Commun. Theor. Phys. (Beijing, China) **32** (1999) pp 521–524

Vol. 32, No. 4, December 15, 1999

Time Evolution for Relative Phase of Two Bose–Einstein Condensates

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(Received November 19, 1998)

Abstract *Considering the collisions between the two condensates and the tunneling effects, we study the time evolution of the relative phase of two Bose–Einstein condensates (BECs). The phase amplitude is given in coherent picture for many cases, which furnishes us a new method to study the interference of two BECs.*

PACS numbers: 42.50.Dv, 05.30.Fk
Key words: relative phase, BEC, interference

The recent experimental realization of a weak-interacting Bose–Einstein condensate[1–3] (BEC) has stimulated various theoretical works on the properties of these condensates. Among them the problem of the phase of an atomic sample has been raised with renewed interest. Theoretically, this phase appears naturally as a result of a broken symmetry in the theory of BEC.[4] The recent study, however, shows that the spontaneous symmetry breaking is unnecessary for the explanation of interference of two BECs.[5–8] Instead, a measurement looking for the interference of two condensates will find the characteristic consequences of the phase, even if there is no phase in the initial state of system.[5,7,9–13]

Tong *et al.*[12] extended the work of Javanainen and Yoo to study the relative phase between two BECs including the effects of collisions (except those between the two condensates). The results show that after many detections the relative phase is very precisely defined and the coherence (built up by the detections) is increasingly degraded with more and more atoms being detected, the later phenomenon is caused by atomic collisions. The evolution of condensates was considered there by using an ***effective Hamiltonian***, which is non-Hermitian. Yvan Castin *et al.*[11] developed an approach to the problem of relative phase of two macroscopic entities that is based on microscopic measurements, the time evolution of the phase distribution shows that it is difficult to establish a long-lived phase coherence between two condensates. Using continuous measurement theory, the dynamics in a single run of an interference experiment between two BECs prepares a state with relative fixed phase, and the relative phase is directly reflected in the spatial distribution of the interference pattern, moreover, the measurement of the position of an atom gives information about the relative phase. Most of these studies, however, have neglected the collisions and tunneling effects between the two condensates, which may occur when the two condensates overlap.[14]

In this paper, we study the time evolution of relative phase between two condensates including the collisions between two condensates and tunneling effects. As shown in the following, they will result in a new conclusion for the interference of two BECs.

The many-body Hamiltonian describing atomic BEC in a double well potential $V(r)$ is[15]

$$H = \int \mathrm{d}^3 r \Big\{ \psi^\dagger(r) \Big[-\frac{\hbar^2 \nabla^2}{2m} + V(r) - \mu \Big] \psi(r) + \frac{U_0}{2} \psi^\dagger(r) \psi^\dagger(r) \psi(r) \psi(r) \Big\}, \tag{1}$$

where m is the atomic mass, $U_0 = 4\pi\hbar^2 a/m$ measures the strength of the two-body interaction, and a is the s-wave scattering length, $\psi(r)$ and $\psi^\dagger(r)$ are the field operators which annihilate and create atoms at position r. In the two-mode approximation, we expend the field operators in terms of local modes and introduce the annihilation and creation operators

$$a_j = \int \mathrm{d}^3 r u_j^*(r) \psi(r), \qquad j = r \text{ (right)}, \quad l \text{ (left)}, \qquad a_j^\dagger = (a_j)^\dagger, \tag{2}$$

which satisfy $[a_j, a_k^\dagger] = \delta_{jk}$ and $u_j(r)$ is the normalized single-particle state in j-potential. The two-mode approximation is valid if the overlap between the modes of opposite wells is very small. Mathematically, this can be stated as follows:

$$\int \mathrm{d}^3 r u_j^*(r) u_k(r) = \delta_{jk} + \epsilon(1 - \delta_{jk}), \tag{3}$$

and $\epsilon \ll 1$. Within the framework of variational principle, it is proved that the single-particle state $u_j(r)$ satisfies equation[16]

$$\Big[-\frac{\nabla^2}{2m} + V(r) - \mu\Big] u_i(r) + 2N_i |u_i(r)|^2 u_i(r) + \sum_{j \neq i} N_j |u_j(r)|^2 u_i(r) = 0. \tag{4}$$

