Commun. Theor. Phys. 13(1990)217-226

Vol. 13, No. 2

Boson Realization of Representations for Quantum Groups and Its Differential Correspondence in Bargmann Space

Chang-pu SUN[1]
Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071 and
Department of Physics, Northeast Normal University
Changchun 130024, China
and
Hong-chen FU
Department of Physics, Northeast Normal University, Changchun 130024, China

(Received November 20, 1989)

Abstract

By constructing the q-analogue of the Heisenberg-Weyl algebra in terms of usual creation and annihilation operators of boson states in the Fock space, the boson realization method recently suggested[7-11] is generalized to obtain a class of representations of quantum group in the Fock space. The q-deformed differential realization of quantum groups proposed by Alvarez-Gaumé is derived by making use of the boson realization in this paper.

I. Introduction

Yang-Baxter equation (YBE) plays a key role in the quantum inverse scattering method and solvable models in statistical mechanics[1]. Drinfield[2] and Jimbo[3] showed that underlying the solutions to the YBE is a new type of mathematical structure, the q-deformation of Lie algebras, which mathematically is a Hopf algebra and loosely called quantum group, and to each representation of the quantum group there is a solution to the YBE. Therefore, it is necessary to study the representations of the quantum group associating with different physical models.

Recently, Reshetikhin[4] studied the representations of quantum group by general mathematical method and derived the fusion rules for the R-matrices associated with solutions to the YBE. Alvarez-Gaume *et al.* gave the q-deformed differential realization of quantum group when they studied the relation between rational conformal field theories and quantum group[5]. On the basis of Ref. [6], the authors generalized the boson (-fermion) realization method used to study the representations of Lie algebras[7,8], Lie superalgebras[9] and Kac-Moody algebras[10,11] to construct the representations of quantum group $(A_n)_q$ in the q-deformed Fock space with non-physical basis[12]. In this paper, we shall construct the representations

[1]Present address: Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, China

of quantum group in the real physical space—Fock space with $(C_n)_q$ as an example by introducing the q-analogue of Heisenberg-Weyl algebra. From the boson realization of quantum group, Alvarez-Gaume's q-deformed differential realization of quantum group is derived.

This paper is organized as follows. After constructing the q-analogue of Heisenberg-Weyl algebra in Sec. II, we turn in Sec. III to study the representations of quantum group by making use of the boson realization with $(C_n)_q$ as an example. In Sec. IV we shall drive Alvarez-Gaume's q-deformed differential realization of quantum group.

The symbol Z^+ denotes the set of non-negative integers. The symbol $\mathbb{C}$ denotes the complex number field.

II. q-Analogue of the Heisenberg-Weyl Algebra

Corresponding to the 1-state Heisenberg-Weyl algebra $\mathcal{H}_1 : \{b^+, b, I\}$ satisfying

$$[b\,,\, b^+] = I\,, \qquad [I\,,\, b^+] = 0\,, \qquad [I\,,\, b] = 0\,, \tag{1}$$

the l-state Fock space $\mathcal{F}_1$ is spanned by the non-normalized basis $|n\rangle$

$$|n\rangle = b^{+n}|0\rangle\,, \qquad (n \in Z^+)\,,$$

where $|0\rangle$ is the vacuum state satisfying $b|0\rangle = 0$. From Eq. (1) it is easy to see that

$$b^+|n\rangle = |n+1\rangle\,, \qquad b|n\rangle = n|n-1\rangle\,. \tag{2}$$

Now we consider such a pair of operators a^+ and a in $\mathcal{F}_1$ such that

$$a^+|n\rangle = |n+1\rangle\,, \qquad a|n\rangle = [n]|n-1\rangle\,, \tag{3}$$

where the definition

$$[f] = \frac{q^f - q^{-f}}{q - q^{-1}}\,, \qquad (q \in \mathbb{C}) \tag{4}$$

holds not only for numbers but also for operators in $\mathcal{F}_1$. In terms of the expression of vacuum projective operator

$$|0\rangle\langle 0| =: e^{-b^+b} := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} b^{+k} b^k\,, \tag{5}$$

the operators a^+ and a are expressed as

$$\begin{aligned} a^+ &= b^+\,, \\ a &= \sum_{m=1}^{\infty} \frac{[m]}{m!}|m-1\rangle\langle m| = \sum_{m=0}^{\infty} \frac{[m]}{m!} b^{+m-1}|0\rangle\langle 0|b^m \\ &= \sum_{m=1}^{\infty} \frac{[m]}{m!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} b^{+m+k-1} b^{m+k}\,. \end{aligned} \tag{6}$$

