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A New q-Deformed Boson Realization of the Quantum Algebra $C_q(l)$ and Its Representations[1]

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Abstract

A new q–deformed boson realization of the quantum algebra $C_q(l)$ is given and some of its finite dimensional representations are explicitly presented when q is a root of unity.

Since the so-called quantum algebra was discovered by Jimbo[1–3] and Drinfeld[4], its representation theory has been the focus of much attention and has been studied from many different directions. One of the methods to obtain the representations of a quantum algebra, which has proved to be powerful, is the q–deformed boson realization method. It was first used to construct the symmetric representations of the quantum algebra $A_q(n)$[5]. Recently, the q–deformed boson realization of the quantum algebra $C_q(l)$ has been obtained[6]. In this letter, we will give a new q-deformed boson realization of $C_q(l)$ based on the new realization of $A_q(n)$[7] and construct its finite dimensional representations in the non–generic case (q is a root of unity).

As we know, the quantum algebra $C_q(l)$ is generated by $X_i^{\pm}$ and H_i $(i = 1, 2, \cdots, l)$ which satisfy the commutation relations

$$\begin{aligned}
&[H_i,\ X_j^{\pm}] = \pm(2\delta_{i\,j} - \delta_{i+1\,j} - \delta_{i\,j+1})X_j^{\pm}\,, \quad i,j = 1,2,\cdots,l-1\,,\\
&[H_l,\ X_i^{\pm}] = \pm(2\delta_{i\,l} - \delta_{i+1\,l})\,, \quad [H_i,\ X_l^{\pm}] = \pm(2\delta_{i\,l} - 2\delta_{i+1\,l})\,, \quad i = 1,2,\cdots,l\,,\\
&[H_i,\ H_j] = 0\,, \qquad i,j = 1,2,\cdots,l\,,\\
&[X_i^{+},\ X_j^{-}] = \delta_{ij}[H_i] = \delta_{ij}\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}\,, \quad i,j = 1,2,\cdots,l\,, \quad \text{except } i = j = l\,,\\
&[X_l^{+},\ X_l^{-}] = [H_l]_{q^2} = \frac{q^{2H_l} - q^{-2H_l}}{q^2 - q^{-2}}\,,
\end{aligned} \tag{1}$$

and Serre relations

$$\begin{aligned}
&X_i^{\pm^2} X_j^{\pm} - (q + q^{-1})X_i^{\pm} X_j^{\pm} X_i^{\pm} + X_j^{\pm} X_i^{\pm^2} = 0\,, \quad |i-j| = 1\,, \quad i,j = 1,2,\cdots,l\,,\\
&X_l^{\pm^2} X_{l-1}^{\pm} - (q^2 + q^{-2})X_l^{\pm} X_{l-1}^{\pm} X_l^{\pm} + X_{l-1}^{\pm} X_l^{\pm^2} = 0\,,\\
&X_{l-1}^{\pm^3} X_l^{\pm} - (q^2 + q^{-2} + 1)X_{l-1}^{\pm^2} X_l^{\pm} X_{l-1}^{\pm} + (q^2 + q^{-2} + 1)X_{l-1}^{\pm} X_l^{\pm} X_{l-1}^{\pm^2} - X_l^{\pm} X_{l-1}^{\pm^3} = 0\,.
\end{aligned} \tag{2}$$

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The new q-deformed boson realization of $C_q(l)$ is given by

$$X_i^+ = g_i^{(-)^i}\,, \quad X_i^- = g_i^{(-)^{i+1}}\,, \quad H_i = (-)^i(N_i + N_{i+1} + 1)\,, \quad i = 1, 2, \cdots, l-1\,,$$
$$X_l^+ = g_l^{(-)^l}\,, \quad X_l^- = g_l^{(-)^{l+1}}\,, \quad H_l = (-)^l\Big(N_l + \frac{1}{2}\Big)\,, \tag{3}$$

where $g_i^\pm = \pm a_i^\pm a_{i+1}^\pm$, $g_l^\pm = \pm[2]^{-1}a_l^{\pm^2}$, and a_i^+, $a_i^- = a_i$ and N_i are q-deformed boson operators which satisfy

$$a_i a_i^+ - q^{-1}a_i^+ a_i = q^{N_i}\,, \quad [N_i,\, a_j^\pm] = \pm\delta_{ij}a_j^\pm\,, \quad [N_i,\, N_j] = 0\,,$$
$$[a_i^+,\, a_j^+] = [a_i^-,\, a_j^\pm] = 0\,, \qquad i \neq j\,. \tag{4}$$

Using Eq. (4), by direct calculation one can easily check that equations (1) and (2) are satisfied by Eq. (3) on the q-deformed Fock space $\mathcal{F}_q(l)$ spanned by

$$\{|m_i\rangle = |m_1 m_2 \cdots m_l\rangle = a_1^{+^{m_1}} a_2^{+^{m_2}} \cdots a_l^{+^{m_l}}|0\rangle\, |m_i \in \mathbb{Z}^+\,,\; i = 1, 2, \cdots, l\}\,,$$

where the vacuum state $|0\rangle$ satisfies $a_i|0\rangle = N_i|0\rangle = 0$.

