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Generalized Quasi-Exactly-Solvable Quantal Problems, the Representations and Differential Realization of General Deformation of su(2)[1]

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Abstract

This paper is devoted to building the representation theory of the general deformation for su(2) and then using it to generalize the quasi-exactly-solvable quantal (QESQ) problems. For the finite dimensional and infinite dimensional representations, two classes of generalized QESQ models are respectively constructed in terms of the differential realization of the general deformation of su(2). When the deformation is the q-deformation in the quantum group theory, the QESQ model is discussed in detail by associating it with the nonlinear precession of high spins in an external field.

I. Introduction

The so-called quasi-exactly-solvable quantal (QESQ) problem (or partial algebraization problem of quantal spectra)[1−3] occupies an intermediate and important position between the exactly solvable ones and others. It has been probed for various areas in mathematical physics, such as the supersymmetry quantum mechanics[4,5], the Heisenberg spin model[6,7], one-dimensional analogue of rational conformal field theories[8] and so on. In order to proceed our discussion, we first describe the central idea of the QESQ problem briefly as follows. Let the operator $\hat{H}$ on the Hilbert space $\mathcal{H}$ be the Hamiltonian of a quantal system. If $\hat{H}$ possesses a diagonal block structure on a proper basis for $\mathcal{H}$, i.e.,

$$\hat{H} = \mathrm{diag\,block}\,(\hat{h}, \hat{h}') \,, \tag{1}$$

where $\hat{h}$ is an $n \times n$ matrix with 'small' n and $\hat{h}'$ is an $m \times m$ matrix with 'large' or infinite m, then $\hat{h}$ can be diagonalized without affecting $\hat{h}'$ and a limited part of the spectra for the quantal system is obtained in a purely algebraic way. The spectral problem $\hat{H}\Psi = E\Psi$ with such a Hamiltonian is called QESQ problem. In order to construct such a Hamiltonian, one tries to express $\hat{H}$ as an element of the universal enveloping algebra (UEA) U(L) of a Lie algebra L, i.e., $\hat{H} = H(x_1, x_2, \cdots, x_M)$ where x_i $(i = 1, 2, \cdots, M)$ are the basis vectors of L. Then, one decomposes $\mathcal{H}$ into $\mathcal{H} = V \oplus \bar{V}$ and $\hat{H}$ takes the form (1) where V is a finite dimensional representation space for L. Since the Hamiltonian $\hat{H}$ in the coordinate representation is a function of coordinates (such as x) and the corresponding differential operator (such as $-i\hbar d/dx$), the inhomogeneous differential realization of L must be needed in the QESQ problems. To this end, the general formula of differential realization of Lie algebras has been given in Refs. [9] and [10].

One purpose of this paper is to make an extensive generalization of the QESQ problem and the main idea is quite natural: As U(L) is an associative algebra over the complex number

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field $\mathbb{C}$ ($\mathbb{C}$-algebra), one can use a more general $\mathbb{C}$-algebra $\mathcal{A}$ to replace $\mathrm{U}(L)$ in the original QESQ problem so that a wide class of generalized QESQ spectral problems is constructed. The more general $\mathbb{C}$-algebra used in this paper for generalized QESQ problems is a general deformation $D = D(f(x);\ \mathrm{su}(2))$ of su(2)[11,12], which is generated by $X^{\pm}$ and h with

$$[X^+,\ X^-] = f(h), \qquad [h,\ X^{\pm}] = \pm X^{\pm}, \tag{2}$$

where $f(h)$ is an operator function defined by a holomorphic function $f(x)$. When $f(h) = 2h$, $D = \mathrm{U}(\mathrm{su}(2))$ is just the UEA of Lie algebra su(2); when $f(x) = [2x] \equiv (q^{2x} - q^{-2x})/(q - q^{-1})$, $D = \mathrm{U}_q(\mathrm{su}(2))$ is just the quantum algebra (loosely called quantum group) of su(2)[13,14].

This paper is arranged as follows. In Sec. II, we generally study the representation theory of the deformation D from its regular representation. In Sec. III, we construct various inhomogeneous differential realizations of D associated with certain representations obtained in Sec. II. In Sec. IV, two classes of generalized QESQ models are constructed in terms of the general $\mathbb{C}$-algebra D and its realizations of representation. The corresponding spectrum problem $\hat{H}\Psi = E\Psi$ which is equivalent to certain (nonlocal) differential equation or integral-differential equation, is explicitly solved with sufficient examples. Finally, in Sec. V, the QESQ problem of quantum group $\mathrm{U}_q(\mathrm{su}(2))=\mathrm{sl}_q(2)$ is discussed in detail and related to the nonlinear precession in an external field.

