One Hair Postulate for Hawking Radiation as Tunneling Process[*]

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Abstract *For Hawking radiation, treated as a tunneling process, the no-hair theorem of black hole together with the law of energy conservation is utilized to postulate that the tunneling rate only depends on the external qualities (e.g., the mass for the Schwarzschild black hole) and the energy of the radiated particle. This postulate is justified by the WKB approximation for calculating the tunneling probability. Based on this postulate, a general formula for the tunneling probability is derived without referring to the concrete form of black hole metric. This formula implies an intrinsic correlation between the successive processes of the black hole radiation of two or more particles. It also suggests a kind of entropy conservation and thus resolves the puzzle of black hole information loss in some sense.*

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1 Introduction

Hawking discovered that the black hole radiation possesses an exactly thermal spectrum of temperature depending on the surface gravity of the black hole.[1] Particularly, the radiation does not depend on the details of the structure of the object that collapsed to form the black hole. Thus, an initially pure quantum state will evolve into a mixed thermal state as the black hole radiates. This phenomenon, known as the paradox of black hole information loss, obviously violates the quantum unitarity for the closed system.

Since its appearing, many attempts[2] have been made to resolve this paradox. In the previous investigations, the radiation is always treated as possessing the thermal spectrum and the space-time geometry is fixed. Recently, based on the WKB approximation, the tunneling probability for the Hawking radiation was derived in the framework of dynamical geometry. It turns out surprisingly that the radiation spectrum is not exactly thermal.[3] For this reason, it is found in Ref. [4] that the successively radiated two particles are correlated, and thus no information is lost in the radiation.[4] Actually, by using the same approach as that in Ref. [3], the Hawking radiation spectra of various black holes have been obtained.[5−10] These results verify the correlation between the successive radiations and the conservation of the information in the radiation.[11−12] We find that the chain rule for the probability is essential for the information conservation in the black hole radiation, and we verify case by case that

the chain rule indeed holds for various Hawking radiations coincidentally.

We observe that the above mentioned coincidence can be exactly explained by the No-hair theorem of black hole together with the law of energy conservation. In fact, from our "one hair" postulate based on the No-hair theorem and the law of energy conservation, we are able to derive a general form of the tunneling probability of Hawking radiation without resorting to the details of the black hole, such as its geometric structure. We are thus able to prove that for the tunneling probability obtained from the WKB approximation, the chain rule is satisfied automatically and the above mentioned coincidence is of physical necessity. It should be clear that our results demonstrate the advantage of treating the black hole radiation as a tunneling process.

This letter is organized as follows. In Sec. 2, our postulate is stated based on the No-hair Theorem. In Sec. 3, a general formula for the tunneling probability is derived from the postulate. In Sec. 4, the tunneling rate for the Schwarzschild black hole is obtained without referring to its geometry. In Sec. 5, the case by case verification of our postulate is given for various black hole radiations.

2 "One Hair" for Hawking Radiation as Tunneling

It is well known that all black hole solutions of the Einstein–Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only three externally observable classical pa-

rameters: mass, electric charge, and angular momentum. This result is referred to as No-hair theorem of steady black hole. For our purpose, we generalize this theorem for the dynamic black hole as follows: the tunneling probability for the Hawking radiation only depends on the final state of the steady black hole and the total energy $E_T = E_1 + E_2 + \cdots + E_N$ after simultaneously radiating N particles with the energies $E_1, E_2, \ldots, E_N$. Here, there is only "one hair" quantity E_T and the tunneling probability has nothing to do with its partition.

To investigate the above "one hair" postulate, let us consider the two processes in the Hawking radiation, illustrated in Fig. 1:

(i) The black hole radiates a single particle with the energy E_T, as illustrated in Fig. 1(a). The mass of the black hole reduces to $M - E_T$. The tunneling probability is defined as $p(\{E_T\}; M)$. The black hole can also simultaneously radiate two particles with the energies E_1 and E_2 respectively. The probability of this process is denoted by $p(\{E_1, E_2\}; M)$. Based on the No-hair Theorem of black hole and the law of energy conservation, we postulate the one-hair Theorem for black hole radiation: if $E_T = E_1 + E_2$, then

$$p(\{E_1, E_2\}; M) = p(\{E_T\}; M). \qquad (1)$$

Actually, we can imagine that after the Hawking radiation the radiated particle immediately splits into two particles with the energy E_x and $E_T - E_x$ respectively, and in the split no particular energy partition between the two particles is preferred. The one-hair Theorem simply means that all the splits satisfying the law of energy conservation possess the same tunneling probability.

