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Effect of Induced Gauge Field in Jaynes–Cummings Atom with Space-Dependent Coupling[1]

En-Ke TIAN and Chang-Pu SUN
Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, China
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Abstract

The effect of the motion of mass center of a two-level atom on its transition in a quantized radiation field is studied in this paper by generalizing the Jaynes–Cummings model to include the coupling depending on space degrees of freedom. As an additional geometrical phase—Berry's phase in the oscillation, the effect of the induced gauge field resulting from the Born–Oppenheimer (BO) approximation explicitly exists in the transition probability between the up and down states of atomic quasispin. It is shown that when the atom's kinetic energy is small enough that the BO adiabatic approximation holds for the case with small atom's momentum or large detuning, the adiabatically approximate solutions are applicable and it is possible to localize an atom in the cavity.

Due to the remarkable advance in cavity electromagnetics, dynamics of atoms within single-mode cavities (the micromaser),[1] it is reasonable to consider the effect of the mass center motion of the atom on its internal transitions. It is shown in this paper that in a quantized cavity with a very small size comparable with the wavelength of atom's emissions. The space-dependent interaction between the motion of mass center and the internal energy levels should play an indelible role for the dynamics of the atom in a quantized radiation field (QRF). In fact, a number of recent studies have involved this kind of space-dependent effect, e.g., in Refs [2]–[5].

To investigate this problem analytically, we invoke the space-dependent generalization of the Jaynes–Cummings (JC) model[4] in this paper. Notice that the originally defined JC model comprised a single two-level atom interacting with a single-mode quantized radiation field (QRF) and its couplings with QRF are grouped into a space-independent constant by considering the field which is homogeneous at the scale of the atom. However, it is necessary to consider an inhomogeneous cavity field coupling with the internal degrees of freedom according to the recent studies in both the experimental and theoretical aspects.[2–5]

Write the Hamiltonian for our model

$$H = \frac{\hat{P}^2}{2m} + \frac{1}{2}\hbar\omega_0\sigma_z + \hbar[\kappa(x)a^\dagger + \kappa^*(x)a]\sigma_x + \hbar\omega a^\dagger a \tag{1}$$

by resembling the alternative treatment for the JC model in Ref. [3]. Here, σ_x and σ_z are the Pauli matrices for the quasispin and denote the internal degree of freedom of the atom; $a^\dagger$ and $a = (a^\dagger)^\dagger$ are the creation and annihilation operators for the single mode of QRF with frequency ω; $\hat{P}^2/2m$ represents the kinetic energy of the mass center motion, and the coupling $\kappa(x)$ depends on the canonical coordinate x that satisfies $[x, \hat{P}] = i\hbar$. In the rotating-wave approximation neglecting the rapidly-varying terms, the generalized JC Hamiltonian with space-dependent coupling

$$H = \frac{\hat{P}^2}{2m} + \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + \hbar g(x)(e^{-i\phi(x)}a^\dagger\sigma_- + e^{i\phi(x)}a\sigma_+) \equiv \frac{\hat{P}^2}{2m} + H_n(x) \tag{2}$$

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immediately follows the original one (1), where $g(x)e^{-i\phi(x)} = \kappa(x)$, $g(x)$ and $\phi(x)$ are the real functions of x.

Notice that the case we deal with here is quite general and its specific case has been considered for different needs in physical situations in terms of various methods[2,3,5–7] such as the adiabatic approximation. For instance, the $\kappa(x)$ takes the following forms for the specific physical problems. 1) $\kappa(x) = \Omega(x)$ is real for the dynamics of trapping atoms by the vacuum field in a cavity.[5] 2)

$$\kappa(x) = u(x) = \begin{cases} 1 & \text{if } 0 \le x \le l \\ 0 & \text{otherwhere} \end{cases}$$

for reflecting slow atoms by a cavity.[6,7] 3) $\kappa(x) = ke^{ikx}$ for the quantum-nondemolition measurement of atomic momentum by a quantized optical ring cavity.[3] This case was proved to be exactly solvable.[8] It should be pointed out that these studies not only concern the effect of the quasispin atom on the field as well as the usual effect of the field on the quasispin atom, but also take into account the effect of space dependence on both the quasispin atom and the radiation field.

In this paper we present a solution to this general model by making use of the generalized *BO approximation*[9] developed by one of the authors (C.P. SUN). If the coupling $\kappa(x)$ depends on the space slightly or the atom moves slowly enough, we can regard the coordinate x and the quasispin dressed by QRF as the slowly- and rapidly-changing variables respectively in the BO approximation. Firstly, we consider H_n in an invariant scope of Hilbert space spanned by

$$|n, \uparrow\rangle = |n\rangle \otimes |\uparrow\rangle\,, \qquad |n+1, \downarrow\rangle = |n+1\rangle \otimes |\downarrow\rangle\,,$$

where

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle\,, \qquad |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