Now, we set to construct the representations of $C_q(l)$ on $\mathcal{F}_q(l)$. For convenience, we choose $\{f(\lambda_1, \lambda_2, \cdots, \lambda_l) = |\lambda_i\rangle = |\lambda_0 + \lambda_1, \lambda_1 + \lambda_2, \cdots, \lambda_{l-1} + \lambda_l\rangle|\lambda_i \in \mathbb{Z},\; i = 1, 2, \cdots, l, \lambda_0 = 0\}$ as the basis of $\mathcal{F}_q(l)$, with which the natural action of $C_q(l)$ on $\mathcal{F}_q(l)$ can be written as

$$\begin{aligned}
g_i^+|\lambda_j\rangle &= |\lambda_j + \delta_{ij}\rangle\,,\\
g_i^-|\lambda_j\rangle &= -[\lambda_{i-1} + \lambda_i][\lambda_{i+1} + \lambda_i]|\lambda_j - \delta_{ij}\rangle\,,\\
H_i|\lambda_j\rangle &= (-1)^i(2\lambda_i + \lambda_{i-1} + \lambda_{i+1} + 1)|\lambda_j\rangle\,,\\
g_l^+|\lambda_j\rangle &= [2]^{-1}|\lambda_j + 2\delta_{jl}\rangle\,,\\
g_l^-|\lambda_j\rangle &= -[2]^{-1}[\lambda_{l-1} + \lambda_l][\lambda_{l-1} + \lambda_l - 1]|\lambda_j - 2\delta_{jl}\rangle\,,\\
H_l|\lambda_j\rangle &= (-1)^l\Big(\lambda_{l-1} + \lambda_l + \frac{1}{2}\Big)|\lambda_j\rangle\,.
\end{aligned} \tag{5}$$

From Eq. (5) we observe that

$$\mathcal{F}_q(l) = \mathcal{F}_q^+(l) \oplus \mathcal{F}_q^-(l)\,,$$

where $\mathcal{F}_q^\pm(l)$ is the $C_q(l)$-invariant subspace of $\mathcal{F}_q(l)$ defined by

$$\mathcal{F}_q^\pm(l) = \{f(\lambda_1, \lambda_2, \cdots, \lambda_l) \in \mathcal{F}_q(l)|(-1)^{\lambda_l} = \pm 1\} = \{f^\pm(\lambda_1, \lambda_2, \cdots, \lambda_l)\}\,.$$

Consequently, the representation Γ given by Eq. (5) is completely reducible:

$$\Gamma = \Gamma^+ \oplus \Gamma^-\,,$$

where $\Gamma^\pm$ is the representation induced by Γ on $\mathcal{F}_q^\pm(l)$.

It is obvious that when q is generic, $\Gamma^\pm$ is an irreducible infinite dimensional representation. But when q is a root of unity, it is not the case. In fact, if $q^p = 1$ ($p \geq 3$ and $q \neq \pm 1$), there exist $C_q(l)$-invariant subspaces of $\mathcal{F}_q^\pm(l)$:

$$S_l^\pm(j, \alpha_j, p) = \{f^\pm(\lambda_i)|\lambda_{j-1} + \lambda_j \geq \alpha_j p\,, \quad \alpha_j = 1, 2, \cdots\}\,, \quad j = 1, 2, \cdots, l\,.$$

So, on the quotient space

$$Q_l^\pm(\alpha_1, \alpha_2, \cdots, \alpha_l, p) = \frac{\mathcal{F}_q^\pm(l)}{\sum_j S_l^\pm(j, \alpha_j, p)}\,,$$

we have

$$\{\bar{f}^{\pm}(\lambda_1,\lambda_2,\cdots,\lambda_l)=f^{\pm}(\lambda_1,\lambda_2,\cdots,\lambda_l) \text{ Mod } \sum_j S_l^{\pm}(j,\alpha_j,p)|0\le\lambda_j+\lambda_{j-1}\le\alpha_j p-1,$$
$$j=1,2,\cdots,l\},$$

we can construct a finite dimensional representation of $C_q(l)$. Explicitly, we have