This is the Hartree–Fock equation, the solution of Eq. (4) represents the amplitude of density profile for the atoms in i-potential. Here, we do not deal with the detailed method of solving the HF equation, the detailed discussion was given in Ref. [17]. Substituting $\psi(r) = a_r u_r(r) + a_l u_l(r)$ into Hamiltonian (1), the many-body Hamiltonian reduces to the following two-mode approximation

$$\begin{aligned} H = & \sum_{i,j=r,l} e_{ij} a_i^\dagger a_j + (U_0 u_0/2) a_r^\dagger a_r a_r^\dagger a_r + (U_0 u_0/2) a_l^\dagger a_l a_l^\dagger a_l \\ & + 2U_0 u_1 a_l^\dagger a_l a_r^\dagger a_r + \tfrac{1}{2} U_0 u_2 a_r^\dagger a_l a_r^\dagger a_l + \tfrac{1}{2} U_0 u_2 a_l^\dagger a_r a_l^\dagger a_r, \end{aligned} \tag{5}$$

where $e_{ij} = U_0 \int \mathrm{d}^3 r u_j^*(r)[-(\hbar^2/2m)\nabla^2 + V - \mu] u_j(r)$, $u_0 = U_0 \int \mathrm{d}^3 r u_r^* u_r^* u_r u_r = U_0 \int \mathrm{d}^3 r u_l^* u_l^* u_l u_l$, $u_1 = U_0 \int \mathrm{d}^3 r u_l^* u_l u_r^* u_r$, and $u_2 = U_0 \int \mathrm{d}^3 r u_r^* u_r^* u_l u_l$. Here, we have considered the self-interaction (terms with u_0) within each well and the cross-interaction (terms with u_1 and u_2) between two wells, which have the order of ϵ^2. The Hamiltonian (5) was widely considered to discuss the problems of discrete self-trapping equation,[18] nonlinear optical directional coupler[19] and quantum dynamics of an atomic BEC in a double-well potential.[14]

To begin with, we consider the following Hamiltonian for simplicity

$$H = \sum_{i,j=r,l} e_{ij} a_i^\dagger a_j, \tag{6}$$

which was used to study the coherent quantum tunneling between two BECs in Ref. [20]. Here we borrow it to discuss the time evolution of relative phase between two condensates.

For an initial state $|\psi\rangle$ with a well-defined total number of particles N, the time evolution of relative phase is conveniently analyzed by expanding $|\psi\rangle$ onto the overcomplete set of phase state $|\phi\rangle_N$,[21]

$$|\phi\rangle_N = (1/\sqrt{2^N N!})\,(a_l^\dagger \mathrm{e}^{\mathrm{i}\phi} + a_r^\dagger \mathrm{e}^{-\mathrm{i}\phi})^N |0\rangle, \tag{7}$$

where $|0\rangle$ stands for the vacuum. If the system is in a given state $|\phi\rangle_N$, there exists a well-defined relative phase ϕ between r (right) and l (left).

Any state $|\psi\rangle$ with N particles can be expanded in a set of phase states

$$|\psi\rangle = \int_{-0.5\pi}^{0.5\pi} \frac{\mathrm{d}\phi}{\pi}\, c(\phi) |\phi\rangle_N, \tag{8}$$

where the phase amplitude $c(\phi)$ is given as

$$c(\phi) = 2^{N/2} \sum_{n_r=0}^{N} \Big(\frac{n_r!(N-n_r)!}{N!}\Big)^{1/2} \mathrm{e}^{\mathrm{i}(N-2n_r)\phi} \langle n_r, N - n_r | \psi \rangle. \tag{9}$$

The phase states are complete for large N and for $-0.5\pi \leq \phi < 0.5\pi$, namely,

$${}_N\langle \phi | \phi' \rangle_N = \cos^N(\phi - \phi') \simeq \sqrt{2\pi/N}\, \delta(\phi - \phi').$$

In what follows, we use the phase state $|\phi\rangle_N$ to study the time evolution of the relative phase.

