Quantifying Spontaneously Symmetry Breaking of Quantum Many-Body Systems*

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Abstract *Spontaneous symmetry breaking is related to the appearance of emergent phenomena, while a non-vanishing order parameter has been viewed as the sign of turning into such symmetry-breaking phase. We study the spontaneous symmetry breaking in the conventional superconductor and Bose–Einstein condensation with a continuous measure of symmetry by showing that both the many-body systems can be mapped into the many spin model. We also formulate the underlying relation between the spontaneous symmetry breaking and the order parameter quantitatively. The degree of symmetry stays unity in the absence of the two emergent phenomena, while decreases exponentially at the appearance of the order parameter which indicates the inextricable relation between the spontaneous symmetry and the order parameter.*

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1 Introduction

Symmetry and its breaking are evidently of significance in physics. Many physical laws originate from symmetry. For every continuous symmetry, it follows from the Noether's theorem[1] that a corresponding conserved law exists; the Kibble-Zurek mechanism,[2−3] on the other hand, allows the dynamical quench through condensed matter phase transitions to be used as a mean to simulate the formation of cosmological defects.[4−5] Actually, symmetry has also been studied in many other subjects such as mathematics,[6] biology,[7] and chemistry.[8]

Spontaneous symmetry breaking (SSB) is a fashion of symmetry breaking of a quantum system S that the Hamiltonian or the motion equation of S possesses some symmetry while its ground state does not.[9] The importance of SSB is fundamental, as well as practical. For example, the Higgs mechanism explains the generation of mass for the gauge bosons in the unified theory for the weak and electromagnetic interactions,[10−11] and owing to the spontaneous chiral symmetry breaking in living organism,[7] synthetic cells with opposite handedness have been considered as an appealing therapeutic tool.[12] It has been known that the emergent phenomena, e.g., superconductivity and Bose–Einstein condensation (BEC)[13−14] are all rooted in SSB. The traditional approach to the SSB based emergent phenomena is the mean field theory (MFT), which is capable of qualitatively explaining phenomena in diverse areas. A more strict method beyond MFT was developed by Yang,[15] based on the consideration of off-diagonal long-range order (ODLRO). In ODLRO approach, a non-vanishing order parameter arises with the emergence of SSB.

The fast developments of quantum information theory not only provide the possibility to improve the performances of certain quantum tasks, e.g., quantum teleportation and quantum cryptography, but also support a new viewpoint towards a further understanding of quantum physics. Although concepts like entanglement and coherence are originally introduced in a dichotomous fashion similar in spirit to the symmetry breaking, the practical need of minimizing the resource costs in various information protocols motivates the introduction of quantitative measures.[16−18] And those measures have in turn been successfully applied to the studies of entanglement dynamics,[19−21] classical and quantum correlations,[22] as well as behaviors of systems near quantum critical points.[23−27]

Analogous to quantitative frameworks of quantum entanglement and coherence, the communication tasks such as sharing a reference frame[28−30] and remote clocks synchronization[31−33] have stimulated the theoretical explorations of possible quantifications of the asymmetric properties of quantum states which could provide novel

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solutions. In consequence, various measures were thus introduced.[34−37] As it is well known that the order parameter usually characterises SSB, it does make sense physically that if their definitions were reasonable, the order parameter should possess an underlying connection with those asymmetry measures. However, to our best knowledge, investigations along this line remain elusive.

The present paper is aimed at establishing such a quantitative relation between the order parameters and a symmetry measure called degree of symmetry (DoS). We consider two representative phenomena that have been well explored following the traditional approach, namely the superconductivity with isotropic pairing and the Bose–Einstein condensation (BEC) in the many-spin model. It will be shown that the DoS for those two types of SSB phenomena could be expressed in terms of the corresponding order parameters in the thermodynamic limit. This result also represents an important step to justify the potential of applying this DoS-based approach to detect unknown symmetry-breaking-related effects in systems whose order parameters are unknown in advance.

This paper is organized as follows: In Sec. 2, it is illustrated that why DoS is suitable for investigating SSB when compared with some other existing asymmetry measures, and a summarize to our main results is also given. In Sec. 3, the mapping from the BCS and BEC systems to the many-spin model is illustrated. In Sec. 4, the DoS for the many-spin model is explicitly evaluated and its relation with SSB is discussed. Then the DoS for the BCS and the BEC systems are separately explored in Secs. 5 and 6, respectively. Finally, we draw the conclusion in Sec. 7.

