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Berry's Phase Effects in Spin Precession of Relativistic Neutral Particle in Slowly Changing Magnetic Field

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Abstract

By making use of the adiabatic approximate method, we derive the spin rotation formula of a relativistic neutral particle propagating in a slowly changing magnetic field, which involves the effects of the Berry's phases. In the non-relativistic limit, the Berry's phases result in an additional rotation in OVV spin rotation.

A few years ago, it was recognized by Berry that, when the Hamiltonian H depends on time through a set of slowly changing parameters, the quantum dynamics is described by the instantaneous eigenstates of H, multiplied by an additional geometrical phase factor and a dynamical one[1]. This geometrical phase is now called Berry's phase and it has recently emerged as a universal element in the topological analysis of various problems in quantum theories[2]. However, only non-relativistic cases were mainly involved and the corresponding Berry's phases were analysed. In this letter we shall pay attention to the Berry's phases in relativistic case.

Our discussion is motivated by the proposal due to Okun, Voloshin and Vysotskii (OVV)[3] for the solar neutrino problem (SNP) and its relativistic generalization by Liu[4]. Assuming that the spin of neutrino with magnetic moment is rotated by a constant magnetic field, they give the SNP a possible solution, but our discussion for the slowly changing magnetic field is rather general, and suitable for evolution of any neutral particle with spin–1/2 and a magnetic moment (e.g. a neutron) in an adiabaticaly varying magnetic field.

In a varying magnetic field

$$\vec{B}(t) = B_0(\sin\theta(t)\cdot\cos\phi(t), \sin\theta(t)\cdot\sin\phi(t), \cos\theta(t))\ , \tag{1}$$

the motion of a neutral particle with spin-1/2 and magnetic moment μ is described by the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)|\psi\rangle = -\frac{1}{2}\mu\sigma^{\alpha\beta}\cdot F_{\alpha\beta}|\psi\rangle\ . \tag{2}$$

By making use of the properties of the Dirac matrices $\gamma^\mu, \sigma^{\alpha\beta}, \vec{\alpha}, \beta$ and $\vec{\Sigma}$, the above equation can be rewritten as a Schrödinger–type equation

$$\begin{aligned} i\frac{\partial}{\partial t}|\psi\rangle &= \hat{H}[\vec{B}(t)]|\psi\rangle\ , \\ \hat{H}[\vec{B}(t)] &= m\beta + \vec{\alpha}\cdot\hat{\vec{P}} + \mu\beta\vec{\Sigma}\cdot\vec{B}(t)\ . \end{aligned} \tag{3}$$

For the particle propagating in the z-direction with momentum p, let $|\psi\rangle = e^{ipz}|\phi\rangle$, we then obtain an effective equation of motion

$$i\frac{\partial}{\partial t}|\phi\rangle = \hat{H}_{\text{eff}}[\vec{B}(t)]|\phi\rangle \ , \tag{4}$$

where the effective Hamiltonian is

$$\begin{aligned} \hat{H}_{\text{eff}}[\vec{B}(t)] &= \omega_0 \begin{bmatrix} a+\vec{\sigma}\cdot\vec{B}(t)/B_0 \ , & b\sigma_z \\ b\sigma_z \ , & a-\vec{\sigma}\cdot\vec{B}(t)/B_0 \end{bmatrix} ; \\ \omega_0 = \mu B_0 \ , \qquad & a = \frac{m}{\omega_0} \ , \qquad b = \frac{p}{\omega_0} \ . \end{aligned} \tag{5}$$

According to the adiabatic approximate method[1,5,6], we first solve the eigen-equation of $\hat{H}_{\text{eff}}[\vec{B}(t)]$ for a given arbitrary $\vec{B}(t)$, and then obtain the instantaneous eigenvectors and the corresponding eigenvalues respectively as

$$\begin{aligned} |u_{E_i}[\vec{B}(t)]\rangle &= \frac{1}{\sqrt{1+u_2[\theta,\lambda_i]^2+u_3[\theta,\lambda_i]^2+u_4[\theta,\lambda_i]^2}} \cdot \begin{bmatrix} 1 \\ u_2[\theta,\lambda_i]e^{i\phi} \\ u_3[\theta,\lambda_i] \\ u_4[\theta,\lambda_i]e^{i\phi} \end{bmatrix} , \\ E_i \equiv E_i(t) &= \lambda_i\cdot\omega_0 \equiv (-1)^i[a^2+b^2+1-(-1)^{[i/2]}\cdot 2(a^2+b^2\cos^2\theta)^{1/2}]^{1/2}\omega_0 \end{aligned} \tag{6}$$

