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q-Analogue Charged Coherent State and SU(3) Charged, Hypercharged Coherent State[1]

Hong-Yi FAN
Department of Material Science and Engineering
China University of Science and Technology, Hefei 230026, China
Chang-Pu SUN
Department of Physics, Northeast Normal University, Changchun 130024, China
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Abstract

We generalize the conception of q-analogue coherent state to the cases of charged coherent state and charged, hypercharged coherent state. Their explicit expressions in the ordinary Fock space are derived.

I. Introduction

Recently, great interest has been paid to q-analogue of the harmonic oscillator[1−3]. Let $a_i^\dagger(a_i)$ be the creation (annihilation) operators of the q-analogue of boson operators, their commutation relation is postulated as (no summation over repeated indices in a term is implied)

$$a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i} , \tag{1}$$

where N_i is defined to satisfy

$$[N_i,\, a_i^\dagger] = a_i^\dagger , \qquad [N_i,\, a_i] = -a_i . \tag{2}$$

The corresponding Fock space is constructed as

$$||n\rangle_i = \frac{(a_i^\dagger)^n}{([n]!)^{1/2}} ||0\rangle_i , \qquad a_i||0\rangle_i = 0 \tag{3}$$

with

$$a_i^\dagger||n\rangle_i = [n+1]^{1/2}||n+1\rangle_i , \qquad a_i||n\rangle_i = [n]^{1/2}||n-1\rangle_i , \qquad N_i||n\rangle_i = n||n\rangle_i , \tag{4}$$

where $||0\rangle_i$ is the q-analogue boson vacuum, annihilated by a_i and

$$[n]! = [n][n-1]\cdots[2][1] , \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} .$$

As pointed out by Biedenharn[1], it is possible to define a coherent state $||\alpha\rangle_i$ for q-harmonic oscillator by the eigenfunction equation

$$a_i||\alpha\rangle_i = \alpha_i||\alpha\rangle_i , \qquad ||\alpha\rangle_i = \exp_q\left(-\frac{1}{2}|\alpha_i|^2\right) \sum_{n=0} \frac{\alpha_i^n a_i^{\dagger n}}{[n]!} ||0\rangle_i , \tag{5}$$

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where $\exp_q(x)$ is the q-analogue of the exponential function. Enlightened by this, we are naturally motivated to recall the so-called charged coherent state (CCS)[4] and SU(3) charged, hypercharged coherent state (SCHCS)[5] for the ordinary boson operators. Thus, in this letter, we construct the q-analogue of CCS and SCHCS in Sec. II and Sec. III, respectively.

II. q-Analogue of CCS

Since the usual CCS is the eigenstate of both charge operator $b_1^\dagger b_1 - b_2^\dagger b_2$ and two-mode annihilator $b_1 b_2$, where $b_i^\dagger(b_i)$ are the creation (annihilation) operators of the ordinary harmonic oscillator satisfying $[b_i, b_j^\dagger] = \delta_{ij}$, one might think that the q-analogue operators $a_1^\dagger a_1 - a_2^\dagger a_2$ and $a_1 a_2$ would be the candidates for constructing the q-analogue CCS, but this is not, because from Eqs. (1) and (4) we know that

$$[a_1^\dagger a_1 - a_2^\dagger a_2,\, a_1 a_2] = [[N_1] - [N_2],\, a_1 a_2]$$
$$= \frac{\{\cosh[(\gamma/2)(2N_2+1)] - \cosh[(\gamma/2)(2N_1+1)]\}a_1 a_2}{\cosh(\gamma/2)} \neq 0\,, \quad \gamma \equiv \log q\,.$$

Therefore, we turn to consider the commutator $[N_1 - N_2, a_1 a_2]$, because of Eq. (2), we have

$$[N_1 - N_2,\, a_1 a_2] = 0\,. \tag{6}$$

Hence, it is feasible to construct an eigenstate $|\xi, \underline{q}\rangle$ which satisfies

$$a_1 a_2 |\xi, \underline{q}\rangle = \xi |\xi, \underline{q}\rangle\,, \tag{7}$$

$$(N_1 - N_2)|\xi, \underline{q}\rangle = \underline{q}|\xi, \underline{q}\rangle\,. \tag{8}$$