Solving the eigenvalue problem of $H_n(x)$ with the representation

$$H_n(x) = \hbar\omega\Big(n + \frac{1}{2}\Big)I + H_n'(x)\,, \qquad H_n'(x) = \hbar\Delta\omega \begin{pmatrix} \cos\theta_n & \sin\theta_n e^{-i\phi} \\ \sin\theta_n e^{i\phi} & -\cos\theta_n \end{pmatrix} \tag{3}$$

in the above invariant scope we obtain the eigenstates

$$\begin{aligned} |X_{n,+}(x)\rangle &= \cos\frac{\theta_n}{2}e^{-i\phi}|n,\uparrow\rangle + \sin\frac{\theta_n}{2}|n+1,\downarrow\rangle\,, \\ |X_{n,-}(x)\rangle &= \sin\frac{\theta_n}{2}e^{-i\phi}|n,\uparrow\rangle - \cos\frac{\theta_n}{2}|n+1,\downarrow\rangle \end{aligned} \tag{4}$$

and the corresponding eigenvalues $E_{n,\pm} = \hbar\omega(n+1/2) \pm \hbar\Delta\omega$. Here, I is a unit matrix, and

$$\delta = \frac{1}{2}(\omega_0 - \omega)\,, \qquad \Delta\omega = \sqrt{\delta^2 + g^2(n+1)}\,, \qquad \cos\theta_n = \frac{\delta}{\Delta\omega}$$

depend on x though $g = g(x)$. So long as the adiabatic conditions hold so that the mass center motion does not cause the transition between $|X_{n,+}(x)\rangle$ and $|X_{n,-}(x)\rangle$, the generalized BO approximation can be used to determine the eigenfunctions of the whole Hamiltonian with eigenvalues $E_\pm$ as

$$|\Psi_{n,\pm}(x,\sigma)\rangle = \Phi_\pm(x)|X_{n,\pm}(x)\rangle \tag{5}$$

approximately. Here, the space wavefunctions $\Phi_\pm(x)$ of the mass center satisfy the effective Schrödinger equations with steady states when the coupling $g(x)$ is a constant

$$-\frac{\hbar^2}{2m}[\nabla - iA_\sigma(n)]^2\Phi_\pm = (E_\pm - E_{n,\pm})\Phi_\pm\,, \tag{6}$$

where

$$A_\sigma(n) = i\langle X_{n,\sigma}|\nabla X_{n,\sigma}\rangle = \begin{cases} \nabla\phi(x)\cos^2(\theta_n/2) & \text{if } \sigma=\uparrow, \\ \nabla\phi(x)\sin^2(\theta_n/2) & \text{if } \sigma=\downarrow \end{cases} \tag{7}$$

are the induced gauge potentials that modify the solution of Eq. (6)

$$\Phi_\pm(x) = \frac{1}{\sqrt{2\pi}} e^{i\int_0^x A_\pm(x)dx} e^{ikx} \tag{8}$$

with the Berry's phases

$$\gamma_+ = \int_0^x A_+(x)dx = \int_0^x \cos^2\frac{\theta_n}{2}\nabla\phi dx\,,$$

$$\gamma_- = \int_0^x A_-(x)dx = \int_0^x \sin^2\frac{\theta_n}{2}\nabla\phi dx\,.$$

It should be pointed out that the above solution (5) is valid only when the motion of the atomic mass center does not cause the transition between $|X_{n,+}(x)\rangle$ and $|X_{n,-}(x)\rangle$. This just requires the adiabatic condition

$$\begin{gathered} \left|\frac{\langle X_{n,+}(x)|\hat{P}^2/(2M)|X_{n,-}(x)\rangle}{E_{n,+}-E_{n,-}}\right|, \\ \left|\frac{2\langle X_{n,+}(x)|\hat{P}X_{n,-}(x)\rangle\langle\Phi_+|\hat{P}|\Phi_-\rangle/(2M)}{E_{n,+}-E_{n,-}}\right| \sim \frac{\epsilon}{\hbar\sqrt{\delta^2+g^2(n+1)}} \ll 1\,, \end{gathered} \tag{9}$$

where ϵ is in rough $\hbar^2 f/2M$ where

$$f = \left(\frac{d}{dx}\theta_n\right)^2,\ \left(\frac{d}{dx}\phi_n\right)^2,\ \left|\frac{d^2}{dx^2}\theta_n\right|,\ \left|\frac{d^2}{dx^2}\phi_n\right|,\ \left|\frac{d}{dx}\theta_n\frac{d}{dx}\phi_n\right|,\ \left|\hbar k\frac{d}{dx}\theta_n\right|,\ \left|hk\frac{d}{dx}\phi_n\right|,$$

where $\hbar k$ denotes the momentum of the atom's mass center due to Eq. (8). The above conditions mean that the adiabatic approximation solutions (6) are only used effectively for physical situation with small atom's kinetic energy ($\sim|\dot{x}|^2$) and large detuning ($\simeq\delta$) or large photon numbers.