II. The Representation Theory of the Deformation D

In the following discussions, we sometimes take

$$f(x) = g(x+1) - g(x) \tag{3}$$

for convenience. In fact, for many concrete cases, taking the special form (3) does not reduce the generality. For example, if $g(h) = (h - \frac{1}{2})$ $(= [h][h-1])$, then $D = \mathrm{U}(\mathrm{su}(2))$ $(= \mathrm{sl}_q(2))$.

The basic representation of the deformation D as a $\mathbb{C}$-algebra is its left regular representation (LRR) $\rho_0 : D \longrightarrow \mathrm{End}\,(D)$ defined by $\rho_0(a)b = a \cdot b$, $\forall a, b \in D$. Because of the general function $F(h)$, the LRR of D cannot be written in a concise form. We left need to consider a left-invariant subspace $W(\lambda) = D(h - \lambda)$ $(\lambda \in \mathbb{C})$. On the quotient space $Q(\lambda) = D/W(\lambda)$:

$$F(m,n) = X^{+m}X^{-n} \bmod W(\lambda)\,, \quad m, n \in \mathbb{Z}^+ = \{0, 1, 2, \cdots,\},$$

the representation induced by ρ_0 explicitly reads

$$\begin{aligned} &X^- F(m,n) = F(m, n+1) + G(\lambda - n, m)F(m-1, n)\,,\\ &X^+ F(m,n) = F(m+1, n)\,, \qquad hF(m,n) = (m - n + \lambda)F(m,n)\,, \end{aligned} \tag{4}$$

where

$$G(x,m) = \sum_{k=0}^{m-1} f(x+k), \ m \ge 1\,, \qquad G(x,0) = 0\,. \tag{5}$$

If $f(x) = g(x+1) - g(x)$, then

$$G(x,m) = g(x+m) - g(x)\,. \tag{6}$$

Now, we consider the deformed Verma representation ρ_V of D, which is a generalization of the Verma representation of su(2)[15] or $\mathrm{sl}_q(2)$[16]. Over a left-invariant subspace $L(\lambda) = Q(\lambda) \cdot X^-$, $Q(\lambda)$ has a quotient space $V(\lambda) = Q(\lambda)/L(\lambda)$ with the basis $\{F(m) = F(m,0) \bmod L(\lambda)\,|\, m \in \mathbb{Z}^+\}$. On $V(\lambda)$, the representation (4) induces an infinite dimensional representation ρ_V

$$X^+F(m) = F(m+1)\,, \quad X^-F(m) = G(\lambda, m)F(m-1)\,, \quad hF(m) = (m+\lambda)F(m)\,. \tag{7}$$

It is the deformed Verma representation of D with the lowest weight, which is a general deformation of the Verma representation of su(2).

Let us discuss how to obtain a finite dimensional representation for the deformed Verma representation (7). If $G(\lambda, m) \neq 0$ for any $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}^+$, the representation (7) is irreducible and there is not a finite dimensional representation obtained from Eq. (7).

However, if $G(\lambda, m = N) = 0$ for certain $\lambda \in \mathbb{C}$ and $N \in \mathbb{Z}^+$, then $W(N\lambda)$: $\{F(N), F(N+1), F(N+2), \cdots\}$ is an invariant subspace and the corresponding quotient space $Q(N,\lambda) = V(\lambda)/W(N,\lambda)$:

$$\{\bar{F}(m) = F(m) \bmod W(N,\lambda) \mid m = 0,1,2,\cdots,N-1\}$$

is finite dimensional. On $Q(N,\lambda)$, ρ_V induces a finite dimensional representation ρ,

$$\rho(g)x = \pi \circ \rho_V(g) \circ \pi^{-1}(x), \quad g \in D, \ x \in Q(N,\lambda), \tag{8}$$

where $\pi : V(\lambda) \to Q(N,\lambda)$, ($\pi F(m) = F(m) \bmod W(N,\lambda)$ is a natural map and $\pi^{-1}(x)$ is the inverse image set of x under π). Let us consider several examples.