$$\begin{aligned}
g_i^+\overline{|\lambda_j\rangle}&=\overline{|\lambda_j+\delta_{ij}\rangle}\theta(\lambda_i+\lambda_{i-1}-\alpha_i p+1)\theta(\lambda_{i+1}+\lambda_i-\alpha_{i+1}p+1),\\
g_i^-\overline{|\lambda_j\rangle}&=-[\lambda_i+\lambda_{i-1}][\lambda_{i+1}+\lambda_i]\overline{|\lambda_j-\delta_{ij}\rangle},\\
H_i\overline{|\lambda_j\rangle}&=(-1)^i(2\lambda_i+\lambda_{i-1}+\lambda_{i+1}+1)\overline{|\lambda_j\rangle},\\
g_l^+\overline{|\lambda_j\rangle}&=[2]^{-1}\overline{|\lambda_j+2\delta_{jl}\rangle}\theta(\lambda_l+\lambda_{l-1}-\alpha_l p+1),\\
g_l^-\overline{|\lambda_j\rangle}&=-[2]^{-1}[\lambda_{l-1}+\lambda_l][\lambda_{l-1}+\lambda_l-1]\overline{|\lambda_j-2\delta_{jl}\rangle},\\
H_l\overline{|\lambda_j\rangle}&=(-1)^l\Big(\lambda_{l-1}+\lambda_l+\frac{1}{2}\Big)\overline{|\lambda_j\rangle},
\end{aligned}\tag{6}$$

where $\overline{|\lambda_j\rangle}=\bar{f}^{\pm}(\lambda_1,\lambda_2,\cdots,\lambda_l)$ and θ is a function defined by

$$\theta(x)=\begin{cases}0, & x\ge 0,\\ 1, & x<0.\end{cases}$$

As for the dimension of the above representation, we have the following theorem:

Theorem: The dimension $D_l^{\pm}$ of $Q_l^{\pm}$ is

$$\frac{1}{2}\Big(\prod_{j=1}^{l}\alpha_j p\pm\prod_{j=1}^{l}\sigma(\alpha_j p)\Big),$$

where

$$\sigma(x)=\frac{1}{2}(1-(-1)^x).$$

Proof: To simplify the process we introduce the sets

$$M_k=\{(\lambda_1,\lambda_2,\cdots,\lambda_k)|0\le\lambda_j+\lambda_{j-1}\le\alpha_j p-1, j=1,2,\cdots,k\},\quad k=1,2,\cdots,l,$$

and denote by $n_k^{\pm}$ the number of those elements of M_k that satisfy $(-1)^{\lambda_k}=\pm1$. With these notations we can write $D_l^{\pm}=n_l^{\pm}$. From the condition $0\le\lambda_j+\lambda_{j-1}\le\alpha_j p-1$, we know that when $(-1)^{\lambda_{j-1}}=\pm1$ the numbers of the even and odd integers that λ_j can take are $(\alpha_j p\pm\sigma(\alpha_j p))/2$ and $(\alpha_j p\mp\sigma(\alpha_j p))/2$ respectively. Accordingly, we have

$$n_j^{\pm}=n_{j-1}^{\pm}\cdot\frac{1}{2}(\alpha_j p+\sigma(\alpha_j p))+n_{j-1}^{\mp}\cdot\frac{1}{2}(\alpha_j p-\sigma(\alpha_j p))\tag{7}$$

with the conditions $n_0^+=1$ and $n_0^-=0$. Now, from Eq. (7) one can obtain $n_l^{\pm}$ by induction and prove the theorem without any difficulty.

Before concluding this short paper, let us take $C_q(2)$ as an example. In this special case, equation (3) becomes

$$X_1^+=-a_1a_2,\quad X_1^-=a_1^+a_2^+,\quad H_1=-(N_1+N_2+1),$$
$$X_2^+=[2]^{-1}a_2^{+^2},\quad X_2^-=-[2]^{-1}a_2^2,\quad H_2=\Big(N_2+\frac{1}{2}\Big).$$

Taking $\alpha_1=\alpha_2=1$ and $p=3$, from Eq. (6) we get the following two representations of $C_q(2)$.

a). 5–dimensional representations on $Q_2^+(1,1,3)$:

$$X_1^+ = -E_{13} - [2]^2 E_{35}\,, \qquad X_1^- = E_{31} + E_{53}\,,$$

$$H_1 = -(E_{11} + 3E_{22} + 3E_{33} + 3E_{44} + 5E_{55})\,,$$

$$X_2^+ = [2]^{-1}(E_{21} + E_{54})\,, \qquad X_2^- = -E_{12} - E_{45}\,,$$

$$H_2 = \frac{1}{2}(E_{11} + 5E_{22} + 3E_{33} + E_{44} + 5E_{55})\,.$$

b). 4–dimensional representations on $Q_2^-(1,1,3)$:

$$X_1^+ = -[2]E_{13} - [2]E_{24}\,, \qquad X_1^- = E_{31} + E_{42}\,,$$

$$H_1 = -(2E_{11} + 2E_{22} + 4E_{33} + 4E_{44})\,,$$

$$X_2^+ = [2]^{-1}E_{32}\,, \qquad X_2^- = -E_{23}\,,$$

$$H_2 = \frac{1}{2}(3E_{11} + E_{22} + 5E_{33} + 3E_{44})\,.$$

Here, E_{ab} is a matrix defined by $(E_{ab})_{ij} = \delta_a^i \delta_b^j$.

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