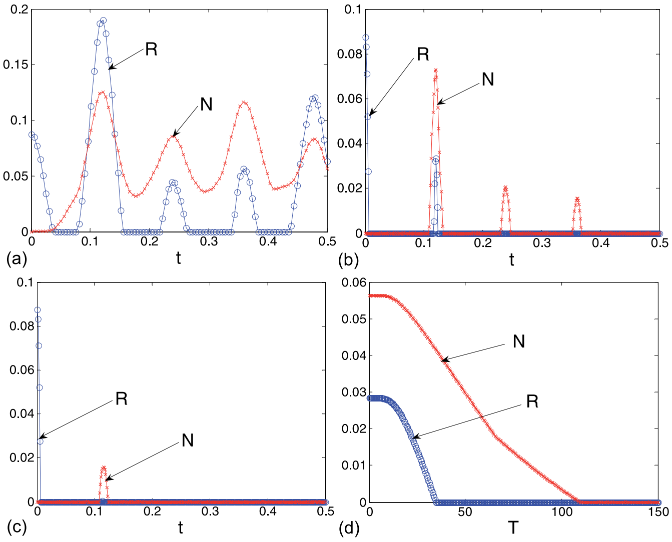


**Fig. 5.** (Color online) We consider the BE and FE together. And in the four subfigures the blue lines with circle markers correspond to $\mathcal{R}$ and the red lines with $\times$ markers denote $\mathcal{N}$. (a) $\mathcal{R}$ and $\mathcal{N}$ versus time with coupling $g = 1$, the size of environment $L = 300$, and the system is at temperature $T = 10$. The frequencies of heat bath are chosen randomly in the region $\omega_j \in [50, 55]$. (b) Only changing the coupling to a larger one $g = 5$, and the other parameters are the same as (a). In (c), parameter $\delta = 9$, the other parameters are the same as (b). In (d), it shows $\mathcal{R}$ versus temperature at $t = 0.005$ and $\mathcal{N}$ versus temperature at $t = 0.115$. $g = 1$ and $L = 300$.

## 3 Bound entanglement in a spin environment

Different environment should cause different decoherence processes with distinct characters for the same qutrit system. In this section, we choose another modeling of environment which consists of $L$ spin halves. The corresponding model Hamiltonian reads [30]

$$H = \frac{g}{2}\left(S_{1z} + S_{2z}\right) \otimes \sum_{k=1}^{L} \omega_k \sigma_z^{(k)}, \tag{36}$$

where $\sigma_z^{(k)}$ denotes the $z$-component of the Pauli vector corresponds to the $k_{th}$ spin half. $g$ and $\omega_k$ together denote the coupling strength between central qutrits and each spin half in the environment. We notice that the above model has been considered by Zurek [30] as a solvable model of decoherence.

The time evolution operator can be expressed as:

$$U\left(t\right) = \prod_{k=1}^{L} \exp(-it\hat{\Lambda}\omega_k \sigma_z^{(k)}), \tag{37}$$

where we define a special operator-valued parameter

$$\hat{\Lambda} = \frac{g}{2}\left(S_{1z} + S_{2z}\right). \tag{38}$$

### 3.1 Horodecki's bound entangled state

In a similar vein as the discussions in the bosonic environment, we first study the disentanglement of Horodecki's bound entangled state (9) and give the analytical results.