From Eqs. (3) and (6) we can check that

$$\begin{aligned} &aa^+ - q^{-1}a^+a = q^N\,, \qquad N = b^+b\,, \\ &[N\,,\, a^+] = a^+\,, \qquad [N\,,\, a] = -a\,, \end{aligned} \tag{7}$$

which is just the relations used to study the representations of the quantum group $SU(2)_q$ in Ref. [6] and $SU(n)_q$ in Ref. [11] in the q–deformed Fock space. When $q \to 1$, equation (7) becomes the commutation relations of the usual Heisenberg–Weyl algebra $\mathcal{H}_1$. The associative algebra $\mathcal{H}_q(1)$ generated by $\{a, a^+, N\}$ that satisfies the relations (7) is called the q–analogue of the Heisenberg-Weyl algebra $\mathcal{H}_1$. From Eq. (7) we also have

$$\begin{aligned} &Na^{+n} = a^{+n}N + na^{+n} , \\ &a^+q^N = q^{-1}q^N a^+ , \qquad aq^N = qq^N a , \\ &aa^{+n} = q^{1-n}[n]q^N a^{+n-1} + q^{-n}a^{+n}a . \end{aligned} \tag{8}$$

The above discussions can be naturally extended to the case of the n–state Heisenberg–Weyl algebra $\mathcal{H}(n) : \{b_i^+ , b_i , I | i = 1, 2, \cdots, n\}$ with commutation relations

$$[b_i , b_j^+] = I\delta_{ij} , \quad [I , b_i^+] = [I , b_i] = 0 . \tag{9}$$

Its corresponding q–analogue $\mathcal{H}_q(n)$ is generated by $\{a_i^+, a_i, N_i = b_i^+ b_i, I | i = 1, 2, \cdots, n\}$ with relations

$$\begin{aligned} &a_i a_j^+ = \begin{cases} a_j^+ a_i & \text{for } i \neq j , \\ q^{-1}a_i^+ a_i + q^{N_i} & \text{for } i = j , \end{cases} \\ &[N_i , a_j^+] = \delta_{ij}a_i^+ , \qquad [N_i , a_j] = -\delta_{ij}a_i . \end{aligned} \tag{10}$$

When $q \to 1$ (or $q = e^{\hbar/2}, \hbar \to 0$), $\mathcal{H}_q(n)$ becomes the usual Heisenberg-Weyl algebra $\mathcal{H}(n)$. The relations (8) can also be extended to the $\mathcal{H}_q(n)$.

The relations (7), (8) and (10) will be used to study the representations of the quantum group with $(C_n)_q$ as an example in Sec. III and derive the Alvarez–Gaumé's q–deformed differential realization of quantum group in Sec. IV.

III. Quantum Group and Its Representations With $(C_n)_q$ as an Example

Let L be a classical Lie algebra with Chevalley basis $\{h_{\alpha_i}, e_{\pm\alpha_i} | i = 1, 2, \cdots, l = \text{rank}L, \alpha_i$ is the simple roots of $L\}$ satisfying the Lie product

$$\begin{aligned} &[h_{\alpha_m}, h_{\alpha_j}] = 0 , \\ &[h_{\alpha_m}, e_{\pm\alpha_j}] = \pm A_{mj} e_{\pm\alpha_j} , \\ &[e_{\alpha_m}, e_{-\alpha_j}] = \delta_{mj} h_{\alpha_m} , \end{aligned} \tag{11}$$

where A_{ij}'s are the matrix elements of Cartan matrix of L. Its q-deformation L_q, or quantum group L_q associated with L, is an associative algebra generated by $\{\hat{h}_{\alpha_i}, \hat{e}_{\pm\alpha_i}\}$ satisfying

