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Indecomposable Representations of the Quantum Group $SU(2)_q$[1]

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Abstract

The infinite-dimensional indecomposable representations are constructed by using the purely algebraic method and the matrix elements for these representations are obtained in an explicit form. Each familiar finite-dimensional irreducible representation is induced on a quotient space.

I. Introduction

The interest in the Yang-Baxter equation (YBE)[1,2] has significantly increased recently[3]. Drinfeld and Jimbo have found the intermate relation between the solutions of the quantum YBE and the irreducible representations of the quantum group[4–7]. At present, considerable researches have been focused on the representation theory of the quantum group. Besides Reshetikhin's work about the irreducible representations of the quantum groups associated with constructing the universal R-matrix[8], there are other methods to realize the quantum group and to obtain its representations. The interesting one of them is the q-deformed Boson realization presented independently by different authors[9–11]. However, in previous works people only paid their attention to the irreducible representations and did not concern with another type representations, e.g. the indecomposable (reducible, but not completely reducible) representation. Since a quantum group is only a q-analogue of Lie algebra and the indecomposable representations of Lie algebra have been well investigated by some authors by using different methods[12–16], in our opinion, it is considerable to study the indecomposable representations of the quantum group so that the representation theory of the quantum group has rather complete development. In this paper, we will construct and study the indecomposable representations of the quantum group $SU(2)_q$ and associate one of them with the usual irreducible representations. The method used here is very similar to that used by Gruber *et al.* for Lie algebra[12,13] and can be directly generalized to investigate other quantum groups. We consider only the case that q is not a root of unity.

II. Quantum Group $SU(2)_q$ as an Associative Algebra

Let $\mathcal{A}$ be an associative algebra generated by the operators J_+, J_- and J_3 satisfying the generating relations

$$[J_+,\ J_-] = [2J_3]\,, \qquad [J_3,\ J_\pm] = \pm J_\pm\,, \tag{1}$$

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where the definition

$$[f]=\frac{q^f-q^{-f}}{q-q^{-1}}\,,\qquad q\in \text{ the field } \mathbb{C} \text{ of complex numbers} \tag{2}$$

holds for any operator f or number f. Considering that, as $q\to 1$, equation (1) becomes the commutation relations

$$[\hat{J}_+,\,\hat{J}_-]=2\hat{J}_3\,,\qquad [\hat{J}_3,\,\hat{J}_\pm]=\pm\hat{J}_\pm\,,\qquad (\hat{J}_a=J_a|_{q\to 1}\,,\quad a=\pm,\,3) \tag{3}$$

of Lie algebra SU(2), we regard $\mathcal{A}$ as a q-deformation of the universal enveloping algebra of Lie algebra SU(2). Now, we call it q-analogue of universal enveloping algebra of SU(2). Choose a basis $\{J_+^\alpha J_-^\beta J_3^\gamma\,|\,\alpha,\,\beta,\,\gamma=0,\,1,\,2,\,\cdots\}$ for the associative algebra $\mathcal{A}$ and define $K=q^{2J_3}$. From Eq. (1), we can obtain the multiplication rules

$$\begin{cases} KJ_+=q^2J_+K\,, & KJ_-=q^{-2}J_-K\,, \quad K^{-1}J_+=q^{-2}J_+K^{-1}\,,\\ K^{-1}J_-=q^2J_-K^{-1}\,, & [J_+,\,J_-]=\dfrac{(K-K^{-1})}{(q-q^{-1})}\,. \end{cases} \tag{4}$$

Equation (4) shows that the operators J_+, J_-, K and K^{-1} generate a subalgebra $SU(2)_q$ of $\mathcal{A}$, which was called the quantum group of Lie algebra SU(2) according to Drinfeld. Here, it is needed to point out that some authors also call $\mathcal{A}$ quantum group $SU(2)_q$, but it does not represent the substantial problem so long as we make the definition clearly.

The following multiplication relations, within the associative algebra $SU(2)_q$, will be needed in order to calculate the matrix element for the various representations of $SU(2)_q$.

$$KJ_+^n=q^{2n}J_+^nK\,,\qquad KJ_-^n=q^{-2n}J_-^nK\,, \tag{5a}$$

$$K^{-1}J_+^n=q^{-2n}J_+^nK^{-1}\,,\quad K^{-1}J_-^n=q^{2n}J_-^nK^{-1}\,, \tag{5b}$$

$$J_-K^n=q^{2n}K^nJ_-\,,\qquad J_-(K^{-1})^n=q^{-2n}(K^{-1})^nJ_-\,, \tag{5c}$$

$$J_-J_+^n=J_+^nJ_-+[n]J_+^{n-1}(q^{-n+1}K^{-1}-q^{n-1}K)(q-q^{-1})^{-1}\,, \tag{5d}$$

$$J_+J_-^n=J_-^nJ_++[n]J_-^{n-1}(q^{-n+1}K-q^{n-1}K^{-1})(q-q^{-1})^{-1}\,. \tag{5e}$$