2 Symmetry and Asymmetry Measures

The symmetry property of states has been studied by several groups with different physical considerations. For example, an asymmetry measure of a quantum state ρ is defined as the entropy difference.[34−35]

$$A_G(\rho) = S(\overline{R(g)\rho R(g)^\dagger}) - S(\rho)\,, \tag{1}$$

where $S(\rho) \equiv -\mathrm{Tr}(\rho\log_2\rho)$ is the von Neumann entropy, $R(g)$ is the matrix representation of element g in group G and $\overline{f(g)} = n_G^{-1}\sum_{g\in G} f(g)$ is the average performed over group G with order n_G. This measure represents the difference between the extractable work using states ρ and $\overline{R(g)\rho R(g)^\dagger}$ and also quantifies the ability of ρ to act as a reference system. Apparently, $A_G(\rho) = 0$ if $\overline{R(g)\rho R(g)^\dagger} = \rho$, which means ρ is symmetric under G. Another quantifier is proposed by Marvian *et al.*,[36−37] which is called the characteristic function of a pure state

$$\chi_\psi(g) = \langle\psi|R(g)|\psi\rangle\,. \tag{2}$$

It has been proved that two pure states ψ and φ who satisfy $\chi_\psi(g) = \chi_\varphi(g)$ for arbitrary $g \in G$ can be converted into each other by G-convariant unitary dynamics.

Although those measures could provide reasonable quantifications of asymmetry property of quantum states, there exist two limitations in connecting straightforwardly to symmetry breaking related phenomena in the traditional fashion: (i) Some of those measures are not normalized, e.g., Eq. (1); (ii) Owing to the particular applications, those asymmetry measures are introduced with respect to quantum states. As a result, they might not be appropriately used to describe spontaneous symmetry breaking, where a comparison between the symmetry properties of Hamiltonian and the underlying ground state is needed.[38−39]

To go beyond these limitations, a continuous measure of symmetry breaking has been proposed by introducing the degree of symmetry (DoS)[40] and then it was applied to the Frobenius-norm-based measures for quantum coherence and asymmetry.[41] Specifically, for a given transformation set G, the DoS of the Hamiltonian H and the density matrix ρ of a quantum state are defined, respectively, as follows

$$S(G,H) = \frac{1}{4|\tilde{H}|^2}\overline{|\{R(g),\tilde{H}\}|^2}\,, \tag{3}$$

$$S(G,\rho) = \frac{1}{4|\rho|^2}\overline{|\{R(g),\rho\}|^2}\,, \tag{4}$$

where G is a given transformation set, $g \in G$, $R(g)$ is its d-dimensional representation, $|A| = \sqrt{\mathrm{Tr}A^\dagger A}$. Especially, $\tilde{H} = H - \mathrm{Tr}\{H\}\mathrm{I}_{d\times d}/d$ is a re-biased Hamiltonian, which possesses a similar energy spectrum as that of H but is free of the choice of the energy zero point. It is worth mentioning that $|\rho|^2 = \mathrm{Tr}(\rho^2)$ denotes the purity of ρ. The definition of the DoS satisfies three nice properties which are physically reasonable: (i) Tighter bound when G forms a transformation group $1/2 \le S(G,H) \le 1$; (ii) Independent of the basis in the system's Hilbert space; (iii) Independent of the choice of the ground state energy; (iv) Scaling invariance.[40]

Apparently, DoS with properties of normalization and compatibility with Hamiltonian and quantum states can be applied to the study of some emergent phenomena, e.g., spontaneously symmetry breaking. Apart from this, one may find that there are some similarities between DoS and other measures mentioned above. First, the average performed over group is introduced by both DoS Eq. (4) and the entropy asymmetry measure Eq. (1). Second, $S(G,\rho)$ reduces to $1/2 + \overline{\chi_\psi(g)\chi_\psi^*(g)}/2$ when $\rho = |\psi\rangle\langle\psi|$.

In order to calculate the DoS for the conventional superconductor and BEC in a unified fashion, we map them to the many-spin model with Hamiltonian

$$H = \sum_{i=1}^{N} \epsilon\sigma_z^i + \lambda\sigma_x^i + \mu\sigma_y^i\,, \tag{5}$$

where ϵ, λ and μ are real and σ_n^i $(n = x, y, z)$ denotes the Pauli operator of the i-th particle. We will show that both the Bardeen–Cooper–Schrieffer (BCS) and BEC Hamiltonians can be mapped into Eq. (5). Then, the DoS of the BCS and BEC systems is given through the study of the many-spin model.

$$S(G,\rho_{T=0})_S = \frac{1}{2}+\frac{1}{2}\exp(-(2-\sqrt{2})\pi g(0)|\Delta(0)|)\,,$$
$$S(G,\rho_{T=0})_B = \frac{1}{2}+\frac{1}{2}\exp(-2\langle a\rangle_0^2)\,, \tag{6}$$

where the subindices S stands for superconductivity and B for BEC, $g(\epsilon)$ is the density of states, and $\Delta(0)$ together with $\langle a\rangle_0$ are the corresponding order parameters at the absolute zero temperature. It follows from Eq. (6) that the DoS reaches 1 (the symmetry is unbroken) and the symmetry is totally recovered as $\Delta(0)$ or $\langle a\rangle_0$ vanishes.