2.1) When $g(h) = h^3$, on the basis

$$\Big\{\phi(n) = \Big\{\prod_{k=0}^{n}(k^2 + 3k\lambda + 3\lambda^2)^{-1}\Big\}F(n) \mid n \in \mathbb{Z}^+\Big\},$$

we have an infinite dimensional representation ρ_V,

$$X^+\phi(n) = [(n+1)^2 + 3\lambda(n+1) + 3\lambda^2]\phi(n+1), \quad X^-\phi(n) = n\phi(n-1), \quad h\phi(n) = (n+\lambda)\phi(n). \tag{9}$$

If the parameter λ satisfies

$$3\lambda^2 + 3(N+1)\lambda + (N+1)^2 = 0 \tag{10}$$

for an integer N ($\neq 0$), or

$$\lambda = \lambda_\pm = \frac{1}{6}(-3 \pm i\sqrt{3})(N+1), \tag{10a}$$

then there exists an invariant subspace $W(N) : \{\phi(n) \mid 0 \le n \le N\}$ defined by such an extreme vector $\phi(N)$ that $X^+\phi(N) = 0$. On $W(N)$, equation (9) subduces an $(N+1)$-dimensional representation

$$\begin{aligned} &X^+\phi(n) = (n-N)\Big[n + 1 - \Big(1 \mp i\frac{\sqrt{3}}{2}\Big)\Big]\phi(n+1), \quad \text{for } \lambda = \lambda_\pm, \\ &X^-\phi(n) = n\phi(n-1), \qquad h\phi(n) = (n + \lambda_\pm)\phi(n). \end{aligned} \tag{11}$$

2.2) When $g(h) = \sin(h\pi/N)$ ($N \in \mathbb{Z}^+$), we have an infinite dimensional representation

$$\begin{aligned} &X^-F(n) = \Big[\sin\Big(\frac{\lambda+n}{N}\pi\Big) - \sin\Big(\frac{\lambda}{N}\pi\Big)\Big]F(n-1), \\ &X^+F(n) = F(n+1), \qquad hF(n) = (n+\lambda)F(n). \end{aligned} \tag{12}$$

For a nonzero integer $N \in \mathbb{Z}^+$ and $\alpha \in \mathbb{Z}^+$, $XF(2\alpha N) = 0$. Then, we have an invariant subspace chain $V(\lambda) = W^0(N) \supset W^1(N) \supset W^2(N) \supset \cdots \supset W^\alpha(N) \supset \cdots$, where $W^\alpha(N)\{F(2\alpha N), F(2\alpha N+1), F(2\alpha N+2), \cdots\}$ is an invariant subspace. On each quotient space $Q^\alpha(N,\lambda) = V(\lambda)/W^\alpha(N)$, we will obtain a finite dimensional representation and its explicit form can be written out in the way similar to Eq. (12).

2.3) In fact, in the quantum group case that $f(h) = [2h]$, we have the representation

$$X^+\phi(n) = [\mu - n]\phi(n+1), \quad X^-\phi(n) = [n]\phi(n-1), \quad h\phi(n) = \Big(n - \frac{\mu}{2}\Big)\phi(n) \tag{13}$$

of the quantum algebra $sl_q(2)$[17]. Here

$$\phi(n) = \Big\{\prod_{k=0}^{n-1}\frac{1}{[\mu-k]}\Big\}F(n).$$

III. The Inhomogeneous Differential Realization of D

The inhomogeneous differential realization (IDR) of Lie algebras plays a key role in construction of the QESQ models. To make a generalization of QESQ models in terms of the general deformation of su(2), we also need the IDR of the $\mathbb{C}$-algebra D. In this section we will build the general scheme to obtain IDR of D based on previous works[9,10].

Let B be the Bargmann space of the holomorphic function $f(z)$ on $\mathbb{C}$ and its basis be chosen as $\{z^n|n \in \mathbb{Z}^+\}$. The coordinate z and the corresponding differential d/dz are fundamental operators on B. A natural homorphism from the deformed Verma space $V(\lambda)$ to B is Φ: $\Phi(F(n)) = z^n$. Then the following commutative diagram

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\Phi} & B \\ \rho_V(g)\downarrow & & \downarrow T(g) \\ V(\lambda) & \xrightarrow{\Phi} & B \end{array} \qquad \forall g \in D,$$

defines an operator representation of T on B : $\{\hat{g} = T(g)|g \in D\}$. We call T the IDR of D associated with ρ_V, which can be written in an algebraic form $\Phi \circ \rho_V(g) = T(g) \cdot \Phi$, or

$$T(g) = \Phi \circ \rho_V(g)\Phi^{-1}. \tag{14}$$

In fact, ρ_V is a $\mathbb{C}$-algebra representation of D, so we can check that

$$T(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 T(g_1) + \alpha_2 T(g_2), \quad T(g_1 g_2) = T(g_1)T(g_2), \qquad \alpha_1, \alpha_2 \in \mathbb{C}, \tag{15}$$

namely, $T(g)$ defines a $\mathbb{C}$-algebra representation by a direct calculation. It follows from Eqs. (14) and (15) that

$$\begin{aligned} T(X^-)z^n &= G(\lambda, n)z^{n-1} = [g(\lambda+n) - g(\lambda)]z^{n-1}, \\ T(X^+)z^n &= Z^{n+1}, \qquad T(h)z^n = (\lambda+n)z^n, \end{aligned} \tag{16}$$

that is to say

$$\hat{X}^+ = z, \quad \hat{X}^- = \frac{1}{z}\Big[g\Big(\lambda + z\frac{d}{dz}\Big) - g(\lambda)\Big], \quad \hat{h} = \lambda + z\frac{d}{dz}, \quad (\hat{g} = T(g),\ g = X^\pm, h). \tag{17}$$