$$\begin{aligned} &[\hat{h}_{\alpha_m} , \hat{h}_{\alpha_j}] = 0 , \\ &[\hat{h}_{\alpha_m} , \hat{e}_{\pm\alpha_j}] = \pm A_{mj} \hat{e}_{\pm\alpha_j} , \\ &[\hat{e}_{\alpha_m} , \hat{e}_{-\alpha_j}] = \delta_{mj} [\hat{h}_{\alpha_m}] , \end{aligned} \tag{12}$$

and the relation

$$\sum_{l=0}^{n_{ij}}(-1)^l \begin{bmatrix} n_{ij} \\ l \end{bmatrix}_{q_i} q_i^{-l(n_{ij}-l)}(e_{\pm\alpha_i})^l e_{\pm\alpha_j}(e_{\pm\alpha_i})^{n_{ij}-l} = 0 \,, \qquad i \neq j, l = \text{rank} L \tag{13}$$

with $q_i = q^{(\alpha_i,\alpha_j)/2}$, $n_{ij} = 1 - (\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$, where (,) is the scalar product of roots.

The boson realization of quantum group L_q is mapping T of L_q onto the operator algebra $\mathcal{A}$ on the Fock space, which satisfies

$$\begin{aligned} &[T(\hat{h}_{\alpha_i}),\ T(\hat{h}_{\alpha_j})] = 0 \,, \\ &[T(\hat{h}_{\alpha m}),\ T(\hat{e}_{\pm\alpha_j})] = \pm A_{mj} T(\hat{e}_{\pm\alpha_j}) \,, \\ &[T(\hat{e}_{\alpha_m}),\ T(\hat{e}_{-\alpha_j})] = \delta_{mj}[T(\hat{h}_{\alpha_m})] \,, \\ &\tilde{x} \equiv T(\hat{x}) \,,\ \ \forall \hat{x} \in \{\hat{h}_{\alpha_i},\ \hat{e}_{\alpha_i}\} \,. \end{aligned} \tag{14}$$

The boson realization of L_q generated by $\{\tilde{h}_{\alpha_i}, \tilde{e}_{\pm\alpha_i} | i = 1, 2, \cdots, l\}$ can be regarded as a subalgebra of $\mathcal{A}$.

For the classical Lie algebra C_n with Chevalley basis

$$\begin{aligned} h_{\alpha_j} &= E_{jj} - E_{j+1,j+1} - E_{n+j,n+j} + E_{n+j+1,n+j+1} \,, \\ &\qquad (j = 1, 2, \cdots, n-1) \,, \\ h_{\alpha_n} &= E_{nn} - E_{2n,2n} \,, \\ e_{\alpha_j} &= E_{j,j+1} - E_{n+j+1,n+j} \,, \qquad (j = 1, 2, \cdots, n-1) \,, \\ e_{-\alpha_j} &= E_{j+1,j} - E_{n+j,n+j+1} \,, \qquad (j = 1, 2, \cdots, n-1) \,, \\ e_{\alpha_n} &= E_{n,2n} \,, \\ e_{-\alpha_n} &= E_{2n,n} \,, \end{aligned} \tag{15}$$

where (E_{ij}) is a $2n \times 2n$ matrix with matrix element $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, we can give a boson realization of quantum group $(C_n)_q$ as

$$\begin{aligned} \hat{h}_{\alpha_j} &= N_j - N_{j+1} - N_{n+j} + N_{n+j+1} \,, \quad (j = 1, 2, \cdots, n-1) \,, \\ \hat{h}_{\alpha_n} &= N_n - N_{2n} \,, \\ \hat{e}_{\alpha_j} &= a_j^+ a_{j+1} - a_{n+j+1}^+ a_{n+j} \,, \qquad (j = 1, 2, \cdots, n-1) \,, \\ \hat{e}_{-\alpha_j} &= a_{j+1}^+ a_j - a_{n+j}^+ a_{n+j+1} \,, \qquad (j = 1, 2, \cdots, n-1) \,, \\ \hat{e}_{\alpha_n} &= a_n^+ a_{2n} \,, \\ \hat{e}_{-\alpha_n} &= a_{2n}^+ a_n \,, \end{aligned} \tag{16}$$

that satisfies the relations (14) with the Cartan matrix (A_{ij}) of C_n on the Fock Space $\mathcal{F}_{2n}$.

