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Coherent State Realization of Exponential Solutions of Yang–Baxter Equation and Its Generalizations[1]

Chang-Pu SUN
Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, China
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Abstract

In this letter, it is shown that the generator of coherent state can be used to realize exponential solutions of Yang–Baxter equation with a spectral parameter. Based on generalization of this observation, the non-unitary finite dimensional solutions of Yang–Baxter equation are constructed in terms of the Heisenberg–Weyl algebra. The unitary realization of the solution is also obtained in connection with the Hamiltonian of the system of forced harmonic oscillators.

Progresses in nonlinear physics such as exactly solvable models in statistical mechanics and low-dimensional quantum field theory have manifested that the Yang–Baxter equation (YBE) is the key to the integrability in these physical problems.[1] Therefore, much attention from both theoretical physicists and mathematicians has been paid to the studies of YBE and its solutions in past years. A standard approach obtaining solutions of YBE is Drinfeld and Jimbo's quantum group theory, in which q-deformations of universal enveloping algebras of simple Lie algebras and their representations are applied to constructing the solutions through the so-called universal R-matrix.[2–3]

Recently, some new quantum doubles, which are not the q-deformation, are found in connection with a new class of solutions for YBE without spectral parameter.[4–6] Notice that these new solutions are of the exponential form and have not spectral parameters. In this letter, we first construct such exponential solutions with a spectral parameter for YBE in terms of the generator of coherent state for harmonic oscillator. It is shown that the YBE is enjoyed by the associativity of the 1-cocycle of the Heisenberg–Weyl group in this construction. For the non-unitary representations of the Heisenberg–Weyl algebra, the finite-dimensional exponential solutions are obtained by using a purely algebraic method.

Let $a^\dagger$ and a be the creation and annihilation operators for a boson state. The generator $D(Z) = e^{Za^\dagger - Z^* a}$ of a coherent state $|Z\rangle = D(Z)|0\rangle$ is defined in terms of a complex number $Z \in C$ (complex number field) $Z = re^{i\theta} = X + iY$ where $X, Y \in R$ (real number field). In fact, $D(Z)$ forms a projection representation of the two-dimensional translation group G (Heisenberg–Weyl group) for

$$D(Z)D(Z') = e^{i\Phi(Z,Z')}D(Z+Z'), \qquad (1)$$

where $\phi(Z, Z') = (i/2)(ZZ'^* - Z^*Z') = rr'\sin(\theta' - \theta)$ is a phase, which is equal to the area of the parallelogram spanned by the vector $\vec{Z} = (X, Y)$ and $\vec{Z}' = (X', Y')$. Obviously, the phase $\Phi(Z, Z')$ satisfies the 1-cocycle relation

$$\Phi(Z, Z') + \Phi(Z+Z', Z'') = \Phi(Z, Z'+Z'') + \Phi(Z', Z''). \qquad (2)$$

Let us try to construct the solutions for YBE $h_k(x)h_{k+1}(x+y)h_k(y) = h_{k+1}(y)h_k(x+y)h_{k+1}(x)$, where $h_k(x)$ is an operator or a matrix parametrized by x. We first consider the two-particle case that the index k only runs from 1 to 2. Define

$$h_1(x) = D(xZ), \qquad h_2(x) = D(xZ') \qquad (3)$$

in terms of the coherent state operator $D(z)$. Then, a direct calculation leads to

$$\begin{aligned} h_1(x)h_2(x+y)h_1(y) &= \exp[i\phi(xZ, (x+y)Z') + i\Phi(xZ + (x+y)Z', yZ)] \\ &\times D((x+y)(Z+Z')) = \exp[i(x^2-y^2)\Phi(Z,Z')]D((x+y)(Z+Z')); \\ h_2(y)h_1(x+y)h_2(x) &= \exp[i\Phi(yZ', (x+y)Z) + i\Phi(yZ' + (x+y)Z, xZ')] \end{aligned}$$

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$$\times D((x+y)(Z+Z')) = \exp[i(x^2-y^2)\Phi(Z,Z')]D((x+y)(Z+Z')) .$$

The above expressions from Eq. (1) just show that $h_1(x)$ and $h_2(x)$ satisfy the YEB. Here, we have considered the linearity of the function $\Phi(z,z')$ in both variables Z and Z', which is related to the associativity of both the translation group G and what YEB implies actually.

