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Topological Action for One-Dimensional Antiferromagnet as Superposition of Berry's Phase for Coherent States[1]

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Abstract

A new complete set of spin coherent states is constructed and applied to study the time evolution matrix of one-dimensional antiferromagnet by means of the path integral method. A topological term expressed as the superposition of the Berry's phase of individual site is obtained. The superposition is not the Berry's phase of the system but the A-A phase of the spin chains, and the equation obtained by our path integral method is consistent with that given by others.

I. Introduction

A path integral method in terms of the generalized coherent state was early introduced by J.R. Klauder to study the transition matrix elements[1–2]. Through it, the conventional, classical Hamiltonian dynamical formalism arises from an analysis of quantum dynamics restricted to a complete set of vectors. In the following we give a brief introduction to the path integral in terms of the generalized coherent states for the need of this paper.

Let $|l\rangle$ denote vectors in Hilbert space $\mathcal{H}$, where $l = (l^1, \cdots, l^L)$ is a real point in a L-dimensional label space, and these vectors are assumed to be continuously differentiable in the variables. In addition, they are normalized

$$||l|| = 1 \,. \tag{1}$$

A resolution of unity is assumed to be in the form

$$\int |l\rangle\langle l|\delta l = I \,, \tag{2}$$

where δl is a suitable measure. The vectors which satisfy Eqs. (1) and (2) are called the generalized coherent states[2].

We want to derive a path integral representation for the propagator

$$\langle l'', t''|l', t'\rangle \equiv \left\langle l'' \right| \exp\left[\frac{-i(t''-t')H}{t}\right] \left| l' \right\rangle , \tag{3}$$

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where H is the Hamiltonian for the physical system.

According to the usual path integral method and the Trotter formula, the propagator can be written as

$$\langle l'', t''|l', t'\rangle = \int \exp\left\{\frac{i}{\hbar}[i\hbar\langle l(t)|\dot{l}(t)\rangle - H(l(t))]dt\right\} Dl\,, \tag{4}$$

where

$$|\dot{l}(t)\rangle \equiv \frac{d}{dt}|l(t)\rangle\,, \quad Dl \equiv \prod_t \delta l(t) \equiv \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \prod_{k=1}^{N} \delta l_k\,.$$

With the help of the general form of the path integral, equation (1) leads to

$$S = \int_{t'}^{t''} [i\hbar\langle l(t)|\dot{l}(t)\rangle - H(l(t))]dt\,. \tag{5}$$

Equation (5) has the form of classical action functional, where $H(l) = \langle l|H|l\rangle$ is the classical Hamiltonian. Thus, if a set of coherent states is chosen, one can always obtain the classical action functional in terms of the integral of the coherent states.

In this paper we first construct a new set of coherent states for one-dimensional antiferromagnetic spin chain, then we apply the path integral of the coherent states to study the topological term and the equation of motion of the chain. We obtain a topological term expressed as the superposition of the Berry's phases of individual site. The superposition is not the Berry's phase of the system, but the A-A phase of the spin chains. The consistency of the equation of motion obtained by our path integral method with that got by F.D.M. Haldane derives the reasonability of the classical representation of the spin by Haldane.

II. Spin Coherent State

In the following discussion, we take $\hbar = 1$.

For a single particle of spin S (see Refs. [3]–[4]), we consider the state

$$|v_1\rangle \equiv N^{-1/2}\exp(v_1, S_-)|\uparrow\rangle = N^{-1/2}\sum_{P=0}^{2S}\left[\frac{(2S)!}{P!(2S-P)!}\right] v_1^P |P\rangle\,, \tag{6}$$

where $|\uparrow\rangle$ denoted the highest weight vector, $\hat{S}_z|\uparrow\rangle = S|\uparrow\rangle$, and $|P\rangle$ is the eigenstate of $\hat{S}_z$ such that

$$\hat{S}_z|P\rangle = (S-P)|P\rangle\,, \qquad P = 0, 1, 2, \cdots, 2S\,, \tag{7}$$