The above equations (5a-5e) can be proved from Eq. (4) by induction. In the following sections, we choose a basis

$$X(m,\,n,\,r)=J_+^mJ_-^nK^r=\begin{cases} J_+^mJ_-^nK^r\,, & \text{for } m,\,n=0,\,1,\,2,\,\dots\,;\ r=1,\,2,\,\dots\\ \mathbb{1}\,, & \text{for } m=n=r=0\\ J_+^mJ_-^n(K^{-1})^{-r} & \text{for } m,\,n=0,\,1,\,2,\,\dots\,;\quad r=-1,\,-2,\,\dots \end{cases} \tag{6}$$

for the associative algebra $SU(2)_q$.

III. Regular Representation of $SU(2)_q$

Since $SU(2)_q$ is an associative algebra, its natural representation can be constructed on its own linear space by the left transformation action ρ: $SU(2)_q\to Edv(SU(2)_q)$

$$\rho(x)\cdot u=x\cdot u\,,\qquad \forall u,\,x\in SU(2)_q\,.$$

On the basis (6) for $SU(2)_q$, this representation is explicitly written as

$$\begin{cases} \rho(J_+)X(m,\,n,\,r) = X(m+1,\,n,\,r)\;, \\ \rho(J_-)X(m,\,n,\,r) = X(m,\,n+1,\,r) + \dfrac{[m]}{q-q^{-1}} \\ \qquad\cdot\{(q^{-m+1+2n}X(m-1,\,n,\,r-1) + q^{m-1-2n}X(m-1,\,n,\,r+1)\}\;, \\ \rho(K)X(m,\,n,\,r) = q^{2m-2n}X(m,\,n,\,r+1)\;, \\ \rho(K^{-1})X(m,\,n,\,r) = q^{2n-1m}X(m,\,n,\,r-1)\;. \end{cases} \tag{7}$$

In comparison with Eqs. (5a-5e) the representation (7) is very similar to the master representation for Lie algebra[11,12] and mathematically called regular representation[17]. The regular representation (7) is an infinite-dimensional and quite general. On the basis of certain invariant subspaces and quotient spaces for $SU(2)_q$, the regular representation can respectively subduce and induce various representations containing the usual irreducible representations. It can be seen from Eq. (7) that the action of ρ can only increase the index of vector $X(m,\,n,\,r)$, whereas indices m and r can be either increased or decreased. Thus, for any given positive integer N, there is an invariant subspace V_N with basis $\{X(m,\,n+N,\,r)|m,\,n,\,r = 0,\,1,\,2,\,\ldots\}$, on which representation (7) subduces a subrepresentation of $SU(2)_q$. On the quotient space $Q_N = SU(2)_q/V_N$ with respect to V_N, a quotient representation of $SU(2)_q$ is induced. Because $V_{N'}$ is an invariant subspace of V_N when $N' > N$, equation (7) also induces a new quotient representation of $SU(2)_q$ on each quotient space $Q(N,\,N') = V_N/V_{N'}$. Then, various representations of $SU(2)_q$ are obtained from the regular representation (7).

Obviously, there exists a sequence of ρ-invariant subspace

$$\Omega = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_N \supset V_{N+1} \supset \cdots$$

and the regular representation has a semidirect sum structure

$$\rho = \rho_0 \bar{\oplus} \rho_1 \bar{\oplus} \rho_2 \bar{\oplus} \ldots \bar{\oplus} \rho_N \bar{\oplus} \ldots = \begin{bmatrix} \rho_0,\ *,\ * \cdots * \cdots \\ 0,\ \rho_1,\ * \cdots * \cdots \\ 0,\ 0,\ \rho_2,\ \cdots * \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0,\ 0,\ 0,\ \cdots\ \rho_N\ \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix}, \tag{8}$$

where each $*$ is a nonzero matrix and ρ_N is a quotient representation induced by ρ on the quotient space $Q(N,\,N+1)$. On the basis $\{X_N(m,\,r) = X(m,\,N,\,r) \text{ Mod } V_{N+1}|m,\,r = 1,\,2,\,\ldots\}$ for $Q(N,\,N+1)$, the explicit expression for ρ_N is given by

$$\begin{cases} \rho_N(J_+)X_N(m,\,r) = X_N(m+1,\,r)\;, \\ \rho_N(J_-)X_N(m,\,r) = \dfrac{[m]}{q-q^{-1}}\{q^{-m+1+2N}X_N(m-1,\,r-1) \\ \qquad +q^{m-1-2N}X_N(m-1,\,r+1)\}\;, \\ \rho_N(K)X_N(m,\,r) = q^{2m-2N}X_N(m,\,r+1)\;, \\ \rho_N(K^{-1})X_N(m,\,r) = q^{-2m+2N}X_N(m,\,r-1)\;. \end{cases} \tag{9}$$

It is easy to check that ρ_N forms an infinite-dimensional nonunitary representation of $SU(2)_q$.