Corresponding to the examples in the last section, three IDR's are given respectively as follows.

3.1) When $g(h) = h^3$ the representation (9) leads to

$$\begin{aligned} \hat{X}^+ &= z\Big[z^2\frac{d^2}{dz^2} + (2+3\lambda)z\frac{d}{dz} + (3\lambda^2 + 3\lambda + 1)\Big], \\ \hat{X}^- &= \frac{d}{dz}, \qquad \hat{h} = z\frac{d}{dz} + \lambda. \end{aligned} \tag{18}$$

3.2) When $g(h) = \sin(\pi h/N)$ the representation (12) leads to

$$\begin{aligned} \hat{X}^- &= \frac{1}{z}\Big\{\sin\Big[\Big(\lambda + z\frac{d}{dz}\Big)N^{-1}\Big] - \sin\Big(\frac{\lambda\pi}{N}\Big)\Big\}, \\ \hat{X}^+ &= z, \qquad \hat{h} = \lambda + z\frac{d}{dz}. \end{aligned} \tag{19}$$

3.3) The IDR of the quantum algebra $sl_q(2)$ is given by the representation (13) as follows:

$$\hat{X}^+ = z\Big[\mu - z\frac{d}{dz}\Big], \qquad \hat{X}^- = \frac{1}{z}\Big[z\frac{d}{dz}\Big] = D_z, \qquad \hat{h} = \lambda + z\frac{d}{dz}, \tag{20}$$

where D_z is the q-(deformed) differential operator so that[17]

$$\begin{aligned} D_z f(z) &= \frac{f(qz) - f(q^{-1}z)}{z(q-q^{-1})} = \frac{1}{2\pi i}\oint_C K(\xi, z)f(\xi)d\xi, \\ K(\xi, z) &= [(\xi - qz)(\xi - q^{-1}z)]^{-1}. \end{aligned} \tag{21}$$

IV. Generalized QESQ Problems

In the original QESQ problem, the Hamiltonian $\hat{H}$ is built in terms of the generators $\{x_i\}$ of a Lie group corresponding to a Lie algebra L, that is to say, H is an element of the UEA $U(L)$ of L. So the inherent dynamic symmetry is just characterized by $U(L)$ or L. Up to now the QESQ problem has been generalized so that it possesses the inherent dynamic symmetry of Lie superalgebra. In this paper we will make a further generalization by replacing $U(L)$ with a general $\mathbb{C}$-algebra—the general deformation D of su(2). The QESQ model built in terms of D naturally possesses a new symmetry defined by D. We will discuss this generalization for two cases, one is for the finite dimensional representations and the other is for the infinite dimensional representations.

4.1 The Case of Finite Dimension

In this case the generalization is quite direct and the Hamiltonian is a combination of the generators for D, i.e., $\hat{H} = H(X^+, X^-, h)$. Through the differential realization, the spectral problem $\hat{H}\Psi = E\Psi$ is equivalent to the eigenvalue problem of the ordinary differential equation such as

$$H\Big(z, z^{-1}\Big(g\Big(\lambda + z\frac{d}{dz}\Big) - g(\lambda)\Big), \lambda + z\frac{d}{dz}\Big)\Psi(z) = E\Psi(z). \tag{22}$$

Provided that the basis for calculating the matrix elements of H is chosen as $\{v_1, v_2, \cdots, v_N,$ other basis orthogonal to v, $i = 1, 2, \cdots, N\}$, where v_i $(i = 1, 2, \cdots, N)$ is the basis for an N-dimensional representation of D, the Hamiltonian $\hat{H}$ is automatically written as the following block diagonal form

$$\hat{H} = \begin{pmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,N} & \\ H_{2,1} & H_{2,2} & \cdots & H_{2,N} & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ H_{N,1} & H_{N,2} & \cdots & H_{N,N} & \\ & 0 & & & \bar{H}_0 \end{pmatrix} \equiv \begin{pmatrix} H_0 & 0 \\ 0 & \bar{H}_0 \end{pmatrix}, \tag{23}$$

where $H_{i,j} = \langle v_i|\hat{H}|v_j\rangle$ and $\bar{H}_0$ is usually an infinite dimensional matrix. Then, we can obtain a part of spectra for Eq. (22) by diagonalizing an N by N matrix H_0 without touching $\bar{H}_0$. Let us use two examples to give a sketch of the above arguments.