3 Many-Spin Model for the Conventional Superconductor and BEC

In the conventional superconductivity theory,[14] the BCS Hamiltonian is obtained by eliminating the phonon variable.

$$H_S = \sum_k \epsilon_k(a_k^\dagger a_k + b_k^\dagger b_k) - V\sum_{k,k'} a_k^\dagger b_k^\dagger b_{k'} a_{k'}\,, \tag{7}$$

where $a_k(b_k)$ denotes the annihilation operator of an electron with momentum $k(-k)$ and spin up (down). In accordance with the BCS assumption,[14,42] the net electron-phonon attractive interaction V is non-zero only for single electron states whose energy satisfy $|\varepsilon_k-\varepsilon_F|\le\hbar\omega_D$, with ω_D the Debye frequency and ε_F the chemical potential in the normal phase. Thus the summation over k in Eq. (7) is correspondingly restricted to a thin shell around the sphere whose radius is given by the Fermi wavevector k_F. Two electrons with opposite momentum and spin create an electron pair which is called the Cooper pair.[42] The Jordan–Wigner transformation[43] exactly maps the fermion model into the pseudo-spin model as

$$\sigma_+^k = b_k a_k\,,\quad \sigma_-^k = a_k^\dagger b_k^\dagger\,,$$
$$\sigma_z^k = 1-(a_k^\dagger a_k + b_k^\dagger b_k)\,. \tag{8}$$

It is easily checked that these pseudo-spin operators defined above satisfy the commutation relations of spin type

$$[\sigma_+^k,\sigma_-^{k'}] = \sigma_z^k\delta_{k,k'}\,,\quad [\sigma_z^k,\sigma_\pm^{k'}] = \pm 2\sigma_\pm^k\delta_{k,k'}\,. \tag{9}$$

Then, the BCS Hamiltonian given in Eq. (7) is re-expressed as

$$H_S = -\Bigl(\sum_k \epsilon_k\sigma_z^k + V\sum_{k,k'}\sigma_-^k\sigma_+^{k'}\Bigr) + \sum_k \epsilon_k\,, \tag{10}$$

where for a given system $\sum_k \epsilon_k$ is determinate and can be dropped. We then obtain the BCS Hamiltonian described in the spin model

$$H_S = -\Bigl(\sum_k \epsilon_k\sigma_z^k + V\sum_{k,k'}\sigma_-^k\sigma_+^{k'}\Bigr)\,. \tag{11}$$

In the BCS theory, one deal with the eigenenergy problem with the mean field approximation (MFA), which assumes that the difference between $\sigma_-^k(\sigma_+^{k'})$ and its expect value is a small quantity. That is to say, $\sigma_-^k = \langle\sigma_-^k\rangle + \lambda$ and $\sigma_+^k = \langle\sigma_+^k\rangle + \tau$ where λ and τ are small quantities. Then, we make another assumption that the summation of the averages of σ_+^k is non-zero,

$$\Delta = V\sum_k\langle\sigma_+^k\rangle\,, \tag{12}$$

where Δ is the energy gap of BCS. Δ serves as the order parameter for the superconducting transition. Δ is zero when T is above the critical temperature T_c, indicating that the effective interaction is no more attractive. As a result, the Cooper pairs around the Fermi surface are seperated. When the temperature is below T_c, Δ becomes non-zero, which implies that the number of electrons is not conserved in H_S and the gauge symmetry $a_k(b_k)\to a_k(b_k)\exp[\mathrm{i}\varphi/2]$ is spontaneously broken. Then, the BCS Hamiltonian reads

$$H_S = -\sum_k(\epsilon_k\sigma_z^k + \mathrm{Re}(\Delta)\sigma_x^k + \mathrm{Im}(\Delta)\sigma_y^k)\,. \tag{13}$$

Obviously, the BCS Hamiltonian H_S possesses the same form as the general many-spin model Hamiltonian, which we present in Eq. (5).