IV. Representations Related to Left Ideals

For a given element b of $SU(2)_q$ and a given complex number ξ, the associative algebra $SU(2)_q$ has a left ideal $L(b,\xi) = \{X(b-\mathbb{1},\xi)|X \in SU(2)_q\}$ as a ρ-invariant subspace. On the corresponding quotient space $Q(b,\xi) = \Omega/L(b,\xi)$, the regular representation ρ can induce a new representation of $SU(2)_q$. In this way we can obtain various representations so long as we properly choose b and ξ so that $Q(b,\xi)$ is not trivial. In the case with $\xi = q^{2\lambda}(\lambda \in \mathbb{C})$ and $b = K$, the quotient space $Q(K, q^{2\lambda})$ with the basis

$$\{X(m,n) = X(m,n,0) \operatorname{Mod} L(K, q^{2\lambda})|m, n = 0, 1, 2 \cdots\}$$

carries a representation of $SU(2)_q$

$$\begin{cases} \rho(J_+)X(m,n) = X_n(m+1,n)\,, \\ \rho(J_-)X(m,n) = X(m,n+1) + [m][2n-m+1-2\lambda]X(m-1,n) \\ \rho(K)X(m,n) = q^{2m-2n+2\lambda}X_N(m,n)\,, \\ \rho(K^{-1})X(m,n) = q^{2n-2m-2\lambda}X(m,n)\,. \end{cases} \tag{10}$$

The representation (10) is also an indecomposable representation of $SU(2)_q$. It is easily seen from Eq. (10) that there exists an invariant space $S_R : \{X(m, R+n)|m, n = 0, 1, 2, \ldots\}$ for given $R = 0, 1, 2, \ldots$. On the quotient space S_R/S_{R+1} with basis $\{X(m) = X(m,R) \text{ Mod } S_{R+1}|m = 0, 1, 2, \ldots\}$, equation (7) induces a new representation

$$\begin{cases} \bar{\rho}_R(J_+)X_R(m) = X_R(m+1)\,, \\ \bar{\rho}_R(J_-)X_R(m) = [m][2R-m+1-2\lambda]X_R(m-1) \\ \bar{\rho}_R(K)X_R(m) = q^{-2R+2m+2\lambda}X_R(m)\,, \\ \bar{\rho}_R(K^{-1})X_R(m) = q^{2R-2m-2\lambda}X_R(m)\,. \end{cases} \tag{11}$$

This is a Verma representation with the lowest weight $\lambda - R$ and the corresponding weight vector is $X_R(0)$. Then, the representation (10), a reducible representation, is decomposed into a semidirect sum of the representations $\bar{\rho}_R(R = 0, 1, 2, \ldots)$,i.e.

$$\rho = \bar{\rho}_0 \bar{\oplus} \bar{\rho}_1 \bar{\oplus} \bar{\rho}_2 \bar{\oplus} \cdots \bar{\oplus} \bar{\rho}_N \bar{\oplus} \cdots$$

When $2(\lambda - R)$=integer, equation (11) is still reducible, but not complete reducible, i.e. representation (11) is indecomposable. In fact, there are two extremal vectors $X_R(2R+1-2\lambda)$ and $X_R(0)$ such that

$$\bar{\rho}_R(J_-)X_R(2R+1-2\lambda) = 0\,, \qquad \bar{\rho}_R(J_-)X_R(0) = 0\,,$$

i.e. $\bar{\rho}_R$ has an invariant subspace

$$W(2R+1-2\lambda) : \{X_R(2R+1-2\lambda+m)|m = 0, 1, 2, \cdots\}$$

with the extremal vector $X(2R+1-2\lambda)$. The quotient space with respect to the invariant space $W(2R+1-2\lambda)$ has a finite dimension and carries a finite-dimensional representation.