On the Fock space $\mathcal{F}_{2n}$ of $2n$ boson states with basis

$$|K_i\rangle = b_1^{+k_1} b_2^{+k_2} \cdots b_{2n}^{+k_{2n}} |0, 0, \cdots, 0\rangle \ , \tag{17}$$

by defining the action of $(C_n)_q$ on $\mathcal{F}_{2n}$

$$\Gamma(x)|u\rangle = x|u\rangle \ , \qquad \forall x \in (C_n)_q \, , \ |u\rangle \in \mathcal{F}_{2n} \ , \tag{18}$$

we obtain a representation Γ of $(C_n)_q$ on $\mathcal{F}_{2n}$ as

$$\begin{cases} \Gamma(\hat{h}_{\alpha_j})|K_i\rangle = (K_j - K_{j+1} - K_{n+j} + K_{n+j+1})|K_i\rangle \ , \\ \Gamma(\hat{h}_{\alpha_n})|K_i\rangle = (K_n - K_{2n})|K_i\rangle \ , \\ \Gamma(\hat{e}_{\alpha_j})|K_i\rangle = [K_{j+1}]|K_i + \delta_{ij} - \delta_{i,j+1}\rangle \\ \qquad\qquad - [K_{n+j}]|K_i + \delta_{i,n+j+1} - \delta_{i,n+j}\rangle \ , \\ \Gamma(\hat{e}_{-\alpha_j})|K_i\rangle = [K_j]|K_i + \delta_{i,j+1} - \delta_{ij}\rangle \\ \qquad\qquad - [K_{n+j+1}]|K_i + \delta_{i,n+j} - \delta_{i,n+j+1}\rangle \ , \\ \Gamma(\hat{e}_{\alpha_n})|K_i\rangle = [K_{2n}]|K_i + \delta_{in} - \delta_{i,2n}\rangle \ , \\ \Gamma(\hat{e}_{-\alpha_n})|K_i\rangle = [K_n]|K_i + \delta_{i,2n} - \delta_{in}\rangle \ . \end{cases} \tag{19}$$

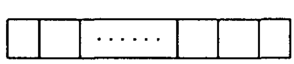
Because the value $\sum_{i=1}^{2n} k_i$ doesn't change in the representation (19) under the action of Γ, every non–negative integer $K \in Z^+$ defines an invariant subspace $\mathcal{F}_{2n}(K)$

$$\begin{aligned} &\mathcal{F}_{2n}(K): \quad \{|k_i\rangle | \sum_{i=1}^{2n} k_i = K \ , k_i \in Z^+\} \ , \\ &\dim \mathcal{F}_{2n}(K) = \frac{(K + 2n - 1)!}{K!(2n-1)!} \end{aligned} \tag{20}$$

and $\mathcal{F}_{2n}$ is decomposed into the direct sum of invariant subspaces $\mathcal{F}_{2n}(K)$

$$\mathcal{F}_{2n} = \sum_K \oplus \mathcal{F}_{2n}(K) \ . \tag{21}$$

The representation (19) subduces on every $\mathcal{F}_{2n}(K)$ a finite dimensional irreducible representation $\Gamma^{[K]}$. In the limit $q \to 1$, the representation $\Gamma^{[K]}$becomes a symmetrized representation of Lie algebra C_n labeled by the Yang diagram



When $n = 1$, on the Fock space $\mathcal{F}_1$ with basis

$$|k_1, k_2\rangle = b_1^{+k_1} b_2^{+k_2} |0, 0\rangle \ , \qquad (k_1, k_2 \in Z^+) \ , \tag{22}$$

the representation of quantum group $(C_1)_q$, namely $SU(2)_q$, is obtained as

$$\begin{cases} \Gamma(J_3)|k_1 k_2\rangle = \frac{1}{2}(k_1 - k_2)|k_1 k_2\rangle \ , \\ \Gamma(J_+)|k_1 k_2\rangle = [k_2]|k_1 + 1, k_2 - 1\rangle \ , \\ \Gamma(J_-)|k_1 k_2\rangle = [k_1]|k_1 - 1, k_2 + 1\rangle \ , \end{cases} \tag{23}$$