The Hamiltonian (6) can be diagonalized through

$$a_l = -\cos\theta A + \sin\theta B, \qquad a_r = \sin\theta A + \cos\theta B \tag{10}$$

to be

$$H = E_A A^\dagger A + E_B B^\dagger B\,, \tag{11}$$

where $E_A = e_{ll}\cos^2\theta + e_{rr}\sin^2\theta - \sin\theta\cos\theta e_{rl}$ and $E_B = e_{rr}\cos^2\theta + e_{ll}\sin^2\theta - \sin\theta\cos\theta e_{rl}$. It is well known that the mean numbers of atoms in the right well and in the left well are equal. For this reason, we consider the state $|N/2, N/2\rangle = |N/2\rangle_r \otimes |N/2\rangle_l$ as the initial state, in other words, there are $0.5N$ atoms trapped in the left well and the same number in the right well at $t = 0$. It is convenient to calculate the time evolution of relative phase by expanding the initial state into coherent state

$$\left|\frac{N}{2}, \frac{N}{2}\right\rangle = \frac{1}{\pi^2}\int \mathrm{d}^2\alpha \mathrm{d}^2\beta c_{\alpha,N/2} c_{\beta,N/2} |\alpha(t)\rangle_l^c \otimes |\beta(t)\rangle_r^c\,, \tag{12}$$

where $c_{\alpha,N/2} = {}_l^c\langle\alpha|N/2\rangle_l$, $|\alpha\rangle_l^c$ denotes the coherent state with mean number of atoms $|\alpha|^2$ in the left well. With this initial condition, at time t, the state $|N/2, N/2\rangle$ evolves to

$$|\varphi(t)\rangle = \frac{1}{\pi^2}\int \mathrm{d}^2\alpha \mathrm{d}^2\beta c_{\alpha,N/2} c_{\beta,N/2} |\alpha(t)\rangle_l^c \otimes |\beta(t)\rangle_r^c\,, \tag{13}$$

where $\alpha(t) = \cos\theta(-\alpha\cos\theta + \beta\sin\theta)\,\mathrm{e}^{-(\mathrm{i}/\hbar)E_A t} + \sin\theta(\beta\cos\theta + \alpha\sin\theta)\,\mathrm{e}^{-(\mathrm{i}/\hbar)E_B t}$ and $\beta(t) = \sin\theta(-\alpha\cos\theta + \beta\sin\theta)\,\mathrm{e}^{-(\mathrm{i}/\hbar)E_A t} + \cos\theta(\beta\cos\theta + \alpha\sin\theta)\,\mathrm{e}^{-(\mathrm{i}/\hbar)E_B t}$. Equation (13) shows that $|\varphi(t)\rangle$ is a coherent state if the initial state is a coherent state. This property is well used to study the atom-laser coupled form BEC.[16] The phase amplitude $c(\phi, t)$ reads

$$c(\phi,t) = 2^{N/2}\frac{1}{\pi^2}\int \mathrm{d}^2\alpha \mathrm{d}^2\beta \sum_{n=0}^{N}\left(\frac{n!(N-n)!}{N!}\right)^{1/2} \mathrm{e}^{\mathrm{i}(N-2n)\phi}\langle n|\alpha(t)\rangle_l^c\,\langle N-n|\beta(t)\rangle_r^c\,. \tag{14}$$

As shown in Refs [5], [7] and [9]–[13], the relative phase of the condensates is established by measurements. To involve the measurement effect, we treat the measure apparatus as a bath which consists of a set of boson modes. The Hamiltonian describing such a system is given as[22]

$$H = \sum_{ij} e_{ij} a_i^\dagger a_j + \sum_i \hbar\omega_i a_i^\dagger a_i + \sum_{j=r,l}\sum_i \hbar g_{ij}(a_i^\dagger a_j + a_i a_j^\dagger)\,, \tag{15}$$

where a_i is the bath annihilation operator with energy $\hbar\omega_i$. Under the Morkov approximation, the operators a_i can be eliminated, and the Hamiltonian is reduced to

$$H_{\mathrm{eff}} = \sum_{ij} e'_{ij} a_i^\dagger a_j\,, \tag{16}$$

where $e'_{ij} = e_{ij} - \frac{\mathrm{i}}{2}\hbar\gamma$ for $i = j$, and $e'_{ij} = e_{ij}$ for $i \neq j$. The evolution governed by H_{eff} is a nonunitary one and arises because the condensate is being continuously monitored by detector. After m detections, the state vector of the condensates will be[12]