v_1 runs over the complex plane and N is a normalization factor. We have

$$\langle v_1|v_1\rangle = N^{-1}(1+|v_1|^2)^{2S} \tag{8}$$

and hence the normalized state is

$$|v_1\rangle = (1+|v_1|^2)^{-S}\exp(v_1\hat{S}_-)|\uparrow\rangle\,. \tag{9}$$

It is easy to get the resolution of unity

$$\frac{2S+1}{\pi}\int d^2v_1 \frac{1}{(1+|v_1|^2)^2}|v_1\rangle\langle v_1| = \sum_{P=0}^{2S}|P\rangle\langle P| = 1\,. \tag{10}$$

So $|v_1\rangle$ constructs a normalized complete set of states and they are coherent states or spin coherent states.

By the simple calculation, we can have the following relations:

$$\langle v_1|\hat{S}_z|v_1\rangle = S - \frac{2S|v_1|^2}{1+|v_1|^2}\,, \qquad \langle v_1|\hat{S}_+|v_1\rangle = \frac{2Sv_1}{1+|v_1|^2}\,, \qquad \langle v_1|\hat{S}_-|v_1\rangle = \frac{2Sv_1^*}{1+|v_1|^2}\,. \tag{11}$$

Let us write

$$v_1 = e^{i\phi}\mathrm{tg}\frac{\theta}{2}\,, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi\,, \tag{12}$$

then the normalized states can be written as

$$|v_1\rangle = \left(\cos\frac{\theta}{2}\right)^{2S} \exp\left(\mathrm{tg}\frac{\theta}{2}e^{i\phi}\hat{S}_-\right)|\uparrow\rangle \tag{13}$$

and the completeness relation is

$$\frac{2S+1}{\pi}\int d\phi d\theta \sin\theta |v_1\rangle\langle v_1| = 1\,. \tag{14}$$

From Eq. (11) we have

$$\langle v_1|\hat{S}_z|v_1\rangle = S\cos\theta\,, \quad \langle v_1|\hat{S}_+|v_1\rangle = S\sin\theta e^{i\phi}\,, \quad \langle v_1|\hat{S}_-|v_1\rangle = S\sin\theta e^{-i\phi}\,, \tag{15}$$

form which we get the result for the expectation value of the spin vector

$$\langle v_1|\hat{\vec{S}}|v_1\rangle = S\vec{n}\,, \qquad \vec{n} = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)\,. \tag{16}$$

It is shown that $|v_1\rangle$ is the eigenstate of $\vec{S}\cdot\vec{n}$,

$$(\hat{\vec{S}}\cdot\vec{n})|v_1\rangle = S|v_1\rangle\,, \tag{17}$$

and if we let $S = \frac{1}{2}$, we can obtain

$$|v_1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \equiv |\Omega_1\rangle\,, \tag{18}$$

where in the spin-$\frac{1}{2}$ case, $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Similar to the same consideration as proceeding, we consider the state

$$|v_2\rangle = (1+|v_2|^2)^{-S}\exp(v_2\hat{S}_+)|\downarrow\rangle\,, \tag{19}$$

where $|\downarrow\rangle$ is the lowest weight vector, $\hat{S}_z|\downarrow\rangle = -S|\downarrow\rangle$. It may be demonstrated that $|v_2\rangle$ is a normalized, complete state and so it is a coherent state. We have the following relations:

$$\begin{aligned}\langle v_2|\hat{S}_z|v_2\rangle &= -S + \frac{2S|v_2|^2}{1+|v_2|^2}\,, \quad \langle v_2|\hat{S}_+|v_2\rangle = \frac{2Sv_2^*}{1+|v_2|^2}\,,\\ \langle v_2|\hat{S}_-|v_2\rangle &= \frac{2Sv_2}{1+|v_2|^2}\,.\end{aligned} \tag{20}$$

We choose alternative parameters

$$v_2 = -\mathrm{tg}\frac{\theta}{2}e^{-i\phi} , \tag{21}$$

then $|v_2\rangle$ can be written as

$$|v_2\rangle = \left(\cos\frac{\theta}{2}\right)^{2S} \exp\left(-\mathrm{tg}\frac{\theta}{2}e^{-i\phi}\hat{S}_+\right)|\downarrow\rangle . \tag{22}$$