where $J_+ = \hat{e}_{\alpha_1}$, $J_- = \hat{e}_{-\alpha_1}$, $J_3 = h_{\alpha_1}/2$. On the invariant subspace $\mathcal{F}_1(K)$ with basis

$$\begin{aligned} &\mathcal{F}_1(K) : \{|k_1, k_2\rangle | k_1 + k_2 = K\,,\ k_1,\, k_2 \in Z^+\}\,, \\ &\dim\mathcal{F}_1(K) = K + 1\,, \end{aligned} \tag{24}$$

the representation (23) subduces a $(K+1)$-dimensional irreducible representation. By defining the angular momentum basis $||j, m\rangle$

$$||j, m\rangle = \frac{|j+m, j-m\rangle}{\sqrt{[j+m]![j-m]!}}\,, \quad ([p]! = [p][p-1]\cdots[1],\ p \in Z^+)\,, \tag{25}$$

where $j = K/2 = 0, 1/2, 1, 3/2, \cdots$, $m = -j, -j+1, \cdots, j-1, j$, the representation $\Gamma^{[K]}$ on $\mathcal{F}_1(K)$ is rewritten as

$$\begin{aligned} &\Gamma(J_\pm)||j, m\rangle = \sqrt{[j \mp m][j \pm m + 1]}\ ||j, m \pm 1\rangle\,, \\ &\Gamma(J_3)||j, m\rangle = m||j, m\rangle\,. \end{aligned} \tag{26}$$

This is just the Jimbo's standard finite dimensional irreducible representations, which becomes the standard irreducible representations of Lie algebra SU(2) in the limit $q \to 1$.

It should be pointed out that we don't use the relation (13) in constructing the above representations. For the concrete problems to construct the solution to YBE, the relation (13) can be realized by selecting the concrete representations.

IV. From the Boson Realization to q–Deformed Differential Realization

In this section we shall give a method to obtain the Alvarez-Gaumé's q-deformed differential realization of quantum group from the boson realization.

To keep the discussion as simple as possible, we shall first consider the Bargmann space $\mathcal{B}(l)$ with l-variable[13], which is the space of complex analytic functions $f(z)$ with simple variable z. The basis for $\mathcal{B}(l)$ can be chosen as the monomials $u(n)$

$$u(n) = z^n\,, \qquad n \in Z^+\,. \tag{27}$$

In fact, for any complex analytic function $f(z)$ in the region about the point $z = \lambda$, we have

$$\begin{aligned} &f(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k\,, \\ &a_k(\lambda) = \sum_{n=k}^{\infty} \frac{f^{[n]}(\lambda)(-\lambda)^{n-k}}{(n-k)!k!}\,, \\ &f^{[n]}(\lambda) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\lambda)^{n+1}} dz\,, \end{aligned} \tag{28}$$

where C is any closed curve about the point λ. (See Appendix A).

It is well-known that the basic operators b^+ and b on the Fock space correspond respectively to the basis operators z and $\partial/\partial z$ on the Bargmann space $\mathcal{B}(l)$ because of the following equation

$$\begin{aligned} zu(n) &= u(n+1) \ , \\ \frac{\partial}{\partial z}u(n) &= nu(n-1) \ . \end{aligned} \tag{29}$$

What is the correspondence on the Bargmann space $\mathcal{B}(l)$ of the operators a^+ and a on $\mathcal{F}_1$? It is obvious that the operators a^+ should correspond to the operator z in $\mathcal{B}(l)$ because of the equation $a^+ = b^+$. The operator D_z that corresponds to $\partial/\partial z$ should obviously be defined as

$$D_z u(n) = [n]u(n-1) \ . \tag{30}$$

Then we have

$$D_z(z-\lambda)^n = \frac{(qz-\lambda)^n - (q^{-1}z-\lambda)^n}{(q-q^{-1})z} \ . \tag{31}$$

From Eq. (31) it follows that

$$D_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z} \ , \tag{32}$$

which is just the q-analogue of differential operator $\partial/\partial z$ given by Alvarez-Gaumé *et al.* In the limit $q \longrightarrow l$, it is easy to see that the operator D_z becomes the usual differential operator $\partial/\partial z$ by using the L' Hospital rule.