For the boson system, the condensation happens if a finite fraction of the particles occupies the lowest single-particle state under the thermodynamic limit.[14] Mathematically, this is expressed by the appearance of the ODLRO,[15] i.e.,

$$\rho(x,y) = \langle\hat\psi^\dagger(x)\hat\psi(y)\rangle \xrightarrow{|x-y|\to\infty} \langle\hat\psi^\dagger(x)\rangle\langle\hat\psi(y)\rangle \ne 0\,, \tag{14}$$

where $\rho(x,y)$ is the single particle reduced density matrix, $\hat\psi(x)$ is the bosonic field operator, and the average is performed under the ground state of the many-body system. $\langle\hat\psi(x)\rangle$ is the order parameter of BEC and BEC occurs when $\langle\hat\psi(x)\rangle$ is non-zero. Actually, the apperance of BEC breaks the U(1) symmetry which corresponds to the conservation of particle number. Therefore, the Hamiltonian for BEC is over-simplified as

$$H \sim a^\dagger a + \alpha(a+a^\dagger)\,. \tag{15}$$

The representation of a group element of U(1) is $R(\theta) = \exp(\mathrm{i}\theta a^\dagger a)$. It could be proved that $[R(\theta),H]=0$ if and only if $\alpha=0$. Actually, the ground state of H is a coherent state $|\alpha\rangle$ when α is nonzero. According to the Penrose–Onsager criterion,[14] $\langle\alpha|a|\alpha\rangle = \alpha$ is the non-vanishing order parameter.

In order to analyze the DoS of BEC with a general many-spin model, we introduce the many-spin model with the Hamiltonian H_B

$$H_B = \sum_{i=1}^N \epsilon\sigma_z^i + \lambda\sigma_x^i\,. \tag{16}$$

By using

$$J_k = \sum_{i=1}^{N} \frac{1}{2}\sigma_k^i (k = x, y, z)\,, \quad J_\pm = J_x \pm \mathrm{i} J_y\,,$$

as the collective angular momentum operators, the above Hamiltonian becomes

$$H_B = 2\epsilon J_z + \lambda(J_+ + J_-)\,. \tag{17}$$

As $[J^2, H_B] = 0$, the total angular momentum conserves. In the limit of the low excitation, we obtain

$$J^2 = \frac{N}{2}\Big(\frac{N}{2} + 1\Big) = J_z^2 + J_+ J_- - J_z\,,$$

$$J_z = \frac{1}{2} \pm \frac{N}{2}\sqrt{(1+\eta)^2 - 4J_+J_-}\,,$$

where $\eta = 1/N$. Now we map the angular momentum operators to boson operators in the large N limit, i.e., $a = J_-/\sqrt{N}$, $a^\dagger = J_+/\sqrt{N}$. Actually, the commutation relation between a and $a^\dagger$ is

$$[a, a^\dagger] = -\eta\Big(1 - \frac{1}{\eta}\sqrt{(1+\eta)^2 - 4\eta a^\dagger a}\Big)\,, \tag{18}$$

where we have taken

$$J_z = \frac{1}{2} - \frac{N}{2}\sqrt{(1+\eta)^2 - 4\eta a^\dagger a}\,.$$

When $N \to \infty$ and $\eta \to 0$, by taking Eq. (18) to the first order, we obtain $[a, a^\dagger] \simeq 1 - 2\eta a^\dagger a \simeq 1$. It is clear that $a(a^\dagger)$ can be treated as the annihilation (creation) operator of bosons in the limit of low excitation and large N. The Hamiltonian H_B can be expressed as

$$H_B \simeq 2\epsilon a^\dagger a + \lambda\sqrt{N}(a + a^\dagger)\,, \tag{19}$$

which is just the simplified BEC Hamiltonian we consider in Eq. (15). Since we have mapped the many-spin model to the boson model, we make use of Eq. (16) to simulate SSB in BEC in the limit of low excitation and large N.

In conclusion, we have just showed above that the many-spin model is valid both in fermion and boson systems.

4 The DoS of Many-Spin System

According to the definition of the DoS given in Eqs. (3) and (4), we calculate the DoS of the many-spin system as follows. The density matrix reads $\rho = \exp(-\beta H)/Z$, where Z is the partition function $\mathrm{Tr}(\exp(-\beta H))$. Particularly, near the absolute zero temperature, i.e., $\beta \to \infty$, the system stays in its ground state. Actually, the linear superposition of σ_x, σ_y and σ_z appeared in Eq. (5) can be treated as σ_z rotated about some certain axis. Thus, the Hamiltonian appeared in Eq. (5) can be simplified to

$$H = \sum_{i=1}^{N} \xi R_i(\theta,\phi)\sigma_z^i R_i^\dagger(\theta,\phi)\,, \tag{20}$$

where

$$\xi = \sqrt{\epsilon^2 + \lambda^2 + \mu^2}\,, \quad \cos\theta = \frac{\epsilon}{\xi}\,, \quad \tan\phi = \frac{\mu}{\lambda}\,,$$

$$R_i(\theta,\phi) = \mathrm{e}^{-\mathrm{i}\phi\sigma_z^i/2}\,\mathrm{e}^{-\mathrm{i}\theta\sigma_y^i/2}\,.$$

Then, the ground state of this Hamiltonian is obtained immediately as $|G\rangle = \prod_i R_i(\theta,\phi)|\downarrow\rangle_i$, where $|\downarrow\rangle$ denotes the state of spin down. Here, we regard λ and μ as the perturbations and σ_z^i remains unchanged under rotations about the z-axis by an arbitrary angle. All of these symmetric transformations form a group called SO(2).