#### 3.1.1 Analytical results

Let us consider the whole system initially starts from a product state

$$\rho_{tot}\left(0\right) = \rho_a \otimes \rho_E, \tag{39}$$

where the initial state of the two qutrits $\rho_a$ is a bound entangled state represented in equation (9), and $\rho_E$ denotes the initial state of the environment which is assumed to be a thermal state described by the density matrix

$$\rho_E = \prod_{k=1}^{L} \frac{e^{\beta\omega_k\sigma_z^{(k)}}}{2\cosh(\beta\omega_k)}. \tag{40}$$

Then the reduce density matrix at time $t$ can be given by the same matrix as equation (14), and now the three decoherence factors in this spin environment can be obtained as

$$\begin{aligned} |F_1(t)| = |F_2(t)| &= \prod_{k=1}^{L} |F_{1,k}| \\ &= \prod_{k=1}^{L} \sqrt{1 - \frac{\sin^2(gt\omega_k)}{\cosh^2(\beta\omega_k)}}, \end{aligned} \tag{41}$$

$$\begin{aligned} |F_3(t)| &= \prod_{k=1}^{L} |F_{3,k}| \\ &= \prod_{k=1}^{L} \sqrt{1 - \frac{\sin^2(2gt\omega_k)}{\cosh^2(\beta\omega_k)}}. \end{aligned} \tag{42}$$

From equations (41) and (42) one can find each decoherence factor $|F_k|$ is less than unity, which implies that in the large $L$ limit, $|F_k(t)|$ will go to zero under some reasonable condition. Now, we make some further analysis by introducing a cutoff number $K_c$ similar to the discussion in reference [31]. We define the partial product as

$$|F_1(t)|_c = \prod_{k>0}^{K_c} |F_{1,k}| \geq |F_1(t)|, \tag{43}$$

from which the corresponding partial sum $\ln|F_1(t)|_c \equiv -\sum_{k>0}^{K_c} |\ln F_{1,k}|$. We can do some heuristic analysis in some special conditions such as confining the coupling frequencies are random in a region $\omega_k \in [0, \omega]$. When the cutoff number $K_c$ is small enough, in a finite long time, with some proper $g$ we can pick out some tiny $\omega_k$ to make

$gt\omega_k$ begin a small one and achieve the approximation $\sin^2(gt\omega_k) \approx (gt\omega_k)^2$. At a finite temperature we can have

$$\ln|F_1(t)|_c = \frac{1}{2}\sum_{k>0}^{K_c} \ln\left(1 - \frac{\sin^2(gt\omega_k)}{\cosh^2(\beta\omega_k)}\right)$$
$$\approx -\frac{1}{2}\left(\sum_{k>0}^{K_c} \frac{\omega_k^2}{\cosh^2(\beta\omega_k)}\right) g^2 t^2$$
$$= -\gamma t^2 \tag{44}$$

where

$$\gamma = \frac{1}{2} g^2 \sum_{k}^{K_c} \frac{\omega_k^2}{\cosh^2(\beta\omega_k)}. \tag{45}$$

From equations (43) and (44), we find that the decoherence factors decay in a Gaussian form with time, therefore from equation (25) it is apparent that the witness-like quantity $\mathcal{R}$ will vanish in a finite time. On the other hand, when we consider the short time case, with a cutoff frequency $\omega_{K_c}$, the approximation condition $gt\omega_k \ll 1$ can be satisfied and then we can also obtain the Gaussian decay [25]. In order to study the effect of the temperature, we decrease the temperature quite nearly to $T = 0$ in equation (44), then $\gamma$ approaches zero thus the decoherence factors will not decay with time. It implies that, in this environment the temperature greatly affect the dynamics of BE. It is a rough calculation in our analysis, nevertheless it gives us some constructive results.

### 3.1.2 Numerical results

If the frequency distribution are random in the region $\omega_k \in [0, \omega]$, the decoherence factors displays a Gaussian decay, which is just as the analytical discussion in the former section. And in reference [32], Gaussian decay was shown numerically in various distributions of $\omega_k$ in the region $[0, \omega]$, thus we do not show the numerical result in this case.

Instead we consider a random distribution in $[50, 55]$ which will introduce periodic properties to $\mathcal{R}$. In Figure 6, the oscillation of $\mathcal{R}$ is observed, and the maximum achievable level of $\mathcal{R}$ is reduced. We show $\mathcal{R}$ versus time with different widths $\delta$ of frequency distribution. Larger width $\delta$ will weaken the oscillation of $\mathcal{R}$.