D_z is virtually an integral operator and can be expressed as

$$D_z f(z) = \frac{1}{2\pi i}\oint_C K(z,\xi) f(\xi) d\xi \ , \tag{33}$$

where the integral kernel is

$$K(z,\xi) = \frac{1}{(\xi - qz)(\xi - q^{-1}z)} \ . \tag{34}$$

(For the detail please see Appendix B).

Above discussions can be extended to the Bargmann space $\mathcal{B}(n)$ with n-variables spanned by

$$\{u(i_k) \equiv z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n} | \quad i_k \in Z^+\} \ . \tag{35}$$

The operator D_{z_j} that corresponds to the operator a_j on the Fock space $\mathcal{F}_n$ is defined as

$$D_{z_j} u(i_k) = [i_j] u(i_k - \delta_{kj}) \tag{36}$$

or

$$D_{z_j} f(z_k) = \frac{f(z_k - \delta_{kj}(q-1)z_j) - f(z_k - \delta_{kj}(q^{-1}-1)z_j)}{(q-q^{-1})z_j} \ , \tag{37}$$

where $f(z_k) \equiv f(z_1, z_2, \cdots, z_n)$ is a complex analytic function in $\mathcal{B}(n)$. D_{z_j} is Alvarez-Gaumé's q-analogue of the differential operator $\partial/\partial z_j$. Then we have the corresponding relations between a_i^+, a_i $(i = 1, 2, \cdots, n)$ in the Fock space $\mathcal{F}_n$ and z_i, D_{z_i} $(i = 1, 2, \cdots, n)$ in the Bargmann space $\mathcal{B}(n)$

$$a_i^+ \Longleftrightarrow z_i \ , \qquad a_i \Longleftrightarrow D_{z_i} \ , \qquad (i = 1, 2, \cdots, n) \ . \tag{38}$$

The corresponding relation (38) implies that one can immediately obtain the q-deformed differential realization of quantum group from a boson realization by replacing respectively the operators a_i^+, a_i in the boson realization of quantum group by z_i and D_{z_i}, which is the q-deformed differential realization given by Alvarez-Gaumé.

For example, the q-deformed differential realization of quantum group $(C_n)_q$ that corresponds to the boson realization (16) is obtained as

$$\begin{aligned}
\tilde{h}_{\alpha_j} &= z_j \frac{\partial}{\partial z_j} - z_{j+1}\frac{\partial}{\partial z_{j+1}} - z_{n+j}\frac{\partial}{\partial z_{n+j}} + z_{n+j+1}\frac{\partial}{\partial z_{n+j+1}} , \\
&\quad (j = 1, 2, \cdots, n) , \\
\tilde{h}_{\alpha_n} &= z_n \frac{\partial}{\partial z_n} - z_{2n}\frac{\partial}{\partial z_{2n}} , \\
\tilde{e}_{\alpha_j} &= z_j D_{z_{j+1}} - z_{n+j+1} D_{z_{n+j}} , \qquad (j = 1, 2, \cdots, n-1) , \\
\tilde{e}_{-\alpha_j} &= z_{j+1} D_{z_j} - z_{n+j} D_{z_{n+j+1}} , \qquad (j = 1, 2, \cdots, n-1) , \\
\tilde{e}_{\alpha_n} &= z_n D_{z_{2n}} , \\
\tilde{e}_{-\alpha_n} &= z_{2n} D_{z_n} ,
\end{aligned} \tag{39}$$

when $n = 1$, we obtain the q-deformed differential realization of quantum group $SU(2)_q$ as

$$\begin{aligned}
J_+ &= z_1 D_{z_2} , \\
J_- &= z_2 D_{z_1} , \\
J_3 &= \frac{1}{2}\left[z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}\right] ,
\end{aligned}$$

where $J_\pm = e_{\pm\alpha}$, $J_3 = h_\alpha/2$. On the subspace of Bargmann space $\mathcal{B}(2)$ with basis

$$\begin{aligned}
|j, m\rangle &= \frac{z_1^{j+m} z_2^{j-m}}{\sqrt{[j+m]![j-m]!}} , \\
&\quad j = 0, \frac{1}{2}, 1, \cdots, \qquad m = -j, -j+1, \cdots, j ,
\end{aligned} \tag{40}$$

we can obtain the finite dimensional irreducible representations as

$$\begin{aligned}
J_\pm |j, m\rangle &= \sqrt{[j-m][j+m+1]}\ |j, m \pm 1\rangle , \\
J_3 |j, m\rangle &= m |j, m\rangle .
\end{aligned}$$