Having the above BO approximate solutions, we are in the position to consider the dynamical feature of our model. If the system is initially in the polarized state at $x=0$

$$|n,\uparrow\rangle = \cos\frac{\theta_n}{2}|X_{n,\uparrow}\rangle + \sin\frac{\theta_n}{2}|X_{n,\downarrow}\rangle\,. \tag{10}$$

The evolution state at time t observed at x is obtained by Eqs (5), (8) and (9) as follows:

$$|\Psi(x,t)\rangle = a_n(t)|n,\uparrow\rangle + b_n(t)|n+1,\downarrow\rangle\,,$$

where

$$\begin{aligned} a_n(t) &= \Big(\cos^2\frac{\theta_n}{2}e^{-i\sin^2(\theta_n/2)\phi(x)}e^{-i\Delta\omega t} \\ &\quad + \sin^2\frac{\theta_n}{2}e^{-i\cos^2(\theta_n/2)\phi(x)}e^{i\Delta\omega t}\Big)e^{ikx}e^{-i[(\hbar k^2/2m)+\omega(n+1/2)]t}\,, \\ b_n(t) &= \sin\frac{\theta_n}{2}\cos\frac{\theta_n}{2}(e^{i\cos^2(\theta_n/2)\phi(x)}e^{-i\Delta\omega t} \\ &\quad - e^{i\sin^2(\theta_n/2)\phi(x)}e^{i\Delta\omega t})e^{ikx}e^{-i[(\hbar k^2/2m)+\omega(n+1/2)]t}\,. \end{aligned} \tag{11}$$

Then we obtain the probability of finding the system with $n+1$ radiation field quanta, and the quasispin-down atomic state $|\downarrow\rangle$ is given by

$$|\langle n+1,\downarrow|\Psi(x,t)\rangle|^2 = |b_n(t)|^2\,, \tag{12}$$

which becomes

$$|b_n(t)|^2 = \frac{g^2(n+1)}{\delta^2+g^2(n+1)}\sin^2\frac{1}{2}(2t\sqrt{\delta^2+g^2(n+1)} - \gamma_+ + \gamma_-) \tag{13}$$

by using Eqs (10) and (11).

Especially, when $g(x) = g$, the additional phase $\gamma = \gamma_+ - \gamma_- = \cos\theta_n \phi(x)$ is a constant and then the above probability is also written as

$$|b_n(t)|^2 = \frac{g^2(n+1)}{\delta^2 + g^2(n+1)} \sin^2 \frac{1}{2}\Big[2t\sqrt{\delta^2 + g^2(n+1)} - \frac{\delta \cdot \phi(x)}{\sqrt{\delta^2 + g^2(n+1)}}\Big]. \tag{14}$$

Obviously, the additional phase $\gamma_+ - \gamma_- = \gamma$ appearing in Eqs (13) and (14) denotes the effect of space motion of the atom on its internal motion dressed by a quantized radiation field like the phase effect in Bitter–Dubbers experiment, testing the Berry's phase.[11] This phase can also be understood as the Aharonov–Bohm phase of the induced gauge field.[10]

Notice that the appearance of the geometric phase γ depends directly on whether the cavity is prepared in nonresonant case or not. Otherwise, in the resonant case, $\delta = 0$, $|b_n(t)|^2 = \sin^2(gt\sqrt{n+1})$, the effect of γ vanishes!

We can also consider another effect of the motion of atomic mass center in the nonresonant case, that is, the modification of population inversion $p_n(t) = |b_n(t)|^2 - |a_n(t)|^2$ or its relevant absorption and radiation of the atomic energy in cavity field. Assume that the cavity field is initially prepared in a coherent state $|\alpha\rangle$ while the atom in an initial state with upward quasispin and a definite momentum k. In this situation, we can write the wavefunction of the atom plus the cavity

$$\langle x|\Psi(t)\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{\sqrt{n!}} [a_n(t)|n,\uparrow\rangle + b_n(t)|n+1,\downarrow\rangle].$$

Then we have the evolution of the average energy of the atom

$$\langle \sigma_z \rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{\sqrt{n!}} \Big\{ \frac{\delta^2}{\delta^2 + g^2(n+1)} + \frac{g^2(n+1)}{\delta^2 + g^2(n+1)} \times \cos\Big[\frac{\delta \cdot \phi(x)}{\sqrt{\delta^2 + g^2(n+1)}} - 2t\sqrt{\delta^2 + g^2(n+1)}\Big]\Big\}. \tag{15}$$

The effect of motion of atomic mass center manifests as the modified oscillation of energy with an additional phase $\gamma = \delta \cdot \phi(x)/\sqrt{\delta^2 + g^2(n+1)}$.

Finally, it should be emphasized that if the coupling $g(x)$ in the space-dependent parameter $\kappa(x) = g(x)e^{-ikx}$ vanishes at the ends of the cavity, and the adiabatic conditions hold, the atom can be trapped in the cavity for the same considerations as in Ref. [5] which do not depend on the properties of the phase $\phi(x)$.

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