(ii) The black hole firstly radiates a particle with the energy E_1 and then radiates another particle with the energy $E_2 = E_T - E_1$, as illustrated in Fig. 1(b). The mass of the black hole also reduces to $M - E_T$. The tunneling probability for this process is

$$p(\{E_1 : E_2\}; M) = p(\{E_1\}; M)p(\{E_2\}; M - E_1),$$

where the conditional probability $p(\{E_2\}; M - E_1)$ reflects the fact that the mass of the black hole reduces to $M - E_1$ after it radiates the particle of energy E_1.

We remark here that, the first radiated particle is correlated to the second one, since the conditional tunneling probability of the second one actually depends on the energy E_1 of the first one. Most recently, this correlation is employed to account for the information loss in the black hole radiation process.[4,11−12]

In the following we only consider the steady state of the black hole. It will take a longer time to reach the steady state than the relaxation time of each radiation. In this case, the one-hair Theorem for black hole radiation can be re-expressed as $p(\{E_1, E_2\}; M) = p(\{E_1 : E_2\}; M)$ or

$$p(\{E_1, E_2\}; M) = p(\{E_1\}; M)p(\{E_2\}; M - E_1). \qquad (2)$$

Here, as only the steady solutions of the black hole radiation are concerned, we have identified the two processes

of simultaneously and successively radiating two particles. For the multi-particle case, we can recover the chain rule as

$$p(\{E_1 : E_2 : \cdots : E_N\}; M) = \prod_p p\Big(E_p; M - \sum_{j=1}^{p-1} E_j\Big). \qquad (3)$$

based on this two-particle case. Thus, to verify the chain rule for various Hawking radiation, we need only to prove the postulation in Eq. (2).

To justify the above observation, let us briefly review some results derived from the dynamic calculation based on the generalized WKB approximation. In Ref. [3], the tunneling probability for a particle out of the black hole is defined as

$$p \sim \mathrm{e}^{-2\mathrm{Im}S}, \qquad (4)$$

where S is the action for an s-wave outgoing positive particle. The exact form of the imaginary part of the action reads

$$\mathrm{Im}S = \mathrm{Im} \int_M^{M-E} \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{\mathrm{d}r}{\dot{r}} \mathrm{d}H. \qquad (5)$$

Here, the Hamiltonian H is defined through the radial null geodesics equation, and particularly $H = M - E'$ for the Schwarzschild black hole. It is easily seen that $\mathrm{Im}S$ naturally satisfies the above stated postulate. Then it can be concluded that the conservation of information will not be broken if Hawking radiation is treated as tunneling process, as has been proved in many references.[4,11−12]
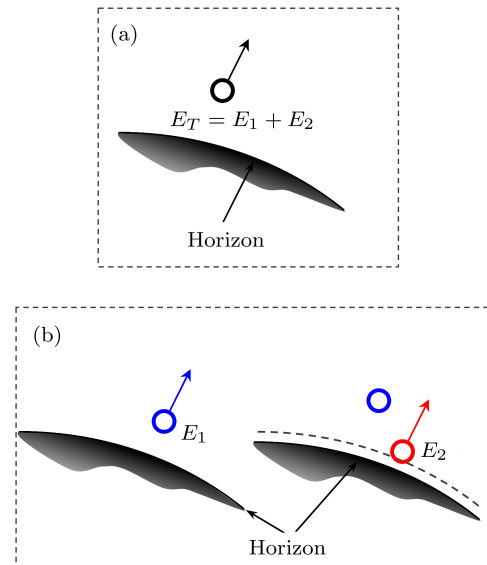

Fig. 1 Radiation. (a) The black hole radiates a particle with energy E_T. (b) The black hole radiates firstly a particle with energy E_1 and successively another particle with energy E_2.