<u>**Example 1**</u>. Let $g(h) = \sin(\pi h)$ and

$$\hat{H} = \sum_{m,n,k=0}^{\infty} C_{mnk}(X^+)^m(X^-)^n(h)^k, \qquad C_{mnk} \in \mathbb{C}. \tag{24}$$

The spectral problem $\hat{H}\Psi = E\Psi$ in the differential realization is

$$\sum_{m,n,k=0}^{\infty} C_{mnk}z^{m-n}\Big\{\sin\Big[\frac{\pi}{N}\Big(z\frac{d}{dz} + \lambda\Big)\Big] - \sin\Big(\frac{\lambda\pi}{N}\Big)\Big\}\Big(z\frac{d}{dz} + \lambda\Big)^k\Psi(z) = E\Psi(z). \tag{25}$$

On the space of the two-dimensional representation, the elements of $\bar{H}_0$ are

$$\begin{aligned} &H_{00} = \sum_{k=0}^{\infty} C_{00k}\lambda^k, \quad H_{10} = \sum_{k=0}^{\infty} C_{10k}\lambda^k, \quad H_{01} = 2\sin(\pi\lambda)\sum_{k=0}^{\infty}(1+\lambda)^k C_{01k}, \\ &H_{11} = H_{00} + \sum_{k=0}^{\infty} 2\sin(\pi\lambda)(1+\lambda)^k C_{11k} + \sum_{k=0}^{\infty}\Big\{\sum_{n=1}^{k}\frac{k!}{n!(n-k)!}\lambda^{k-n}\Big\} = H_{00} + H'_{11}. \end{aligned} \tag{26}$$

Diagonalizing $\bar{H}_0$, we obtain the eigenvalues and the corresponding eigenfunctions

$$E_\pm = H_{00} + \frac{1}{2}\Big(H'_{11} \pm \sqrt{H'^2_{11} - 4H_{01}H_{10}}\Big), \qquad \Psi_\pm = \bar{F}(0) + \frac{E_\pm - H_{00}}{H_{01}}\bar{F}(1). \tag{27}$$

<u>**Example 2**</u>. Let $g(h) = h^3$, $\lambda = (-3 \pm i\sqrt{3})/2$, $(N = 2)$ and

$$\hat{H} = \alpha X^+ + \alpha^* X^- + \beta h, \qquad \alpha, \beta \in \mathbb{C}. \tag{28}$$

The differential realization of $\hat{H}\Psi = E\Psi$ is

$$\Big[\alpha z^3\frac{d^2}{dz^2} + (3\alpha z^2 + \beta z + 3\lambda z + \alpha^*)\frac{d}{dz} + \alpha z + 3\lambda + 3\lambda^2 + \beta\lambda\Big]\Psi(z) = E\Psi(z). \tag{29}$$

Considering the submatrix of $\hat{H}$ on the invariant subspace $\{\phi(0), \phi(1), \phi(2)\}$,

$$\bar{H}_0 = \begin{pmatrix} \beta\lambda & \alpha^* & 0 \\ \alpha(1 + 3\lambda + 3\lambda^2) & \beta(1+\lambda) & 2\alpha^* \\ 0 & \alpha(4 + 6\lambda + 3\lambda^2) & \beta(2+\lambda) \end{pmatrix}, \tag{30}$$

we obtain a part of eigenvalues and the corresponding eigenfunctions

$$E_{1,2,3} = \chi_{1,2,3} - \frac{a}{3}, \qquad \Psi_i(z) = \phi(0) + a_i\phi(1) + b_i\phi(2),$$
$$a_i = \frac{E_i - H_{00}}{\alpha^*}, \qquad b_i = \frac{(H_{00} - E_i)(H_{11} - E_i) - \alpha^* H_{10}}{2\alpha^{*2}}, \tag{31}$$