To find out the explicit form of $|\xi, \underline{q}\rangle$ we may work in Fock space of the ordinary harmonic oscillators, that means we are using the following realization[6–8] of q-analogue boson operators

$$N_1 = b_1^\dagger b_1\,, \qquad N_2 = b_2^\dagger b_2\,, \tag{9}$$

$$a_i^\dagger = \sum_{n=0}^{\infty} \sqrt{[n+1]}\,|n+1\rangle_{ii}\langle n|\,, \qquad a_i = \sum_{n=1}^{\infty} \sqrt{[n]}\,|n-1\rangle_{ii}\langle n|\,, \tag{10}$$

where $|n\rangle_i$ is the number state of a harmonic oscillator,

$$|n\rangle_i = \frac{b_i^{\dagger n}}{\sqrt{n!}}|0\rangle_i\,. \tag{11}$$

It then follows that in this realization $||n\rangle_i$ is just expressed as $|n\rangle_i$, as shown in Refs. [6]–[8].

Using Eq. (11) and the normal product form of the vacuum projection operator

$$|0\rangle_{ii}\langle 0| =: \exp[-b_i^\dagger b_i]:\,, \tag{12}$$

where :: denotes the normal product, we can express a_i and $a_i^\dagger$ in terms of the polynomials of b_i and $b_i^\dagger$, e.g.,

$$\begin{aligned} a_i^\dagger &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left\{\frac{[n+1]}{(n+1)!n!}\right\}^{1/2}\frac{(-)^k}{k!}(b_i^\dagger)^{k+n+1}b_i^{k+n}\,, \\ a_i &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left\{\frac{[n]}{(n-1)!n!}\right\}^{1/2}\frac{(-)^k}{k!}(b_i^\dagger)^{k+n-1}b_i^{k+n}\,. \end{aligned} \tag{13}$$

Further, using the operator identity

$$b_i^{\dagger m} b_i^m = b_i^\dagger b_i (b_i^\dagger b_i - 1)(b_i^\dagger b_i - 2) \cdots (b_i^\dagger b_i - m + 1) , \tag{14}$$

we may put a_i into

$$a_i = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left\{ \frac{[n]}{(n-1)!n!} \right\}^{1/2} \frac{(-)^k}{k!} b_i^\dagger b_i (b_i^\dagger b_i - 1)(b_i^\dagger b_i - 2) \cdots (b_i^\dagger b_i - k - n + 2) b_i . \tag{15}$$

Substituting Eqs. (9) and (15) into $[N_1 - N_2,\, a_1 a_2]$, we see that equation (6) still holds. With the help of the two-mode Fock space's completeness relation

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1 n_2\rangle\langle n_1 n_2| = 1 , \qquad |n_1 n_2\rangle \equiv |n_1\rangle_1 |n_2\rangle_2 , \tag{16}$$

we obtain the solution of the eigenstate of $N_1 - N_2$ and $a_1 a_2$

$$|\xi, \underline{q}\rangle = C_{\underline{q}} \sum_{n=0}^{\infty} \frac{\xi^n}{\{[n+\underline{q}]![n]!\}^{1/2}} |n + \underline{q},\, n\rangle , \tag{17}$$

where the q-dependent normalization factor is given by

$$C_{\underline{q}} = \Big(\sum_n \frac{|\xi|^{2n}}{[n+\underline{q}]![n]!} \Big)^{-1/2} . \tag{18}$$

The above expressions are for $\underline{q} > 0$ and analogous expressions for $\underline{q} < 0$ are derived by replacing N_1 by N_2. The q-analogue charged coherent state may also be derived by projected out of the two-mode unnormalized coherent state

$$||\alpha_1 \alpha_2\rangle = \sum_{n_1, n_2 = 0}^{\infty} \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\{[n_1]![n_2]!\}^{1/2}} |n_1 n_2\rangle . \tag{19}$$

This is accomplished by setting

$$\alpha_1 = \lambda e^{-i(\theta+\varphi)} , \qquad \alpha_2 = \mu e^{-i(\theta-\varphi)} \tag{20}$$

and performing the integration over $||\alpha_1 \alpha_2\rangle$ in the following way

$$(\lambda^{-1} e^{i\theta})^{\underline{q}} \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{i\underline{q}\varphi} ||\alpha_1 \alpha_2\rangle = \sum_{n=0}^{\infty} \frac{(\lambda\mu e^{-2i\theta})^n}{\{[n+\underline{q}]![n]!\}^{1/2}} |n + \underline{q},\, n\rangle , \tag{21}$$

which, up to a normalization factor, equals Eq. (17) if we make the identification $\xi = \lambda\mu e^{-2i\theta}$.