Now, we consider the generalizations of the above discussion for the case with many particles and the case with finite-dimensional solution. Since the coherent state of boson is closely associated with the Heisenberg–Weyl algebra H, the boson operators $a^\dagger$, a and 1, as certain representatives of the elements in H, should be replaced by the general abstract basis elements A, B and C of H satisfying

$$[A,\,B]=C\,, \qquad [C,\,A]=0=[C,\,B]\,. \tag{4}$$

Natice that equations (4) enjoy a non-semisimple Lie algebra, and its unitary representation must be infinite-dimensional. Thus, its finite-dimensional representation is only non-unitary.

In fact, a non-unitary finite-dimensional representation of H

$$\begin{aligned} \rho(A)F(m,\,n) &= \mu F(m,\,n) + mF(m-1,\,n+1)\,, \\ \rho(B)F(m,\,n) &= \theta(N-(m+1+n))F(m+1,\,n)\,, \\ \rho(C)F(m,\,n) &= \theta(N-(m+1+n))F(m,\,n+1) \end{aligned} \tag{5}$$

for $m+n \le N-1$ can be induced as a quotient representation on the quotient space $Q=V/W$: Span$\{F(m,\,n)=f(m,\,n)\,\mathrm{Mod}\,(W)|m+n\le N-1\}$. Here, $\theta(x)=0$ or 1 for $x\le 0$ or $x>0$ respectively; the space W spanned by $\{f(m,\,n)|m+n\ge N\}$ is an invariant subspace of the linear space V spanned by $\{f(m,\,n)=B^mC^n|\mu\rangle|,\ A|\mu\rangle=\mu|\mu\rangle,\ m,\,n=0,\,1,\,2,\ ;\ \mu\in C\}$. A direct computation from Eq. (3) shows that

$$h_1(x)=e^{xA}\,, \qquad h_2(x)=e^{yB} \tag{6}$$

also form a set of solution for YBE. Thus $H_1(x)=e^{x\rho(A)}$ and $H_2(x)=e^{x\rho(B)}$ gives a non-unitary finite-dimensional solution for YBE with the dimension $d_\rho=(N+2)(N+1)/2$. Notices that equations (5) are also a two-particle solution for YBE.

To find an N-particle ($N\ge 3$) unitary solution of YBE, we should use many-state bosonic operators a_i and $a_i^\dagger$ ($i=0,\,1,\,2,\,\cdots,\,N$) to define $A_k=\alpha_k^* a_{k-1}-\alpha_k a_{k-1}^\dagger+\beta_k^* a_k-\beta_k a_k^\dagger$ and $2\le k\le N$. Then, we have non-vanishing commutators among $\{A_i\}$ $[A_k,\,A_{k+1}]=\beta_k\alpha_{k+1}^*-\beta_k^*\alpha_{k+1}\equiv\lambda_k$, Then, it follows from Eq. (6) the $h_k(x)=e^{xA_k}$ satisfies the YBE (2). Notice that

$$H=\sum_{k=1}^{N}\frac{\partial h_k(x)}{\partial x}\Big|_{x=0}=\alpha_0^* a_0-\alpha_0 a_0^\dagger+\sum_{k}^{N}[(\alpha_k+\beta_k)a_k^\dagger+(\alpha_k^*+\beta_k^*)a_k]$$

is just the Hamiltonian of the system of forced harmonic oscillators in interaction picture.

Finally, it is pointed out that, unlike the usual case, the above constructed solution for YBE can not be written intwining a direct product of two linear spaces and so they have not the classical limits as the representations of the braid group. Thus, it is unreasonable to say that they are available to quantum integrable models in statistical mechanics and field theory. However, this letter manifests a remarkable fact that the YBE may be rooted in modern quantum mechanics quite directly because its solutions can be realized in terms of some important concepts such as coherent state and Heisenberg's commutation relation.

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