In the many-spin system here, the symmetric group is $\mathrm{SO}(2)^{\otimes N}$, with the elements $R(g) = \prod_i \exp(-\mathrm{i}\omega_i\sigma_z^i/2)$. The DoS of Hamiltonion is given by

$$S(\mathrm{SO}(2)^{\otimes N}, H) = \frac{1}{2} + \frac{\epsilon^2}{2\xi^2}\,, \tag{21}$$

where the group average here is

$$\frac{1}{(2\pi)^N}\int_{-\pi}^{\pi}\mathrm{d}\omega_i \cdots \int_{-\pi}^{\pi}\mathrm{d}\omega_N\,,$$

which means that N particles are rotated separately.

On the other hand, the DoS of the ground state is obtained as

$$S(\mathrm{SO}(2)^{\otimes N}, \rho_{T=0}) = \frac{1}{2} + \frac{1}{2}\Big(1 - \frac{\lambda^2 + \mu^2}{2\xi^2}\Big)^N\,. \tag{22}$$

This gives the DoS of the thermal equilibrium state as

$$S(\mathrm{SO}(2)^{\otimes N}, \rho_T) = \frac{1}{2} + \frac{1}{2}(1 - \Lambda)^N\,, \tag{23}$$

where $\Lambda = [(\lambda^2 + \mu^2)\sinh^2(\beta\xi)]/[\xi^2\cosh(2\beta\xi)]$.

All Eqs. (21), (22), and (23) decrease as the perturbations λ and μ grow. It means that the symmetry is broken by the perturbations. Especially,

$$\lim_{\lambda,\mu\to 0}\lim_{N\to\infty} S(\mathrm{SO}(2)^{\otimes N}, H) = 1\,,$$

$$\lim_{N\to\infty}\lim_{\lambda,\mu\to 0} S(\mathrm{SO}(2)^{\otimes N}, H) = 1\,, \tag{24}$$

while at sufficiently low temperature,

$$\lim_{N\to\infty}\lim_{\lambda,\mu\to 0} S(\mathrm{SO}(2)^{\otimes N}, \rho_T) = 1\,,$$

$$\lim_{\lambda,\mu\to 0}\lim_{N\to\infty} S(\mathrm{SO}(2)^{\otimes N}, \rho_T) = \frac{1}{2}\,. \tag{25}$$

The non-commutativity of the limits $N \to \infty$ and λ, $\mu \to 0$ in Eq. (25) indicates the emergence of SSB.[39]

5 Quantifying SSB in Fermion System

The Hamiltonian of the conventional superconductor in BCS theory is re-expressed as that of the many-spin model

$$H_S = -\sum_k \xi_k R(\theta_k,\phi)\sigma_z^k R^\dagger(\theta_k,\phi)\,, \tag{26}$$

where

$$\xi_k = \sqrt{\epsilon_k^2 + |\Delta|^2}\,, \quad \tan\phi = \frac{\mathrm{Im}(\Delta)}{\mathrm{Re}(\Delta)}\,,$$

$$\tan\theta_k = \frac{|\Delta|}{\epsilon_k}\,, \quad R(\theta_k,\phi) = \mathrm{e}^{-\mathrm{i}\phi\sigma_z^k/2}\,\mathrm{e}^{-\mathrm{i}\theta_k\sigma_y^k/2}\,.$$

The ground state of this superconductor is $|G\rangle_S = \prod_k R(\theta_k,\phi)|\uparrow\rangle_k$. The quasiparticle excitation energy

$\xi_k = \sqrt{\epsilon_k^2 + |\Delta|^2} \geq |\Delta|$. To excite a quasiparticle around the Fermi surface, one needs at least an energy scale of $|\Delta|$, which ensures the stability of the superconductor. As a consequence, $|\Delta|$ describes the energy gap in BCS.