III. q-Analogue of SCHCS

The usual SCHCS is simultaneously the eigenstate of $b_1 b_2 b_3$, $Q = (2b_1^\dagger b_1 - b_2^\dagger b_2 - b_1^\dagger b_1)/3$, and $Y = (b_1^\dagger b_1 + b_2^\dagger b_2 - 2b_3^\dagger b_3)/3$, where Q and Y are boson realizations of the charge and hypercharge generators of SU(3) group. In similar to the above discussions, we notice

$$\Big[\frac{1}{3}(2N_1 - N_2 - N_3),\, a_1 a_2 a_3\Big] = 0 , \qquad \Big[\frac{1}{3}(N_1 + N_2 - 2N_3),\, a_1 a_2 a_3\Big] = 0 , \tag{22}$$

where equation (2) has been used. Hence, we are able to construct the q-analogue SCHCS in the following way

$$a_1 a_2 a_3 |zy\bar{q}\rangle = z|zy\bar{q}\rangle\ , \tag{23}$$

$$\tilde{Q}|zy\bar{q}\rangle = \bar{q}|zy\bar{q}\rangle\ , \qquad \tilde{Q} = \frac{1}{3}(2N_1 - N_2 - N_3)\ ; \tag{24}$$

$$\tilde{Y}|zy\bar{q}\rangle = y|zy\bar{q}\rangle\ , \qquad \tilde{Y} = \frac{1}{3}(N_1 + N_2 - 2N_3)\ . \tag{25}$$

Working in the ordinary Fock space, we can express $\tilde{Q}$ and $\tilde{Y}$ by Q and Y, respectively. Then it is not difficult to derive the solution of Eqs. (23)–(25)

$$|zy\bar{q}\rangle = C_{\bar{q}y} \sum_{l=0}^{\infty} \frac{z^l}{\{[l]![l+y+\bar{q}]![l+2y-\bar{q}]!\}^{1/2}}\, |l+y+\bar{q},\, l+2y-\bar{q},\, l\rangle\ , \tag{26}$$

where the normalization coefficient

$$C_{\bar{q}y} = \left\{ \sum_{l=0}^{\infty} \frac{|z|^{2l}}{[l]![l+y+\bar{q}]![l+2y-\bar{q}]!} \right\}^{-1/2} \tag{27}$$

We can also project the state (26) out of the three-mode q-analogue unnormalized coherent state $||\alpha_1\alpha_2\alpha_3\rangle$ by doing the following integration

$$\begin{aligned} &\frac{1}{4\pi^2}\int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi e^{-3i\bar{q}\varphi} e^{-3iy\psi} ||\alpha_1\alpha_2\alpha_3\rangle \\ &= \sum_{l=0}^{\infty} e^{3i\theta(l+y)} \frac{\lambda_1^{l+y+\bar{q}} \lambda_2^{l+2y-\bar{q}} \lambda_3^l}{\{[l]![l+y+\bar{q}]![l+2y-\bar{q}]!\}^{1/2}} |l+y+\bar{q},\, l+2y-\bar{q},\, l\rangle \\ &(\text{let} \quad \alpha_1 = \lambda_1 e^{i(\theta+2\varphi+\psi)}\ , \quad \alpha_2 = \lambda_2 e^{i(\theta-\varphi+\psi)}\ , \quad \alpha_3 = \lambda_3 e^{i(\theta-\varphi-2\psi)}) \end{aligned}$$

which, after setting $z = \lambda_1\lambda_2\lambda_3 e^{i\theta}$, becomes an unnormalized SU(3) charged, hypercharged coherent state.

In summary, we have generalized the conception of q-analogue coherent state to the cases of charged coherent state and SU(3) charged, hypercharged coherent state. Their explicit expressions in the ordinary Fock space are obtained.

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