From Eq. (20) we have

$$\begin{aligned} &\langle v_2|\hat{S}_z|v_2\rangle = -S\cos\theta , \qquad \langle v_2|\hat{S}_+|v_2\rangle = -S\sin\theta e^{i\phi} , \\ &\langle v_2|\hat{S}_-|v_2\rangle = -S\sin\theta e^{-i\phi} , \end{aligned} \tag{23}$$

from which we get

$$\langle v_2|\hat{\vec{S}}|v_2\rangle = -S\vec{n} . \tag{24}$$

It can be shown that $|v_2\rangle$ is also the eigenstate of $\hat{\vec{S}}\cdot\vec{n}$, but the eigenvalue is $-S$,

$$(\hat{\vec{S}}\cdot\vec{n})|v_2\rangle = -S|v_2\rangle , \tag{25}$$

and if we let $S=\frac{1}{2}$, we can obtain

$$|v_2\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} \equiv |\Omega_2\rangle , \tag{26}$$

where for spin-$\frac{1}{2}$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

III. One-Dimensional Spin S Antiferromagnet

The Hamiltonian for the one-dimensional antiferromagnetic Heisenberg model, which only deals with the interaction of the neighboring sites, is

$$H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1} . \tag{27}$$

Let the State

$$\begin{aligned} |N\rangle &= \prod_n \otimes |v_2\rangle_n \otimes |v_1\rangle_{n+1} \\ &= \prod_n \otimes (1+|v_n|^2)^{-S}\exp(v_n\hat{S}_+)|\downarrow\rangle \otimes (1+|v_{n+1}|^2)^{-S}\exp(v_{n+1}\hat{S}_-)|\uparrow\rangle \\ &= \prod_n \otimes \left(\cos\frac{\theta_n}{2}\right)^{2S}\exp\left(-\mathrm{tg}\frac{\theta}{2}e^{-i\phi}\hat{S}_+\right)|\downarrow\rangle \otimes \left(\cos\frac{\theta_{n+1}}{2}\right)^{2S}\exp\left(\mathrm{tg}\frac{\theta_{n+1}}{2}e^{i\phi_{n+1}}\hat{S}_-\right)|\uparrow\rangle \end{aligned} \tag{28}$$

describe the Heisenberg model (27), where $\otimes$ means the direct product. With the help of the discussion about the single particle coherent state, we easily get

$$\langle N|N\rangle = 1 , \quad \int |N\rangle\langle N| \prod_n \frac{2S+1}{\pi(1+|v_n|^2)^2}d^2v_n = \int |N\rangle\langle N| \prod_n \left(\frac{2S+1}{4\pi}\sin\theta_n d\theta_n d\phi_n\right) . \tag{29}$$

Thus, $|N\rangle$ constructs a normalized, complete set of states or coherent states for one-dimensional antiferromagnet. From the introduction the propagator is

$$\langle N|e^{-i(t''-t')H}|N\rangle = \int \exp\Big\{i\int\Big[\langle N|i\frac{d}{dt}|N\rangle - \langle N|H|N\rangle\Big]dt\Big\}D(\theta_n,\phi_n)\ , \tag{30}$$

and the classical action functional is

$$S = \int\Big[\langle N|i\frac{d}{dt}|N\rangle - \langle N|H|N\rangle\Big]dt\ . \tag{31}$$

The first term in S is very interesting. After a complicated calculation, we get

$$\langle N|i\frac{d}{dt}|N\rangle = \sum_n iS\frac{\dot{v}_n v_n^* - v_n\dot{v}_n^*}{1+|v_n|^2}\ . \tag{32}$$