where

$$\chi_1 = \left[-\frac{d}{2} + \left(\frac{d^2}{4} + \frac{f^3}{27}\right)^{1/2}\right]^{1/3} + \left[-\frac{d}{2} - \left(\frac{d^2}{4} + \frac{f^3}{27}\right)^{1/2}\right]^{1/3},$$
$$\chi_2 = \omega\left[-\frac{d}{2} + \left(\frac{d^2}{4} + \frac{f^3}{27}\right)^{1/2}\right]^{1/3} + \omega\left[-\frac{d}{2} - \left(\frac{d^2}{4} + \frac{f^3}{27}\right)^{1/2}\right]^{1/3},$$
$$\chi_3 = \omega^2\left[-\frac{d}{2} + \left(\frac{d^2}{4} + \frac{f^3}{27}\right)^{1/2}\right]^{1/2} + \omega\left[-\frac{d}{2} - \left(\frac{d^2}{4} + \frac{f^3}{27}\right)^{1/2}\right]^{1/3},$$

where $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$, $d = -\frac{1}{3}a^2 + b$, $f = \frac{2}{27}a^3 - \frac{1}{3}ab + c$, and

$$a = \alpha - 2\lambda - 3, \qquad b = \beta^2\left[\lambda^2 - (2\alpha - 3)\lambda + 2 - 3\alpha\right] - \alpha^2(9\lambda^2 + 15\lambda + 9),$$
$$c = 2|\alpha|^2\alpha\beta(4 + 6\lambda + 3\lambda^2) - \alpha\beta^2(1 + \lambda)(2 + \lambda) - \alpha^2\beta(1 + 3\lambda + 3\lambda^2)(2 + \lambda).$$

4.2 The Case of Infinite Dimension

The QESQ problem for infinite dimensional representation of Lie algebra has been discussed in Ref. [7] and the method used is quite similar to that in Ref. [6]. This subsection will be proceeded based on Refs. [6] and [7].

Define the elements g_i $(i = 1, 2, \cdots, K)$ of the K-multiple product of D:

$$D^{\otimes K} = \overbrace{D \otimes D \otimes D \otimes \cdots D \otimes D}^{K \text{ times}},$$

by

$$g_i = \underset{1,}{1} \otimes \underset{2,}{1} \otimes \cdots \underset{i-1,}{1} \otimes \underset{i,}{g} \otimes \underset{i+1,}{1} \otimes \underset{\cdots,}{\cdots} \otimes \underset{K}{1} \qquad g_i \in D.$$

Accordingly, the representation space $V(\vec{\lambda})$ of $D^{\otimes K}$ is also a K-multiple product

$$V(\lambda_1, \lambda_2, \cdots, \lambda_K) \equiv V(\vec{\lambda}) = V(\lambda_1) \otimes V(\lambda_2) \otimes \cdots V(\lambda_K)$$

with basis $\{F(n_1, n_2, \cdots, n_K) = F(\vec{n}) = F(n_1) \otimes F(n_2) \otimes \cdots \otimes F(n_K),\ F(n_i) \in V(\lambda_i),\ i = 1, 2, \cdots, K\}$.

Associated with $D^{\otimes K}$, the generalized QESQ Hamiltonian is constructed as

$$\hat{H} = \sum_{\sum_{i=1}^K m_i = \sum_{i=1}^K m_i'} A_{\vec{m},\vec{m}'} \prod_{i=1}^{K} \{(X_i^+)^{m_i}(X_i^-)^{m_i}\} + \sum_{n_i=0}^{\infty} B_{\vec{n}} \prod_{i=1}^{K} \{h_i^{n_i}\}, \tag{32}$$
$$A_{\vec{m},\vec{m}'} = A_{m_1,\cdots,m_K;m_1',\cdots,m_K'} \in \mathbb{C}, \qquad B_{\vec{n}} = B_{n_1,\cdots,n_K} \in \mathbb{C}.$$

Because

$$\left\{\prod_{i=1}^{K}(X_i^+)^{m_i}(X_i^-)^{m_i'}\right\}F(n_1, n_2, \cdots, n_K) \propto F(n_1 + m_1 - m_1', n_2 + m_2 - m_2', \cdots, n_K + m_K - m_K'),$$

the summation $\sum_{i=1}^K n_i$ is invariant under the action of $\hat{H}$ for $\sum_{i=1}^K m_i = \sum_{i=1}^K m_i'$. Thus, there exists an $\hat{H}$-invariant subspace $W(\vec{\lambda}, N) : \{F(\vec{n}) | n_1 + n_2 + \cdots + n_K = N\}$ for a fixed $N \in \mathbb{Z}^+$. Then, we can partially diagonalize $\hat{H}$ on the subspace $W(\vec{\lambda}, N)$ and obtain a part of the spectra of $\hat{H}$.