$$|\varphi_m\rangle = \psi(x_m)\,\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}t_m/\hbar}\ldots\psi(x_1)\,\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}t_1/\hbar}|N/2, N/2\rangle\,, \tag{17}$$

where $|N/2, N/2\rangle$ is the initial state and $\{t_1, t_2, \ldots, t_m\}$ is the sequence of time intervals between the detections. $\psi(x)$ is introduced to denote the field operator $\psi(x) = (1/\sqrt{2})(a_l + a_r\,\mathrm{e}^{\mathrm{i}\phi})$ with $\phi(x) = (k_r - k_l)x$, k_r and k_l are the momenta of atom in the right and left wells respectively. This theory was borrowed from the well-known theory of photon detection.[23] It is evident that the state after m detections $|\varphi_m\rangle$ is different from that of Ref. [12], the difference is only from H_{eff}.

These results are obtained in the case of neglecting the atomic collisions. However, the collision terms such as the last five terms in Eq. (5) play a virtual role in recent experiment on BEC.

In order to study the effect of atomic collisions on the time evolution of the relative phase, we rewrite the Hamiltonian in terms of $J_3 = a_l^\dagger a_l - a_r^\dagger a_r$, $J_x = a_l^\dagger a_r + a_r^\dagger a_l$ and $N = a_l^\dagger a_l + a_r^\dagger a_r$,

$$H = \hbar\Omega N + 2\hbar\delta J_3 + 2e_{rl}J_x + \tfrac{1}{2}U_0 u_0 N^2\,. \tag{18}$$

Here the terms proportional to u_1 and u_2 are dropped, since they are smaller than the terms proportional to u_0. It is interesting to find that the Hamiltonian may be diagonalized by a rotation of θ about y axis. Mathematically, it is given that

$$H' = \mathrm{e}^{\mathrm{i}J_y\theta} H \mathrm{e}^{-\mathrm{i}J_y\theta} = \hbar\Omega N + 2\hbar\sqrt{\delta^2 + e_{rl}^2}\, J_3 + \tfrac{1}{2}U_0 u_0 N^2 , \tag{19}$$

where $\theta = \arccos(\delta/\sqrt{\delta^2 + e_{rl}^2})$. It is well known that $[N, J_3] = 0$, thus the eigenstates and eigenvalues of the Hamiltonian H' are then given as

$$|j,m\rangle \equiv \frac{(a_l^\dagger)^{j-m}(a_r^\dagger)^{j+m}}{\sqrt{(j-m)!(j+m)!}}|0,0\rangle , \qquad E_m^j(\theta) = 2j\hbar\Omega + 2m\hbar\sqrt{\delta^2 + e_{lr}^2} + 2U_0 u_0 j . \tag{20}$$

The eigenstates of the Hamiltonian H are then given as

$$|E_m^j(\theta)\rangle = \mathrm{e}^{-\mathrm{i}J_y\theta}|j,m\rangle = \sum_{m'} d_{mm'}^j |j,m'\rangle , \qquad d_{mm'}^j = \langle jm'|\mathrm{e}^{-\mathrm{i}J_y\theta}|jm\rangle , \tag{21}$$

the corresponding eigenvalues are the same as those of H'. Following the same procedure as stated above, we easily arrive at

$$c(\phi,t) = \sum_{m',n} d_{0,m'}^{N/2}(\theta) d_{n,m'}^{N/2}(\theta)\, \mathrm{e}^{-(\mathrm{i}/\hbar)E_m^{N/2}(\theta)t} c(\phi) , \tag{22}$$

where $c(\phi) = 2^{N/2}\left[(0.5N-n)!(0.5N+n)!/N!\right]^{1/2}\mathrm{e}^{2\mathrm{i}n\phi}$.

In summary, we have studied the time evolution of the relative phase of two BECs. We need to point out that there are a lot of papers devoted to this problem in literatures, but most of them neglect the time evolution of the relative phase during every interval of detection or consider it by using effective Hamiltonian. In this sense, the results of this paper provide a new method for studying the interference of two BECs.

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