Straightforward calculation shows that the DoS of the Hamiltonian and the ground state of the conventional superconductor are given as

$$S(\mathrm{SO}(2)^{\otimes N}, H_S) = \frac{1}{2} + \frac{1}{2}\frac{\sum_k \epsilon_k^2}{\sum_k \xi_k^2}\,, \tag{27}$$

$$\begin{aligned} S(\mathrm{SO}(2)^{\otimes N}, |G\rangle_S) &= \frac{1}{2} + \frac{1}{2}\prod_k \Big(1 - \frac{1}{2}\frac{|\Delta(0)|^2}{\epsilon_k^2 + |\Delta(0)|^2}\Big) \\ &\simeq \frac{1}{2} + \frac{1}{2}\,\mathrm{e}^{-\kappa|\Delta(0)|}\,, \end{aligned} \tag{28}$$

where $\kappa = (2-\sqrt{2})\pi g(0)$ and $g(\epsilon)$ is the density of states. For more details, see Appendix A. Like Eq. (22) in the many-spin model, the DoS of the ground state in fermion system also possesses the non-commutativity of two limits. This is one of the main results of this paper which reflect a direct correspondence between the DoS and the order parameter. The DoS of the conventional superconductor is less than unity as long as there exists a non-vanishing energy gap $\Delta(0)$. It agrees with the fact that SSB occurs when $\Delta(0)$ is non-zero. The energy gap at the absolute zero temperature is defined as

$$\Delta(0) = V\sum_k \langle G|\sigma_+^k|G\rangle_S = \frac{V}{2}\sum_k \frac{\Delta(0)}{\sqrt{\epsilon_k^2 + |\Delta(0)|^2}}\,. \tag{29}$$

At a finite temperature T, the density matrix of the system reads $\rho_S^T = \exp(-\beta H_S)/Z$, with the partition function $Z = 4^N \prod_k \cosh^2(\beta\xi_k/2)$. The corresponding DoS of ρ_S^T is

$$S(\mathrm{SO}(2)^{\otimes N}, \rho_S^T) = \frac{1}{2} + \frac{1}{2}\prod_k G(\epsilon, \Delta(T))\,, \tag{30}$$

where $G(\epsilon, \Delta(T)) = 1 - |\Delta(T)|^2\tanh^2(\beta\xi_k)/2\xi_k^2$. Further simplification shows

$$S(\mathrm{SO}(2)^{\otimes N}, \rho_S^T) \simeq \frac{1}{2} + \frac{1}{2}\,\mathrm{e}^{2g(0)|\Delta(T)|K(T)}\,, \tag{31}$$

where

$$K(T) = \int_0^\infty \ln\Big[1 + \frac{1}{1+t^2}\Big(-\frac{1}{2} + \frac{1}{k(T,t)+1}\Big)\Big]\,\mathrm{d}t\,,$$

$$k(T,t) = \cosh(2\beta|\Delta(T)|\sqrt{1+t^2})\,,$$

and in the above calculation we have assumed $\hbar\omega_D \gg |\Delta(T)|$ for simplicity. It follows from Eq. (31) that $K(T) = \ln(2S-1)/(2g(0)|\Delta(T)|)$. Hence in fact, $K(T) \propto \ln(2S-1)/|\Delta(T)|$.

As analyzed above, the DoS of ρ_S^T at temperature T in the conventional superconductor increases monotonically as the energy gap $|\Delta(T)|$ decreases. Figure 1(a) shows $\Delta(T)$ in unit of $\Delta(0)$ as a function of T/T_c. The energy gap $\Delta(T)$ decreases as T increases and stays zero when $T > T_c$, which means that the system returns to its normal phase. The temperature dependence of the DoS is shown in Fig. 1(b). As can be seen from the figure, the DoS grows as the temperature increases in the region of $0 \leq T \leq T_c$, and reaches its maximum value at the critical temperature T_c. Then, the increasement of the temperature above T_c does not modify the DoS. In comparison with the temperature-dependent behavior shown by the energy gap, it follows that the monotonic increasing of the DoS serves as a quantification for the restoring of the broken symmetry that is traditionally depicted by the decrease of $\Delta(T)$.

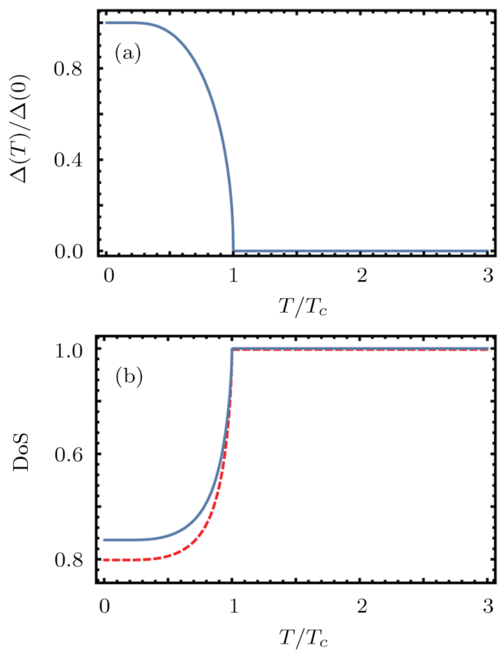

Fig. 1 (Color online) (a) Energy gap Δ vs. temperature T for the conventional superconductor. The energy gap is normalized as $\Delta(T)/\Delta(T_c)$ while T is measured in the unit of the critical temperature T_c. (b) The degree of symmetry of ρ_S^T with respect to the SO(2) transformation group. The blue solid and red dashed curves are corresponding to $g(0)k_BT_c = 0.4$ and 0.5, respectively.