For example, in the case of $g(h) = h$, $K = 2$ and $\hat{H} = X_1^+X_2^- + h_1h_2$, the invariant subspace is chosen as $W(\lambda_1, \lambda_2, 2) : \{F(n_1, n_2) |\ n_1 + n_2 = 2\}$. On $W(\lambda_1, \lambda_2, 2)$, the submatrix H_0 of $\hat{H}$

is

$$H_0 = \begin{pmatrix} \lambda_1(2+\lambda_2) & 0 & 0 \\ 2(1+3\lambda_1+3\lambda_1^2) & (1+\lambda_1)(1+\lambda_2) & 0 \\ 0 & 4+6\lambda_1+3\lambda_1^2 & \lambda_2(2+\lambda_1) \end{pmatrix}. \tag{33}$$

Then, we partially get the spectra of $\hat{H}\Psi = E\Psi$:

$$\begin{aligned} \Big[z_1^3\frac{\partial^3}{\partial z_1^2\partial z_2} + (3z_1^2+3\lambda_1 z_1^2+z_1z_2)\frac{d^2}{dz_1dz_2} + (3\lambda_1^2 z_1+3\lambda_1 z_1+\lambda_1 z_2)\frac{d}{dz_2} \\ + \lambda_2 z_1\frac{d}{dz_1} + (\lambda_1\lambda_2 - E)\Big]\Psi(z_1,z_2) = 0\,. \end{aligned} \tag{34}$$

The eigenvalues and the corresponding eigenfunctions are respectively

$$\begin{aligned} &E_1 = \lambda_2(2+\lambda_1)\,, \qquad E_2 = (1+\lambda_1)(1+\lambda_2)\,, \qquad E_3 = \lambda_1(2+\lambda_2)\,, \\ &\Psi_1 = F(2,0)\,, \qquad \Psi_2 = F(1,1) + \frac{4+6\lambda_1+3\lambda_1^2}{\lambda_1-\lambda_2+1}F(2,0)\,, \\ &\Psi_3 = F(0,2) + \frac{2(1+3\lambda_1+3\lambda_1^2)}{\lambda_1-\lambda_2-1}F(1,1) + \frac{(4+6\lambda_1+3\lambda_1^2)(1+3\lambda_1+3\lambda_1^2)}{(\lambda_2-\lambda_1+1)(\lambda_2-\lambda_1)}F(2,0)\,. \end{aligned} \tag{35}$$

V. Generalized QESQ Models in Terms of Quantum Group

In this section, we will construct a QESQ model with the quantum group (algebra) $\mathrm{sl}_q(2)$ dynamic symmetry. The spectral problem is equivalent to the eigenvalue problem of a class of nonlocal differential equations (integral-differential equations). It is also shown that this QESQ model describes the spin precession with highly nonlinear interaction in an external field.

Considering Eq. (20) and

$$\Big[\mu - z\frac{d}{dz}\Big]z^n = [\mu-n]z^n\,, \qquad \Big[\mu - z\frac{d}{dz}\Big]f(z) = (q^{\mu}f(q^{-1}z) - q^{-\mu}f(qz))(q-q^{-1})^{-1}\,, \tag{36}$$

we write down the realization of $\mathrm{sl}_q(2)$ in an integral form

$$X^+f(z) = \frac{1}{2\pi i}\oint_C A(y,z)f(y)dy, \quad X^-f(z) = \frac{1}{2\pi i}\oint_C B(y,z)f(y)dy, \quad hf(z) = \Big(z\frac{d}{dz} - \mu\Big)f(z), \tag{37}$$

where C is a closed curve around the points qz and $q^{-1}z$ in $\mathbb{C}$, and

$$A(y,z) = (y-q^{-1}z)^{-1}(y-qz)^{-1}\,, \qquad B(y,z) = z\,([\mu]y - [\mu+1]z)\,A(y,z) \tag{38}$$

are the integral kernels.

Now, we consider the spectral problem of the Hamiltonian

$$\hat{H} = \hat{H}(q) = \alpha X^+ + \alpha^* X^- + \beta h\,, \qquad \alpha,\beta \in \mathbb{C}\,. \tag{39}$$

