Commun. Theor. Phys. **22** (1994) pp. 383–384
© International Academic Publishers Vol. 22, No. 3, October 30, 1994

Exactly-Analytical Approach for Neutrons in Magnetic Field and Berry's Phase Effect in Moving Translation Reference Frame[1]

Xiao-Guang WANG and Chang-Pu SUN
Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, China

(Received February 23, 1994)

Abstract

In connection with Bitter–Dubbers experiment, an exact solution for the quantum dynamics of neutrons in magnetic field which is harmonically inhomogeneous (helical) and changes harmonically with time is obtained by invoking a time-dependent transformation with spin–space coupling. It predicts that a beam of neutrons with monotonous momentum will be split into two beams by the magnetic field. As its limiting case, the Berry's geometric phase in moving reference frame appears while the field is very strong.

For neutrons moving in magnetic field which changes harmonically with time, the Schrödinger equation can be solved exactly according to Rabi's method,[1,2] the solution and its higher spin generalization[3,4] have been used to test Berry's geometric phase.[5] In fact, the harmonically inhomogeneous (helical) magnetic field was also used in Bitter–Dubbers' studies for the experimental test of Berry's phase.[6] When we "see" the neutron from the moving translation reference frame with the velocity much less than that of light, the Hamiltonian will depend on both space and time if the magnetic field depends on space coordinate. One has studied it through the generalized Born–Oppenheimer approximation.[7,8] But now we give the exact solution by invoking a unitary transformation depending on time, space and spin.

First, let us review the Galileo transformation in quantum mechanics (Galileo boost) briefly. If the reference frame S' moves uniformly with velocity $\vec{v}$ ($|\vec{v}| \ll c$) relative to the laboratory S, the wave function $|\psi'(t)\rangle$ in S' is related to a wave function $|\psi(t)\rangle$ through the time-dependent unitary transformation, the Galileo boost $U = U(\vec{v}, t)$:[9,10]

$$|\psi'(t)\rangle = U(\vec{v}, t)|\psi(t)\rangle\,, \qquad U(\vec{v}, t) = e^{-imv^2t/\hbar} \cdot e^{im\vec{v}\cdot\vec{r}/\hbar} \cdot e^{-i\vec{p}\cdot\vec{v}t/\hbar}\,. \tag{1}$$

If the Hamiltonian in S is $\hat{H}(\vec{p}, \vec{r})$, the corresponding effective Hamiltonian in S' is

$$\hat{H}' = \hat{H}(\vec{p} - m\vec{v}, \vec{r} - \vec{v}t) + \vec{v}\cdot\vec{p} - \frac{1}{2}mv^2\,. \tag{2}$$

In Bitter–Dubbers experiment, the Hamiltonian in the laboratory reference frame S is

$$\hat{H} = \frac{\hat{p}_z^2}{2m} + \frac{1}{2}g\vec{B}(z)\cdot\vec{\sigma}\,, \tag{3}$$

for the helical field, $\vec{B}(z) = (B\sin\theta\cos(kz),\ B\sin\theta\sin(kz),\ B\cos\theta)$, where $k = 2\pi/L$, L is the length of domain of $\vec{B}(z)$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, σ_x, σ_y, σ_z are Pauli matrices. From Eq. (2) the Hamiltonian in the translation reference frame S' moving along the z-axis with velocity v is

$$\hat{H}' = \frac{\hat{p}_z^2}{2m} + \frac{1}{2}g\vec{B}(z - vt)\cdot\vec{\sigma}\,. \tag{4}$$

We may write $\hat{H}'$ in terms of σ_+ and σ_- ($\sigma_\pm = \sigma_x \pm i\sigma_y$), obtaining

$$\hat{H}' = \frac{\hat{p}_z^2}{2m} + \frac{1}{2}gB[\sin\theta(\sigma_+ e^{-i(kz-\omega t)} + \sigma_- e^{i(kz-\omega t)}) + \cos\theta\sigma_z]\,, \tag{5}$$

[1]The project supported partially by the National Natural Science Foundation of China and the Fok Yin-Tung Education Foundation.

where $\omega = kv$. In order to give the exact solution, we introduce a unitary transformation which depends on time, space and spin $W(z, t, \sigma_z) = e^{-i(kz-\omega t)\cdot\sigma_z/2}$, then the effective Hamiltonian

$$\hat{H}_e = \frac{\hat{p}_z^2}{2m} + \frac{\hbar^2 k^2}{8m} + \hbar\Omega(\cos\alpha\sigma_z + \sin\alpha\sigma_x)\,, \tag{6}$$

where

$$\begin{aligned} \hbar\Omega = \hbar\Omega(\hat{p}_z) &= \frac{1}{2}gB\Big(1 - \frac{2\hbar(\hat{p}_z k - m\omega)\cos\theta}{mgB} + \frac{\hbar^2(\hat{p}_z k - m\omega)^2}{m^2 g^2 B^2}\Big)\,, \\ \operatorname{ctg}\alpha &= \operatorname{ctg}\theta - \frac{\hbar(\hat{p}_z k - m\omega)}{mgB\sin\theta}\,. \end{aligned} \tag{7}$$