Considering the two kinds of coherent states for single particle of spin S and noting

$$v_n = -\mathrm{tg}\frac{\theta_n}{2}e^{i\phi}\ ,\quad v_{n+1} = \mathrm{tg}\frac{\theta_{n+1}}{2}e^{i\phi_{n+1}}\ ,$$

we have respectively

$$\frac{iS}{1+|v_n|^2}[\dot{v}_n v_n^* - v_n\dot{v}_n^*] = S\dot{\phi}_n(1-\cos\theta_n)\ , \tag{33}$$

$$\frac{iS}{1+|v_{n+1}|^2}[\dot{v}_{n+1}v_{n+1}^* - v_{n+1}\dot{v}_{n+1}^*] = -S\dot{\phi}_{n+1}(1-\cos\theta_{n+1})\ , \tag{34}$$

from which we obtain

$$\langle N|i\frac{d}{dt}|N\rangle = \sum_n[-(-1)^n]S\dot{\phi}_n(1-\cos\theta_n)\ . \tag{35}$$

Let $\Phi(\vec{n}(\theta,\phi),\vec{n}(\theta',\phi'),\vec{n}_0)$ be the area of the spherical triangle with vertices $\vec{n}(\theta,\phi),\vec{n}(\theta',\phi')$ and $\vec{n}_0$, then we have

$$\cos\Phi = 1 - \frac{(1-\cos\theta)(1-\cos\theta')}{1+\cos\theta\cos\theta'+\cos(\phi'-\phi)\sin\theta\sin\theta'}\sin^2(\phi'-\phi)\ , \tag{36}$$

where

$$\vec{n}(\theta,\phi) = (\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)\ ,$$

$$\vec{n}(\theta',\phi') = (\sin\theta'\cos\phi',\sin\theta'\sin\phi',\cos\theta')\ ,\quad \vec{n} = (0,0,1)$$

are three points on unit sphere. If $\vec{n}(\theta',\phi')$ approaches $\vec{n}(\theta,\phi)$ infinitely or $\theta' = \theta + d\theta$, $\phi' = \phi + d\phi$, we can get

$$\Phi = (1-\cos\theta)d\phi = (1-\cos\theta)\dot{\phi}dt\ . \tag{37}$$

From Eq. (35) we can write

$$\int\langle N|i\frac{d}{dt}|N\rangle dt = \sum_n\int\langle v_n|i\frac{d}{dt}|v_n\rangle dt = \sum_n\Theta_n\ , \tag{38}$$

where $\Theta_n = \int\Phi_n = \int\langle v_n|i\frac{d}{dt}|v_n\rangle dt$ is the area of the cap Σ bounded by the wire Γ given by the trajectory $\vec{n}_n = (\sin\theta_n\cos\phi_n,\sin\theta_n\sin\phi_n,\cos\theta_n)$ and it can be written as[5]

$$\Theta_n = \int_0^1 d\tau\int_0^T dt\ \vec{n}(t,\tau)\cdot(\partial_t\vec{n}(t,\tau)\times\partial_\tau(t,\tau))\ , \tag{39}$$

where $\vec{n}(t,\tau)$ is an arbitrary extension of the $n(t)$ variable into the rectangle defined by the limits of integration and satisfies the boundary conditions $\vec{n}(t,0)=\vec{n}(t), \vec{n}(t,1)=\vec{n}_0$, $\vec{n}(0,\tau)=\vec{n}(T,\tau)$. If we take two neighboring sites, then we get a local expression

$$\Phi_n+\Phi_{n+1}\approx a\int_0^T dt\,\vec{n}_n\cdot(\partial_t\vec{n}_n\times\partial_x\vec{n}_n)\,, \tag{40}$$

where a is a lattice distance.

Assuming a smooth variation of the $\vec{n}$, the sum of the individual Θ_n gives

$$\int\langle N|i\frac{d}{dt}|N\rangle dt=\sum_n\Theta_n \stackrel{a\to 0}{=} \frac{1}{2}\int dt\int d\vec{x}\,\vec{n}\cdot(\partial_x\vec{n}\times\partial_t\vec{n})\,. \tag{41}$$

Obviously, it is a topological term. Thus we obtain a topological term in the action functional for one-dimensional antiferromagnet in terms of the coherent state $|N\rangle$.