The corresponding function $\hat{H}\Psi = E\Psi$ is equivalent to the nonlocal differential equation

$$\begin{aligned} &(\alpha z^2 q^{\mu} - \alpha^*)\Psi(q^{-1}z) - (\alpha z^2 q^{-\mu} - \alpha^*)\Psi(qz) + \beta z^2(q-q^{-1})\,, \\ &\frac{d}{dz}\Psi(z) = z\Big(E + \beta\frac{\mu}{2}\Big)(q-q^{-1})\Psi(z)\,, \end{aligned} \tag{40}$$

or an integral-differential equation

$$\frac{1}{2\pi i}\oint_C K(y,z)\Psi(y)dy + \beta z\frac{d}{dz}\Psi(z) = \Big(E + \beta\frac{\mu}{2}\Big)\Psi(z) \tag{41}$$

with the integral kernel $K(y,z) = \alpha A(y,z) + \alpha^* B(y,z)$. In fact, we easily observe that $\hat{H} \longrightarrow \hat{H}_0 = \vec{B}\cdot\vec{L}$ as $q \to 1$, where L_1, L_2 and L_3 are usual angular momentum operators satisfying $[L_i,\ L_j] = \epsilon_{ijk}L_k$, $k = 1,2,3$, and $B_1 = \alpha+\alpha^*$, $B_2 = i(\alpha-\alpha^*)$, $B_3 = \beta$. Therefore, the Hamiltonian $\hat{H}(q=1)$ just describes the linear precession of a spin in a magnetic field B. According to Ref. [18], the quantum group operators $X^{\pm}$ can be regarded as the quantum deformations $D(L_{\pm})$ of $L_{\pm} = L_1 \pm iL_2$ in a given irreducible presentation of su(2),

$$L_{\pm}|j,m\rangle = \{(j\mp m)(j\pm m+1)\}^{1/2}|j,m\pm 1\rangle\,, \qquad L_3|j,m\rangle = m|j,m\rangle\,. \tag{42}$$

The explicit forms of these deformations are

$$X^{\pm}=D(L_{\pm})=L_{\pm}\Big(\frac{[j\mp L_3][j\pm L_3+1]}{(j\mp L_3)(j\pm L_3+1)}\Big)^{1/2}\,, \qquad h=D(L_3)=L_3\,. \tag{43}$$

We expand $\{[j\mp L_3][j\pm L_3+1]\}^{1/2}$ about $q=1$ and obtain

$$\begin{aligned} X^{\pm} &= L_{\pm}+\frac{1}{6}(\ln q)^2\{[j(j+1)-1]L_{\pm}+L_{\pm}L_3(L_3\pm 1)\}+\cdots\,,\\ \hat{H} &= \hat{H}_0+\Delta\hat{H}=\hat{H}_0+\frac{1}{6}(\ln q)^2\{[j(j+1)-1](\alpha L_{+}+\alpha^{*}L_{-})\\ &\quad +(\alpha L_{+}+\alpha^{*}L_{-})L_3^2\}+\cdots\,. \end{aligned} \tag{44}$$

Because $\Delta\hat{H}=\hat{H}-\hat{H}_0$ contains higher order interaction of spin coupling, $L_{\pm}L_3$, $L_{\pm}L_3^2$ and so on, the q-deformed Hamiltonian H describes highly the nonlinear interaction. The QESQ scheme built in this paper provides us with a method to partially obtain the excited spectra for such a quantal spectral problem.

On the basis for the irreducible representation of $\mathrm{sl}_q(2)$

$$\Psi_{jm}(z)=\Big\{\prod_{k=m}^{1-j}\Big(\frac{[j+k+1]}{[j-k]}\Big)^{1/2}\Big\}z^{j+m}\,, \quad j=\frac{\mu}{2}\,;\ m=j,j-1,\cdots,-j+1\,, \tag{45}$$

and $\Psi_{j-j}(z)=1$, where μ is taken as an integer, the Hamiltonian H has a block diagonal structure $H=\mathrm{block\,diag}(H^{1/2},H^{3/2},\cdots,H^{\mu/2},\cdots)$, where $H^j=(\langle j,m|\hat{H}|j,m'\rangle|\ m,m'=j,j-1,\cdots,-j)$ is a $(2j+1)\times(2j+1)$ matrix. Without affecting $H^{j'}(j'\neq j)$, we diagonalize H^j to get the spectra of Eq. (40) or Eq. (41) in part. For example, when $j=1$,

$$H^1=\begin{pmatrix}\beta & \sqrt{[2]}\alpha & 0\\ \sqrt{[2]}\alpha^{*} & 0 & \sqrt{[2]}\alpha\\ 0 & \sqrt{[2]}\alpha^{*} & -\beta\end{pmatrix}. \tag{46}$$

Then, we obtain the spectra $E=E_m=m[2(q+q^{-1})|\alpha|^2+\beta^2]^{1/2}$, $m=0,\pm1$, and the corresponding eigenfunctions

$$\Psi_m(z)=\frac{(q+q^{-1})^{1/2}\alpha}{E_m-\beta}\Psi_{11}(z)+\frac{(q+q^{-1})^{1/2}}{E_m+\beta}\alpha^{*}\Psi_{11}(z)+\Psi_{10}(z)\,. \tag{47}$$

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