In the case with $b = J_-$ and $\xi = \Lambda \in \mathbb{C}$, the quotient space $Q(J_-, \Lambda)$ with the basis

$\{Y(m, r) = X(m, 0, r)$ Mod $L(J_-, \Lambda)|m, r = 0, 1, 2, \cdots\}$ carries a representation of $SU(2)_q$

$$\begin{cases} \rho(J_+)Y(m, r) = Y(m+1, r) , \\ \rho(J_-)Y(m, r) = \Lambda q^{2r} Y(m, r) + \frac{[m]q^{-m+1}}{(q-q^{-1})} Y(m-1, r-1) \\ \qquad + \frac{[m]q^{m-1}}{(q-q^{-1}}\} Y(m-1, r+1) , \\ \rho(K)Y(m, r) = q^{2m} Y(m, r+1) , \\ \rho(K^{-1})Y(m, r) = q^{-2m} Y(m, r-1) . \end{cases} \tag{12}$$

This is an infinite-dimensional irreducible representation of $SU(2)_q$.

V. Finite-Dimensional Irreducible Representation

Let us consider the representation induced by Eq. (10) on the quotient space $Q(K, q^{2\lambda})/I_+$ with respect to the left ideal $I_+ = \{X(J_+ - \mu \mathbb{1})|X \in Q(K, q^{2\lambda})\}$ for $\mu \in \mathbb{C}$. Choosing a basis $\{X(n) = X(n, 0)$ Mod $I_+|n = 0, 1, 2, \cdots\}$ for the space $Q(K, q^{2\lambda})/I_+$, we obtain a representation

$$\begin{cases} \rho(J_+)X(n) = \mu X(n) + [n][2\lambda - n + 1]X(n-1) , \\ \rho(J_-)X(n) = X(n+1) \\ \rho(K)X(n) = q^{-2n+2\lambda} X(n) , \\ \rho(K^{-1})X(n) = q^{2n-2\lambda} X(n) , \end{cases} \tag{13}$$

from Eq. (10). When $2\lambda \neq$positive integer, this representation is irreducible because there is not invariant subspace. On the contrary, in the case of 2λ=positive integer and let $\mu = 0$, we have

$$\rho(J_+)X(2\lambda + 1) = 0 , \tag{14}$$

i.e. $X(2\lambda+1)$ is an extremal vector of invariant subspace $U_{2\lambda}$: $\{X(n+2\lambda+1)|n = 0, 1, 2, \ldots\}$. Like the regular representation, the representation (13) also has semidirect sum structure and is indecomposable.

Now, we consider the representation on the quotient space $(Q(K, q^{2\lambda})/I_+)/U_{2\lambda}$ with the basis $\{\phi_\lambda(n) = X(n)$ Mod $U_{2\lambda}|n = 0, 1, 2, \ldots, 2\lambda\}$. According to Eqs. (13) and (14), this representation is written in the following explicit forms

$$\begin{cases} \rho(J_+)\phi_\lambda(n) = [n][2\lambda - n + 1]\phi(n-1) , \\ \rho(J_-)\phi_\lambda(n) = \begin{cases} 0 , & \text{for} \quad n = 2\lambda ; \\ \phi(n+1) & \text{for} \quad n < 2\lambda , \end{cases} \\ \rho(K)\phi_\lambda(n) = q^{-2n+2\lambda}\phi(n) , \\ \rho(K^{-1})\phi_\lambda(n) = q^{2n-2\lambda}\phi(n) , \qquad n = 0, 1, 2, \ldots, 2\lambda , \end{cases} \tag{15}$$

which has $2\lambda + 1$ dimensions. Defining the "angular momentum" basis

$$|\lambda, m\rangle = \prod_{k=0}^{m-1} ([\lambda - K]![\lambda + K + 1]!)^{1/2} \phi_\lambda(\lambda - m) , \tag{16}$$

$$[n]! \equiv [n][n-1] \ldots [2][1] ; \quad m = \lambda, \lambda - 1, \ldots, -\lambda ,$$

for the representation space $(Q(K, q^{2\lambda})/I_+)/U_{2\lambda}$, equation (15) is rewritten as

$$\begin{cases} \rho(J_\pm)|\lambda, m\rangle = ([\lambda \mp m][\lambda \pm m + 1])^{1/2}|\lambda, m+1\rangle\,, \\ \rho(K)|\lambda, m\rangle = q^{2m}|\lambda, m\rangle\,, \\ \rho(K^{-1})|\lambda, m\rangle = q^{-2m}|\lambda, m\rangle\,. \end{cases} \tag{17}$$

This is just the standard form of the irreducible representation of $SU(2)_q$. The expressions (17) are also obtained from infinite-dimensional representation (11).

VI. Remarks

Recently, many works concerned the representation theory of the quantum group in both cases that q is a root of unity[18] and q is not a root of unity[19], but the finite-dimensional case was discussed only. In this paper we only studied the case that q is not a root of unity, because some representations (e.g., equations (7), (9), (10) and (12)) are infinite-dimensional and non-unitary, the theorems about finite dimensional and unitary case by Rosso and others[18] do not directly work. Thus we need further discussions for the general theory based on this paper.

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