Obviously, the corresponding evolution operator for H_e is

$$U_e(t) = e^{-i\hat{H}t/\hbar} = e^{-i\hat{p}_z^2 t/2m\hbar}\cdot e^{-i\hbar k^2 t/8m}\cdot[\cos\Omega t - i(\cos\alpha\sigma_z + \sin\alpha\sigma_x)\sin\Omega t]\,. \tag{8}$$

Therefore we can write the evolution operator for the Hamiltonian (4)

$$\begin{aligned} U(t) = e^{-i\hat{H}t/\hbar} &= e^{-i(kz-\omega t)\sigma_z/2}\cdot e^{-i\hat{p}_z^2 t/2m\hbar}\cdot e^{-i\hbar k^2 t/8m} \\ &\times[\cos\Omega t - i(\cos\alpha\sigma_z + \sin\alpha\sigma_x)\sin\Omega t]\cdot e^{ikz\sigma_z/2}\,. \end{aligned} \tag{9}$$

If the particle is initially polarized in spin-up state with momentum p_0, the wave function at time $t = 0$, $\psi(0) = |\uparrow\rangle \otimes |p_0\rangle$, then equation (8) determines the exact wave function at time t

$$\begin{aligned} \psi(t) = U(t)\psi(0) &= e^{-i(p_0+\hbar k/2)^2 t/2m\hbar}\cdot e^{-i\hbar k^2 t/8m}[(\cos\Omega' t - i\cos\alpha'\sin\Omega' t) \\ &\cdot e^{i\omega t/2}\cdot|\uparrow\rangle\otimes|p_0\rangle - i\sin\Omega' t\sin\alpha'\cdot e^{-i\omega t/2}|\downarrow\rangle\otimes|(p_0+\hbar k)\rangle]\,, \end{aligned} \tag{10}$$

where $\Omega' = \Omega(p_0 + \hbar k/2)$, $\alpha' = \alpha(p_0 + \hbar k/2)$. The momentum shifts accompanying the spin states is an interesting result. The momentum shift is $\hbar k$ for spin-down state, and no shift for spin-up state. Physically, the exact result predicts that when a beam of neutrons enters the magnetic field, the beam will be split into two beams.

Finally, we calculate the polarization rate of neutrons from the exact solution (10), obtaining $P_z = |a|^2 - |b|^2 = 1 - 2\sin^2\omega' t\sin^2\alpha'$. Here, a and b are the probability amplitudes for spin-up and spin-down states. Considering the adiabatic case when the Rabi frequency $\omega_0 = gB/2\hbar$ is much higher than the frequency $\omega_1 = k(p_0 - mv)/m$ of the changing magnetic field "seen" by the moving neutrons, we can obtain the adiabatic polarization rate

$$P_z \simeq 1 - 2\sin^2\theta\cdot\sin^2\Big(\frac{1}{2\hbar}gBt - \frac{p_0 k\cos\theta t}{2m} + \frac{vk\cos\theta t}{2}\Big). \tag{11}$$

From Eq. (11), except for the dynamically phase $gBt/2\hbar$, Berry's phases appear with two terms. The first one $-p_0 k\cos\theta t/2m$ is purely geometrical part, and the second $vkt\cos\theta/2$ is induced by the time-dependent unitary transformation, the Galileo boost. This just agrees with the general theory about the Berry's phase in a moving reference frame.[10]

References

[1] I.I. Rabi, Phys. Rev. **51** (1937) 652.
[2] W.H. Louisell, *Quantum Statistical Properties of Radiation*, John Wiley, New York (1973).
[3] D.R. Richardson, *et al.*, Phys. Rev. Lett. **61** (1988) 2030.
[4] L.Z. ZHANG and C.P. SUN, preprint ITP-NENU 94-4.
[5] M.V. Berry, Proc. R. Soc. Lond. **A392** (1984) 45.
[6] T. Bitter and D. Dubbers, Phys. Rev. Lett. **59** (1987) 251.
[7] C.P. SUN and M.L. GE, Phys. Rev. **D41** (1990) 1349.
[8] C.P. SUN and M.L. GE, Commun. Theor. Phys. (Beijing China) **13** (1990) 63.
[9] R. Jackiw, *High Energy Physics 1985*, World Scientific, Singapore (1986).
[10] C.P. SUN and L.Z. ZHANG, High Energy Phys. Nucl. Phys. **14** (1990) 136.