The individual term in Eq. (38), $\int\langle v_n|i\frac{d}{dt}|v_n\rangle dt$, is the Berry phase[6] of the system whose Hamiltonian is $\hat{\vec{S}}\cdot\vec{n}$. So the topological term of one-dimensional antiferromagnet is the superposition of the Berry's phase of individual spin in $\hat{\vec{S}}\cdot\vec{n}$.

On the other hand, we may ask whether the superposition of the Berry's phase is a Berry's phase of the whole chain. We say it is not, but we can show that it is the A-A phase for the whole chain, we will give the evidence in the following.

Let us assume

$$|N\rangle=e^{if(t)}|M\rangle\,, \tag{42}$$

where $f(t)$ is a function of t, and $|M\rangle$ is a state in the Hilbert space or it satisfies the Schrödinger equation of motion

$$i\frac{d}{dt}|M\rangle=H|M\rangle\,. \tag{43}$$

Here, H is the Hamiltonian for the one-dimensional antiferromagnet. From Eqs. (42) and (43), we have

$$-\frac{df}{dt}=-\langle N|H|N\rangle-\langle N|i\frac{d}{dt}|N\rangle\,, \tag{44}$$

$$-f=\int\Big[\langle N|i\frac{d}{dt}|N\rangle-\langle N|H|N\rangle\Big]dt=S\,. \tag{45}$$

Hence equation (42) becomes

$$|N\rangle=e^{-iS}|M\rangle\,. \tag{46}$$

Let Eq. (31) be acted by H, we get

$$\frac{dS}{dt}\Big|N\rangle=i\frac{d}{dt}|N\rangle-H|N\rangle\,. \tag{47}$$

If we are able to demonstrate Eq. (47) by means of our coherent state $|N\rangle$ (Eq. (28)), it manifests that our assumptions on Eqs. (42) and (43) are right. But it is difficult to show Eq. (47) directly. On the other hand, from Eq. (47) we can reduce

$$\frac{dS}{dt}=\langle N|i\frac{d}{dt}|N\rangle-\langle N|H|N\rangle\,, \tag{48}$$

$$S=\int\Big[\langle N|i\frac{d}{dt}|N\rangle-\langle N|H|N\rangle\Big]dt\,. \tag{49}$$

Obviously, equation (49) is right and this manifests that equation (47) is right indirectly[7]. So $|N\rangle$ is the state in the projective Hilbert space P[8] and $\int\langle N|i\frac{d}{dt}|N\rangle dt$ is the A-A phase of the antiferromagnet.

IV. The Equation of Motion

From Eqs. (11) and (20), we can get

$$\begin{aligned}\langle N|H|N\rangle &= \langle N|\sum_n \vec{S}_n\cdot\vec{S}_{n+1}|N\rangle \\ &= \langle N|\sum_n\Big\{S_{nz}S_{n+1,z}+\frac{1}{2}(S_{n+}S_{n+1}+S_{n-}S_{n+1+})\Big\}|N\rangle \\ &=\sum_n \frac{S^2}{(1+|v_n|^2)(1+|v_{n+1}|^2)}\Big[(-1+|v_n|^2)(1-|v_{n+1}|^2)+2v_n^*v_{n+1}^*+2v_nv_{n+1}\Big]\,.\end{aligned} \tag{50}$$

Regarding v_n and v_n^* as the independent "classical" dynamical variables, a general variation of v_n and v_n^* in the action functional

$$S=\int\Big\{\sum_n iS\frac{\dot{v}_nv_n^*-v_n\dot{v}_n^*}{1+|v_n|^2}-\langle N|H|N\rangle\Big\}dt \tag{51}$$

yields the equations of motion

$$\dot{v}_n=\frac{(1+|v_n|^2)^2}{2Si}\frac{\partial\langle N|H|N\rangle}{\partial v_n^*}\,,\quad \dot{v}_n^*=-\frac{(1+|v_n|^2)^2}{2Si}\frac{\partial\langle N|H|N\rangle}{\partial v_n}\,. \tag{52}$$