6 Quantifying SSB in Boson System

The boson system can be mapped into the many-spin model with the Hamiltonian

$$H_B = \xi_B\sum_{i=1}^N \mathrm{e}^{-\mathrm{i}(\theta/2)\sigma_y^i}\sigma_z^i\,\mathrm{e}^{\mathrm{i}(\theta/2)\sigma_y^i}\,, \tag{32}$$

where

$$\xi_B = \sqrt{\epsilon^2 + \lambda^2}\,, \quad \sin\theta = \lambda/\xi_B\,.$$

In the limit of large N and low excitation, the ground state $|G\rangle_B = \prod_{i=1}^N \mathrm{e}^{-\mathrm{i}(\theta/2)\sigma_y^i}|\downarrow\rangle_i$ is approximately equivalent to a coherent state, i.e., $|G\rangle_B \simeq |\alpha\rangle$, where $\alpha = -\sqrt{N}\theta/2$. It sounds meaningful in physics that the ground state of BEC is also a coherent state as shown in Eq. (15).

Similar to the case of the BCS theory, we can obtain

$$S(\mathrm{SO}(2)^{\otimes N}, H_B) = 1 - \frac{\lambda^2}{2(\epsilon^2 + \lambda^2)}\,. \tag{33}$$

As $\mathrm{Tr}((\rho_B^{T=0})^2) = 1$, the DoS of the ground state reads

$$S(\mathrm{SO}(2)^{\otimes N}, |G\rangle_B) = \frac{1}{2} + \frac{1}{2}\Big(1 - \frac{\lambda^2}{2\xi_B^2}\Big)^N\,. \tag{34}$$

The order parameter at $T = 0$ can be calculated as

$$\langle a\rangle_0 = \frac{1}{\sqrt{N}}\,{}_B\langle G|\sum_i \sigma_-^i|G\rangle_B = -\frac{\sqrt{N}}{2}\sin\theta\,. \tag{35}$$

Furthermore, the DoS of the ground state can be re-expressed in term of the order parameter,

$$S(\mathrm{SO}(2)^{\otimes N}, |G\rangle_B) = \frac{1}{2} + \frac{1}{2}\Big(1 - \frac{2\langle a\rangle_0^2}{N}\Big)^N\,. \tag{36}$$

As $\lim_{x\to\infty}(1 + 1/x)^x = e$, Eq. (36) is simplified in the limit of large N as

$$\lim_{N/2\langle a\rangle_0^2\to\infty} S(\mathrm{SO}(2)^{\otimes N}, |G\rangle_B) = \frac{1}{2} + \frac{1}{2}\,\mathrm{e}^{-2\langle a\rangle_0^2}\,. \tag{37}$$

This is the second main result of this paper. Just as the case for the conventional superconductor, the maximum of DoS is directly associated with the vanishing of the order parameter $\langle a\rangle_0$. Thus the DoS of the ground state which is less than unity can also indicate the SSB in the boson system. The same results also hold for the system at finite temperature, since by evaluating Eq. (4) with ρ_B^T we found

$$S(\mathrm{SO}(2)^{\otimes N}, \rho_B^T) = \frac{1}{2} + \frac{\overline{\mathrm{Tr}(R(\omega)\rho R^\dagger(\omega)\rho)}_{\mathrm{SO}(2)^{\otimes N}}}{2\mathrm{Tr}(\rho^2)}\,. \tag{38}$$

The DoS of ρ_B^T of BEC is given as

$$S(\mathrm{SO}(2)^{\otimes N}, \rho_B^T) = \frac{1}{2} + \frac{1}{2}\Big(1 - \frac{\lambda^2}{2\xi_B^2}\frac{\cosh(2\beta\xi_B) - 1}{\cosh(2\beta\xi_B)}\Big)^N\,. \tag{39}$$

Using the approach similar to the above, we rewrite Eq. (39) in terms of the order parameter in a finite temperature $\langle a\rangle_T = \mathrm{Tr}(\rho_B^T a) = -\lambda\sqrt{N}\tanh(\beta\xi_B)/(2\xi_B)$.