where

$$\begin{aligned} \left[\frac{i}{2}\right] &= \begin{cases} \dfrac{i+1}{2} \ , & \text{for} \quad i=1,3 \ ; \\ \dfrac{i}{2} \ , & \text{for} \quad i=2,4 \ , \end{cases} \\ u_2[\theta,\lambda_i] &= 2\sin\theta(\cos\theta-\lambda_i)\cdot f(\theta,\lambda_i) \ , \\ u_3[\theta,\lambda_i] &= \frac{1}{b}[(a+\cos\theta-\lambda_i)+2\sin^2\theta\cdot f(\theta,\lambda_i) \ , \\ u_4[\theta,\lambda_i] &= \frac{1}{b}[1+2(\cos\theta-\lambda_i)(a-\cos\theta-\lambda_i)f(\theta,\lambda_i)]\sin\theta \ , \\ f(\theta,\lambda_i) &= [(a^2-\lambda_i^2)+b^2+\cos 2\theta-2a\cdot\cos\theta]^{-1} \ . \end{aligned} \tag{7}$$

Under the adiabatic conditions[1,5,6]

$$\left|\frac{\langle u_{E_i}[\vec{B}(t)]|\dot{u}_{E_j}[\vec{B}(t)]\rangle}{E_i-E_j}\right| \ll 1 \ , \qquad i,j=1,2,3,4; i\neq j \ , \tag{8}$$

equation (4) has an adiabatic approximate solution

$$|\phi(T)\rangle = \sum_{i=1}^{4} C_i(0)\exp\left[-i\int_0^T E_i(t)dt\right]\cdot\exp[i\gamma_i(T)]|u_{E_i}[\vec{B}(t)]\rangle \ , \tag{9}$$

where the additional phases

$$\begin{aligned}\gamma_i(T) &= i\int_0^T \langle u_{E_i}[\vec{B}(t)]|\frac{d}{dt}u_{E_i}[\vec{B}(t)]dt = \int_0^T \tilde{A}(\theta, E_i)\dot{\phi}(t)dt\,, \\ \tilde{A}(\theta, E_i) &= -\frac{u_4(\theta,\lambda_i)^2 + u_2(\theta,\lambda_i)^2}{1 + u_2(\theta,\lambda_i)^2 + u_3(\theta,\lambda_i)^2 + u_4(\theta,\lambda_i)^2}\end{aligned} \tag{10}$$

are the Berry's phases. For a cycle evolution $\vec{B}(0) = \vec{B}(T)$, when $\theta(t)$=constant, we have $\phi(T) = \phi(0) + 2\pi n$ and

$$\gamma_i(T) = \tilde{A}(\theta, E_i)\cdot 2\pi n\,. \tag{11}$$

Here, n is the winding number of the mapping from the parameter space $R^3 : \{\vec{B}\}$ to the sphere $S^2 : \{\vec{B} : (\theta,\phi)|\quad |\vec{B}| = 1, \vec{B}\in R^3\}$.

The particle in an initial positive energy state

$$|\psi(0)\rangle = e^{ipz}|\phi(0)\rangle \equiv \frac{1}{2}(|u(2)\rangle + |u(4)\rangle)\cdot e^{ipz} \tag{12}$$

has a helicity expectation value

$$\begin{aligned}\lambda(t=0) &= \langle\psi_{(t=0)}|\Sigma_z|\psi_{(t=0)}\rangle = \frac{1}{2}(\langle u(2)|\Sigma_z|u(2)\rangle \\ &\quad + \langle u(4)|\Sigma_z|u(4)\rangle + 2\langle u(2)|\Sigma_z|u(4)\rangle) \\ &= \frac{1}{2}\sum_{k=2,4}\sum_{i=1}^{4}(-1)^{i+1}\cdot u(k)_i^2 + \sum_{i=1}^{4}(-1)^{i+1}u(2)_i u(4)_i\,,\end{aligned} \tag{13}$$