From Eq. (50) we have

$$\begin{aligned}\dot{v}_n&=\frac{S}{i}\Big[\frac{v_n(1-|v_{n+1}|^2)+v_{n+1}^*-v_n^2v_{n+1}}{1+|v_{n+1}|^2}+\frac{v_n(1-|v_{n-1}|^2)+v_{n-1}^*-v_n^2v_{n-1}}{1+|v_{n-1}|^2}\Big]\,,\\ \dot{v}_n^*&=\frac{S}{-i}\Big[\frac{v_n^*(1-|v_{n+1}|^2)+v_{n+1}-v_n^{*2}v_{n+1}^*}{1+|v_{n+1}|^2}+\frac{v_n^*(1-|v_{n-1}|^2)+v_{n-1}-v_n^{*2}v_{n-1}^*}{1+|v_{n-1}|^2}\Big]\,.\end{aligned} \tag{53}$$

Substituting the alternative parameter of v_n into Eq. (53) we get

$$\begin{aligned}\dot{\theta}_n&=S[-(-1)^n]\sum_\pm \sin\theta_{n\pm1}\sin(\phi_{n\pm1}-\phi_n)\,,\\ \dot{\phi}_n&=S[-(-1)^n]\sum_\pm[\cos\theta_{n\pm1}-\mathrm{ctg}\,\theta_n\sin\theta_{n\pm1}\cos(\phi_{n\pm1}-\phi_n)]\,.\end{aligned} \tag{54}$$

As classical equations of motion, equations (54) should be related to the Hamilton equations in classical dynamics (the principle of correspondence).

Learning from the quantum mechanics, Heisenberg equations should be changed into Hamilton equations under the transformation for the quantum operators becoming the corresponding expectation values. From Eq. (27), we can get the Heisenberg equation

$$\dot{\hat{\vec{S}}}=\frac{1}{i}[\hat{\vec{S}}_n\cdot\hat{H}]=(\hat{\vec{S}}_{n+1}+\hat{\vec{S}}_{n-1})\times\vec{S}_n\,. \tag{55}$$

Haldane has used the spin angular variables to describe the expectation of spin in antiferromagnet[9]

$$\vec{S}_n=(-1)^nS(\sin\theta_n\cos\phi_n,\sin\theta_n\sin\phi_n,\cos\theta_n)\,. \tag{56}$$

Substituting Eq. (56) into Eq. (55) we have

$$\begin{aligned} \dot{\theta}_n &= -(-1)^n S \sum_{\pm} [\cos\theta_{n\pm1} \sin(\phi_{n\pm1} - \phi_n)] \,, \\ \dot{\phi}_n &= -(-1)^n S \sum_{\pm} [\sin\theta_{n\pm1} - \mathrm{ctg}\,\theta_n \sin\theta_{n\pm1} \cos(\phi_{n\pm1} - \phi_n)] \,. \end{aligned} \tag{57}$$

We see that equations (57) are the same as Eqs. (54). It manifests that Haldane's description about the expectation values of spin in antiferromagnet is all right and reasonable.

V. Discussion

Up to now, we choose the eigenstates of $\hat{\vec{S}} \cdot \vec{n}$ whose eigenvalues are $+S$ respectively to construct the coherent state $|N\rangle$ for one-dimensional antiferromagnet. In the same way, we may choose other eigenstates of $\vec{S} \cdot \vec{n}$ to construct $|N\rangle$, we know that the eigenstates of $\hat{\vec{S}} \cdot \vec{n}$ with $\pm m$ eigenvalues are $e^{\pm im\phi} e^{-is_z\phi} e^{-is_y\theta} |S, \pm m\rangle \equiv |L_{1,2}\rangle$ $(m \neq 0)$, where $\hat{S}_z|S, \pm m\rangle = \pm m|S, \pm m\rangle$. Let them substitute for $|v_1\rangle$ and $|v_2\rangle$ and construct $|N\rangle$ by direct product of individual site in the same way, then we can obtain the same result.

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