$$S(\mathrm{SO}(2)^{\otimes N}, \rho_B^T) = \frac{1}{2} + \frac{1}{2}\Big[1 - \frac{2\langle a\rangle_T^2}{N}\frac{1 + \cosh(2\beta\xi_B)}{\cosh(2\beta\xi_B)}\Big]^N\,. \tag{40}$$

Similar to the absolute zero temperature case, the DoS of ρ_B^T in boson system depends on both the order parameter $\langle a\rangle_T$ and the temperature T. Just like the Penrose–Onsager criterion,[14] the BEC occurs if and only if $\langle a\rangle_T$ is nonzero. When there exists BEC, a large fraction of particles (to the order of N) occupy the ground state with zero momentum. When $T > T_c$, where T_c is the critical temperature of BEC, no condensation occurs. Thus $\langle a\rangle_{T>T_c} = 0$, and the symmetry of the boson system is unbroken.

7 Conclusion

In this paper, we have exploited a measure of symmetry — the degree of symmetry (DoS) to describe the SSB in the conventional superconductor and BEC. We have established rigorous relations between the DoS and the order parameters at the absolute zero temperature and finite temperature. It has been demonstrated that for both the fermion and the boson systems, (i) At $T = 0$, the order parameter takes its maximum and the symmetry of the system is maximally broken; (ii) At $0 < T < T_c$, the order parameter is still non-vanishing and the extent of the SSB can be quantified by the DoS; (iii) When T grows beyond T_c, the order parameter vanishes and the symmetry of the system is fully restored.

In fact, the DoS approach that we applied in this paper can be generalized to other circumstances. We can explore symmetry breaking in other quantum many-body systems employing the DoS quantifier and expect to obtain new order parameters when SSB appears. What is worth mentioning is that the new order parameter must be measurable and reasonable in physics.

Appendix A: The DoS of the Ground State in the Fermion System

As shown in Eq. (28), the DoS of the ground state in the fermion system is

$$\begin{aligned} S(\mathrm{SO}(2)^{\otimes N}, |G\rangle_S) &= \frac{1}{2} + \frac{1}{2}\prod_k\Big(1 - \frac{1}{2}\sin^2\theta_k\Big) = \frac{1}{2} + \frac{1}{2}\exp\Big[\sum_k \ln\Big(1 - \frac{1}{2}\sin^2\theta_k\Big)\Big] \\ &= \frac{1}{2} + \frac{1}{2}\exp\Big[\int_{-\hbar\omega_D}^{\hbar\omega_D} g(\epsilon)\ln\Big(1 - \frac{1}{2}\sin^2\theta_k\Big)\mathrm{d}\epsilon\Big] = \frac{1}{2} + \frac{1}{2}\exp[|\Delta(0)|G(\omega_D, |\Delta(0)|)]\,, \end{aligned} \tag{A1}$$

where

$$G(\omega_D, |\Delta(0)|) = \int_{-\hbar\omega_D/|\Delta(0)|}^{\hbar\omega_D/|\Delta(0)|} g(t|\Delta(0)|)\ln\Big(1 - \frac{1}{2}\frac{1}{t^2 + 1^2}\Big)\mathrm{d}t\,,$$

and ω_D is the Debye frequency. As in general case $\hbar\omega_D/|\Delta(0)| \gg 1$ and $g(\epsilon)$ changes slowly in the range of $(-\hbar\omega_D, \hbar\omega_D)$, we can take the integral limits to $\pm\infty$ and replace $g(t|\Delta(0)|)$ with $g(0)$. Therefore, we obtain

$$\begin{aligned} G(\omega_D, |\Delta(0)|) &= \int_{-\hbar\omega_D/|\Delta(0)|}^{\hbar\omega_D/|\Delta(0)|} g(t|\Delta(0)|)\ln\Big(1 - \frac{1}{2}\frac{1}{t^2 + 1^2}\Big)\mathrm{d}t \\ &\simeq g(0)\int_{-\infty}^{\infty}\ln\Big(1 - \frac{1}{2}\frac{1}{t^2 + 1^2}\Big)\mathrm{d}t = -(2 - \sqrt{2})\pi g(0)\,. \end{aligned} \tag{A2}$$

In this sense, the DoS of the ground state in the fermion system is simplified as

$$S(\mathrm{SO}(2)^{\otimes N}, |G\rangle_S) = \frac{1}{2} + \frac{1}{2}\exp[|\Delta(0)|G(\omega_D, |\Delta(0)|)] \simeq \frac{1}{2} + \frac{1}{2}\exp[-(2-\sqrt{2})\pi g(0)|\Delta(0)|]\,. \tag{A3}$$

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