where $u(k)_i$ is the i-th component of the instantaneous spinor $|u(k)\rangle \equiv |u_{E_k}(\vec{B}(t=0))\rangle$. This particle is then evolved into the state at time T

$$\begin{aligned}|\phi(T)\rangle &= \frac{1}{2}\exp\Big[ipz - i\int_0^T E_2(t)dt + i\gamma_2(T)\Big] \\ &\quad \cdot(|u(2)\rangle + \exp[i\Gamma(T)]|u(4)\rangle)\,; \\ \Gamma(T) &= \int_0^T (E_2(t) - E_4(t))dt + \gamma_4(T) - \gamma_2(T)\end{aligned} \tag{14}$$

for the cycle evolution and has the helicity expectation

$$\begin{aligned}\lambda(T) &= \langle\psi(T)|\Sigma_z|\psi(T)\rangle \\ &= \frac{1}{2}\sum_{k=2,4}\sum_{i=1}^{4}(-1)^{i+1}\cdot u(k)_i^2 + \Big[\sum_{i=1}^{4}(-1)^{i+1}u(2)_i\cdot u(4)_i\Big]\cdot\cos[i\,\Gamma(T)]\,.\end{aligned} \tag{15}$$

Considering

$$\Delta\lambda(T) = \lambda(0) - \lambda(T) = 2\Big[\sum_{i=1}^{4}(-1)^{i+1}u(2)_i u(4)_i\Big]\cdot\sin^2\Big[\frac{1}{2}i\,\Gamma(T)\Big]\,, \tag{16}$$

we finally obtain the relativistic spin rotation formula in the slowly changing magnetic field

$$\Omega(L) = \frac{1}{2}\Gamma\Big(\frac{L}{v}\Big)\,, \tag{17}$$

where L is the propagating distance of the particle with a velocity v at time T.

In the non-relativistic limit that $p = 0$, we have

$$
\begin{aligned}
&E_2 \longrightarrow m + \omega_0 \,, \qquad E_4 \longrightarrow m - \omega_0 \,; \\
&u_2(\theta,\lambda_2) \longrightarrow \hat{u}(2) = \frac{2\sin\theta(\cos\theta - a - 1)}{\cos 2\theta + 4a\sin^2(\theta/2) - 1} \,, \qquad u_4(\theta,\lambda_2) \longrightarrow 0 \,, \\
&u_2(\theta,\lambda_4) \longrightarrow \frac{2\sin\theta(\cos\theta - a + 1)}{\cos 2\theta + 4a\sin^2(\theta/2) - 1} \equiv \hat{u}(4) \,, \qquad u_4(\theta,\lambda_4) \longrightarrow 0 \,.
\end{aligned} \tag{18}
$$

Then, the non-relativistic spin rotation formula given from Eq. (17) is

$$
\begin{aligned}
&\Omega = \mu B_0 \cdot \frac{L}{v} + \Delta\gamma\Big(\frac{L}{v}\Big) \,, \\
&\Delta\gamma\Big(\frac{L}{v}\Big) = \lim_{p\to 0}\Big(\gamma_4\Big(\frac{L}{v}\Big) - \gamma_2\Big(\frac{L}{v}\Big)\Big) = \left\{\frac{-\hat{u}^2(2)}{\sqrt{1+\hat{u}^2(2)}} + \frac{\hat{u}^2(4)}{\sqrt{1+\hat{u}^2(4)}}\right\} \cdot 2\pi n \,,
\end{aligned} \tag{19}
$$

which gives the OVV-spin rotation formula[3]

$$
\Omega = \mu B_0 \cdot \frac{L}{v} \tag{20}
$$

again for the cases with constant magnetic field due to $\phi(t) = 0$. The additional rotation angle is caused by the time–dependence of the magnetic field under adiabatic conditions. We can regard it as an effect of the Berry's phase.

When the discussion of this letter is specialized for neutrino, it is shown from Eqs. (17) and (19) that the Berry's phase has something to do with the SNP. One of the authors even analysed some relations between Berry's phase and SNP from the MSW mechanism for solving the SNP[7–8].

If the adiabatic conditions (8) do not hold, the high order adiabatic approximate method suggested by one of the authors can be applied to dealing with the non–adiabatic effects of the problems in this letter.

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