3 Energy Dependence of Non Thermal Hawking Radiation

In this section, we will present a derivation of the general form of the tunneling probability based only on the

"one hair" postulate. Without losing the generality, we assume

$$p(\{E\}; M) = \exp[f(\{E\}; M)],$$

where $f(\{E\}; M)$ is actually the tunneling entropy for the black hole radiation. It then follows from Eq. (2) that

$$f(\{E_T\}; M) = f(\{E_1\}; M) + f(\{E_2\}; M - E_1). \quad (6)$$

Substituting the Taylor expansion form $f(\{\omega\}; M) = \sum_{n=0} A_n(M)\omega^n$ of the function f into this equation and comparing the coefficients of the terms with the same orders of E_2, we obtain the following system of recursive equations

$$0 = A_0(M - E_1),$$

$$\sum_{n=1} A_n(M) C_n^1 E_1^{n-1} = A_1(M - E_1),$$

$$\sum_{n=2} A_n(M) C_n^2 E_1^{n-2} = A_2(M - E_1),$$

$$\vdots$$

$$\sum_{n=m} A_n(M) C_n^m E_1^{n-m} = A_m(M - E_1,),$$

$$\sum_{n=m+1} A_n(M) C_n^{m+1} E_1^{n-(m+1)} = A_{m+1}(M - E_1),$$

$$\vdots$$

Differentiating the left hand right hand sides of the above equations with respect to E_1 then results in the equation

$$(m+1)A_{m+1}(M - E_1) = \frac{\mathrm{d} A_m(M - E_1)}{\mathrm{d} E_1}$$

$$= -\frac{\mathrm{d} A_m(M - E_1)}{\mathrm{d} M},$$

for each m. Thus we have the recursion formula

$$A_m(M) = -\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d} M} A_{m-1}(M)$$

$$= \frac{(-1)^{m-1}}{m!}\frac{\mathrm{d}^{m-1}}{\mathrm{d} M^{m-1}} A_1(M),$$

and the black hole entropy can be rewritten as

$$f(\{E\}; M) = \sum_{m=1} \frac{(-1)^{m-1}}{m!}\frac{\mathrm{d}^{m-1}}{\mathrm{d} M^{m-1}} A_1(M) E^m. \quad (7)$$

Define the entropy $G(M)$ for the black hole radiation through

$$A_1(M) = -\frac{\mathrm{d} G(M)}{\mathrm{d} M},$$

the black hole entropy then reads

$$f(\{E\}; M) = G(M - E) - G(M). \quad (8)$$

This is the main result of this paper. Obviously, $G(M)$ in Eq. (8) is a conservation quantity. According to the above result, after a black hole of mass M radiates a tunneling particle with energy E, its entropy decrease is

$$S(E, M) = -\ln p(\{E\}; M) = G(M) - G(M - E). \quad (9)$$

In deriving the above result, it is tacitly assumed that the black hole does not carry charge. For charged black hole a similar result can easily be obtained by the above method. In fact, when a charged black hole with charge Q radiates a particle with charge q, the tunneling probability can be derived as

$$S(E, q; M, Q) = G(M, Q) - G(M - E, Q - q). \quad (10)$$

4 Tunneling Probability for Schwarzschild Black Hole

In this section, we will derive the tunneling probability for the Hawking radiation of the Schwarzschild black hole without referring to its dynamic geometry.

We assume that the entropy for black hole radiation is corrected to the second order of the tunneling energy E, namely

$$f(\{E\}; M) = A(M) + B(M)E + C(M)E^2, \quad (11)$$

where $A(M)$ $B(M)$, and $C(M)$ are mass-dependent functions to be determined. Then Eq. (6) takes the form

$$A(M) + B(M)(E_1 + E_2) + C(M)(E_1 + E_2)^2$$
$$= A(M) + B(M)E_1 + C(M)E_1^2$$
$$\quad + A(M - E_1) + B(M - E_1)E_2 + C(M - E_1)E_2^2$$

gives the following equations about $A(M)$, $B(M)$, and $C(M)$:

$$A(M - E_1) = 0,$$
$$B(M) - 2C(M)E_1 = B(M - E_1),$$
$$C(M) = C(M - E_1).$$

It then follows that $C(M) = k$ and $B(M) = \xi - 2kM$, and the entropy of black hole radiation is obtained as

$$f(\{E\}; M) = (\xi - 2kM)E + kE^2, \quad (12)$$

where k and ξ are constants. If we take $k = 4\pi$ and $\xi = 0$, then we recover the well-known result by Parikh and Wilczek:

$$f(\{E\}; M) = 4\pi[(M - E)^2 - M^2]. \quad (13)$$

We would like to emphasize again that, in obtaining the above result, we only make the assumption that the entropy of the black hole is a polynomial of the radiated energy E, and the details of the dynamic geometry do not come into the derivation. If the entropy is a polynomial of E of degree 1, then we have $f(\{E\}; M) = \xi E$ where ξ is a constant independent of the mass M. Thus, the conventional thermal spectrum $p'(E, M) = \exp(-8\pi EM)$ does not satisfy Eq. (6) about the conditional probability. In that case, $G(M) = 4\pi M^2 = A/4$ is the usual entropy for the Schwarzschild black hole, and is usually called Bekenstein–Hawking entropy of black hole.

According to Ref. [4], the above spectrum function (13) indicates that the two successively radiated particles are actually correlated. Since Hawking radiation can carry information through this correlation between the radiated particles, the conservation of total information can be restored by taking this correlation into account.

5 Verification of One-Hair Postulate for Other Black Holes

In this section, we will check the radiation spectra of some well known black holes to see if they satisfy the one-hair postulate expressed by Eq. (6).

(i) ***Reissner–Nordström Black Hole*** The tunneling probability of a charged particle with energy E and charge q for the Reissner–Nordström black hole has been obtained in Ref. [5] as

$$p(\{E,q\};M,Q) = \frac{\exp[G_{\mathbf{RN}}(M-E,Q-q)]}{\exp[G_{\mathbf{RN}}(M,Q)]}, \qquad (14)$$

where

$$G_{\mathbf{RN}}(M,Q) = \pi(M + \sqrt{M^2 - Q^2}).$$

Clearly, the radiation spectrum for the Reissner–Nordström black hole is not thermal, and satisfies our one-hair postulate.

(ii) ***Kerr Black Hole*** For the rotating black hole (Kerr black hole), the tunneling probability is found in Ref. [6] as

$$p(\{E\},M) = \exp[G_{\mathbf{K}}(M-E) - G_{\mathbf{K}}(M)], \qquad (15)$$

where

$$G_{\mathbf{K}}(M) = 2\pi(M^2 + M\sqrt{M^2 - a^2}).$$

Obviously, its spectrum structure is in accordance with our one-hair postulate.

(iii) ***Kerr–Newman Black Hole*** For the Kerr–Newman black hole, the tunneling probability for a particle with charge q is obtained in Refs. [6–7] as

$$p(\{E,q\};M,Q) = \frac{\exp[G_{\mathbf{KN}}(M-E,Q-q)]}{\exp[G_{\mathbf{KN}}(M,Q)]}, \qquad (16)$$

where

$$G_{\mathbf{KN}}(M,Q) = \pi(M + \sqrt{M^2 - Q^2 - a^2})^2.$$

It also satisfies our postulate.

(iv) ***Quantum Corrected Hawking Radiation*** Last, we consider the tunneling with quantum correction for the Schwarzschild black hole. For the quantum corrected Hawking radiation, the tunneling probability reads

$$p(\{E\};M) = \left(1 - \frac{E}{M}\right)^{2\alpha} \exp\left[8\pi E\left(M - \frac{E}{2}\right)\right]$$
$$= \exp[G(M-E) - G(M)], \qquad (17)$$

where

$$G(M) = 4\pi M^2 + 2\alpha \ln M.$$

This tunneling probability still satisfies our postulate, thus the information conservation is quite natural. For a detailed discussion about the information conservation, one can refer to the Refs. [11–12].

6 Summary

In this letter, we suggest the one-hair Postulate to describe Hawking radiation as tunneling process based on the No-hair theorem and the energy conservation law. This postulate for tunneling probability naturally leads to the information conservation for the total system formed by the radiated particles plus the remnant black hole. Especially, this postulate is used to determine the tunneling rate by the information (probability) theory method rather than the dynamic geometry method. Finally, some well known examples are presented to support the postulate. We expect the viewpoint developed in this letter will shed light on the parabox of black hole information loss.

*Authors' notes: After we put the manuscript of this work in arXiv with reference number 0907.2085 in 2009, we found a paper [Phys. Rev. Lett. **107** (2011) 071302] by S.L. Braunstein and M.K. Patra that was put as arXiv:1102.2326 and reported the similar results in 2011.*

Acknowledgments

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