In Figure 7, effects of the thermal fluctuation on the dynamic of BE are studied. The 3D plot (a) displays a flat at low temperatures about $T < 10$, which implies that the $\mathcal{R}$ is stable against temperature and time in this region. When temperature is high enough, $\mathcal{R}$ rapidly decreases to zero. In Figure 7b, we can see explicitly that temperature plays an important role in the dynamics of $\mathcal{R}$ in this spin environment. It is just like a control process that when temperature is higher than some proper value, $\mathcal{R}$ will decay sharply with time. This is a quite different property from the case of bosonic environment. At a very low temperature, from equations (41) and (42), we know that decoherence factors tend to 1 and nearly not change with time. Thus $\mathcal{R}$ is stable at low temperatures. Physically speaking, when the temperature is very low, the spins in environment are almost at the ground state of $\sigma_z^{(k)}$. In this case the time evolution operator (37) can not make the whole system entangled, thus the effect of the environment on the two-qutrit entanglement is weak at low temperature. In Figure 7c, we also plot $\mathcal{R}$ against temperature for different size of environment. As we expected, the larger size of the spin environment accelerates the decay of $\mathcal{R}$.

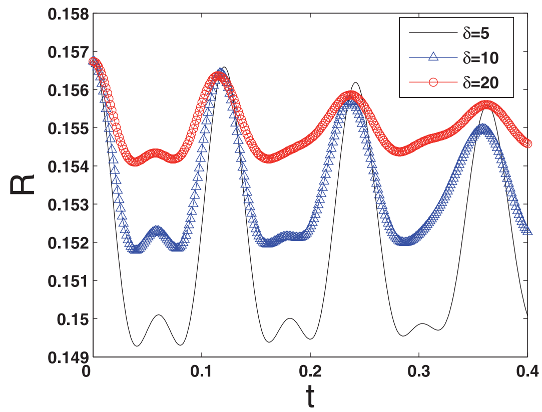


**Fig. 6.** (Color online) $\mathcal{R}$ versus time with different $\delta$ which is the region widths of the couplings $\omega_k \in [50, 50+\delta]$ and $g = 0.5$. The size of the spin environment is $L = 300$ and the system is at the temperature $T = 10$.

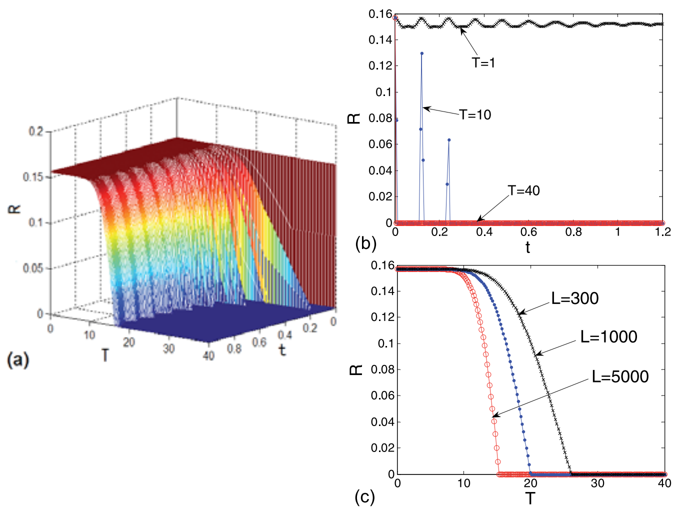


**Fig. 7.** (Color online) (a) Three dimensional (3D) diagram of the $\mathcal{R}$ versus time and temperature, with $L = 300$, $g = 0.5$ and $\omega_k \in [50, 55]$. (b) Several part sections of the 3D diagram. Explicitly, it presents $\mathcal{R}$ versus time at different temperatures of $T = 1$, $T = 10$ and $T = 40$. Panel (c) shows $\mathcal{R}$ decays with temperature at a fixed time $t = 0.005$ with different sizes of environment $L = 300, 1000$, and $5000$.