Appendix A: Derivation of Eq. (28)

Because the function $f(z)$ is analytic in the region about the point $z = \lambda$, we have

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{f^{[n]}(\lambda)}{n!}(z-\lambda)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{[n]}(\lambda)}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}(-\lambda)^{n-k} z^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{f^{[n]}(\lambda)(-\lambda)^{n-k}}{(n-k)!k!} z^k \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{f^{[n]}(\lambda)(-\lambda)^{n-k}}{(n-k)!k!} z^k ,
\end{aligned}
\tag{A1}
$$

where the following equation

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n,k) \equiv \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} F(n,k) \tag{A2}
$$

is used. Equation (A1) can also be expressed as

$$
\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} a_k(\lambda) z^k , \\
a_k(\lambda) &= \sum_{n=k}^{\infty} \frac{f^{[n]}(\lambda)(-\lambda)^{n-k}}{(n-k)!k!} ,
\end{aligned}
\tag{A3}
$$

where $f^{[n]}(\lambda)$ can be expressed in an integral form

$$
f^{[n]}(\lambda) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-\lambda)^{n+1}} dz , \tag{A4}
$$

which is just Eq. (28).

Appendix B: Derivation of Eqs. (33) and (34)

From Eq. (32) we have

$$
\begin{aligned}
D_z f(z) &= \frac{1}{(q-q^{-1})z}[f(qz) - f(q^{-1}z)] \\
&= \frac{1}{(q-q^{-1})z}\left[\frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - qz)} d\xi - \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - q^{-1}z)} d\xi\right] \\
&= \frac{1}{2\pi i}\oint_C \left(\frac{1}{(\xi - qz)(q-q^{-1})z} - \frac{1}{(\xi - q^{-1}z)(q-q^{-1})z}\right) f(\xi) d\xi ,
\end{aligned}
\tag{B1}
$$

where C is an arbitrary closed curve about the point z. Equation (B1) can also be expressed in the form

$$
D_z f(z) = \frac{1}{2\pi i} \oint_C K(z,\xi) f(\xi) d\xi , \tag{B2a}
$$

$$
K(z,\xi) = \frac{1}{(\xi - qz)(\xi - q^{-1}z)} , \tag{B2b}
$$

which is just Eqs. (33) and (34).

References

[1] C.N. Yang, Phys. Rev. Lett. **19**(1967)1312; R.J. Baxter, Ann. Phys. **70**(1972)193; A.B. Zamolodchikov *et al.*, Ann. Phys. **120**(1979)253.

[2] V.G. Drinfeld, *Quantum Group*, in Proc. ICM., Berkeley (1986), p.798.

[3] M. Jimbo, Lett. Math. Phys. **10**(1985)63; M. Jimbo, Lett. Math. Phys. **11**(1986)247.

[4] N.Y. Reshetikhin, LOMI preprint E-4-87 (Leningrad, 1988).

[5] L. Alvarez-Gaume *et al.*, preprint CERN-TH 5369/89; ibid CERN-TH 5367/89.

[6] A.J. Macfarlane, preprint DAMTP/88-28.

[7] Chang-pu Sun, J. Phys. A: **20**(1987)4551.

[8] Hong-chen Fu and Chang-pu Sun, *Inhomogeneous Boson Realization of Indecomposable Representations of Lie Algebras*, J. Math. Phys., in press.

[9] Chang-pu Sun, J. Phys. **A20**(1987)5823.

[10] Chang-pu Sun, J. Phys. **A20**(1987)L1157.

[11] Chang-pu Sun and Hong-chen Fu, *New Boson Realization of Loop Algebras and Their Representations*, IL Nuovo Cimento, in press.

[12] Chang-pu Sun and Hong-chen Fu, J. Phys. **A22**(1989)L983.

[13] V. Bargmann, Comm. Pure Appl. Math. **14**(1961)187; V. Bargmann, *Groups Representations on Hilbert Spaces of Analytic Functions*, in "*Analytic Methods in Mathematical Physics*", Gordan and Breach, New York, pp. 27-63.