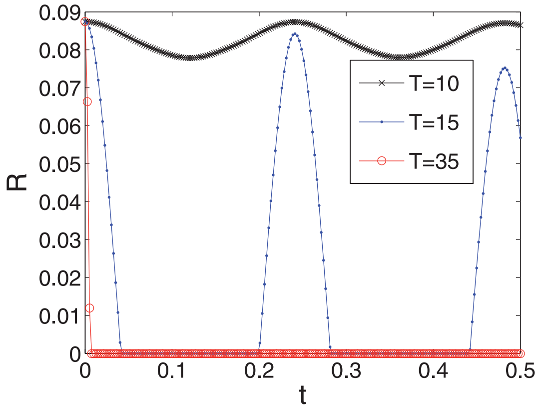


**Fig. 8.** (Color online) $\mathcal{R}$ versus time at different temperatures, $T = 10$, $T = 15$ and $T = 35$. With the coupling $g = 0.5$, environment size $L = 300$ and $\omega_k \in [50, 55]$.

### 3.2 Second bound entangled state

We now consider that the two qutrits initially in the second bound entangled state (35). We choose $\omega_k \in [50, 55]$ and study the finite-temperature effects on entanglement. In Figure 8, we can see the similar phenomena with Figure 6, namely, at low temperature $\mathcal{R}$ only oscillates weakly around the initial value. At higher temperature BE decays sharply with time, and complete destruction and revivals of $\mathcal{R}$ are also observed. We also study negativity, however, it always keeps zero which means there does not exist FE all the time.

## 4 Conclusion

In summary, we have studied the decoherence phenomena of two-qutrit system couple to an environment when the qutrit system is initially prepared in a bound entangled states. Two typical pure dephasing systems, the bosonic and the spin systems are considered in order to show the different dynamical properties of entanglement induced by different environments.

We used the realignment criterion to characterize BE, and the PPT criterion to study FE. Beyond the negativity, we have introduced a novel witness-like quantity $\mathcal{R}$ to study BE. Since the cross-norm criterion is only a necessary criterion of separability, we only studied the bound entanglement which can be detected by $\mathcal{R}$. Two typical bound entangled states are considered as the initial states of the qutrits.

One of our central result is to express the quantity $\mathcal{R}$ and negativity $\mathcal{N}$ in terms of three decoherence factors, and these factors are analytically obtained. In the case of bosonic environments, the Gaussian decay and the exponential decay of $\mathcal{R}$ can be analytically obtained in some cases. And in the spin environment, we also find that $\mathcal{R}$ can display a Gaussian decay under some approximation. In both environments, the collapse and revivals of the BE detected by $\mathcal{R}$ can be observed when we choose the frequency distribution of the environment in a region with the mean frequency $\bar{\omega}$ is larger than the region width $\Delta\omega$. Larger coupling strength, larger environment size and higher temperature will enhance the disentanglement process.

We find that different environment can induce great differences in the time evolution of entanglement. When the two qutrits start from the second bound entangled state, we find the BE and FE appear alternately in the bosonic environment with appropriate frequency region, which is different from the case of spin environment. When the qutrits start from the Horocecki's bound entangled state and in the spin environment, we find that $\mathcal{R}$ keeps stable until the temperature is higher than some proper value, and the remarkable effect of temperature on the dynamical property of entanglement is greatly different from the case of bosonic environment.

Finally we have to point out that since we are lack of entanglement measure for two qutrits, the study of decoherence of entanglement here is incomplete. Nevertheless, the realignment criterion and the PPT criterion are very efficient to characterize the dynamical properties of entanglement. It will be interesting to consider decoherence of BE under other decoherence processes such as dissipation, and investigate the robustness of the BE.

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