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Multistability and Critical Fluctuation in a Split Bose–Einstein Condensate*

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Abstract *By using a two-mode description, we show that there exist the multistability, phase transition and associated critical fluctuations in the macroscopic tunnelling process between the halves of a double-well trap containing a Bose–Einstein condensate. The phase transition that two of the triple stable states and an unstable state merge into one stable state or a reverse process takes place whenever the ratio of the mean field energy per particle to the tunnelling energy goes across a critical value of order one. The critical fluctuation phenomenon corresponds to squeezed states for the phase difference between the two wells accompanying with large fluctuations of atom numbers.*

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Over the last few years, there has been considerable interest in the subjects related to the macroscopic tunnelling processes between the halves of a double-well trap containing a Bose–Einstein condensate (BEC). Here we only mention a few ones closely related to the topic concerned in the present paper, for instance, experimentally realized squeezed states,[1] the scheme for demonstrating nonlinear Josephson-type oscillations of a driven, two-component BEC,[2] coherent oscillations concerning Josephson effects, π oscillations, and macroscopic quantum self-trapping.[3,4] Javanainen and Ivanov[5] have claimed that these studies[2–4] are based on essentially classical models. On the other hand, the fluctuations of atom numbers and phases have been intensively investigated quantum-mechanically mainly by two seemingly quite different approaches: two-mode approximation[5–7] and the one based on taking atom number and phase difference as conjugate quantum variables.[8–10] The corresponding investigations in an array of traps containing BEC have also been carried out recently.[1–6] Although extensively studied, there still exist some important open issues in the the macroscopic tunnelling processes between the halves of a double-well trap containing a BEC. Some of these open issues are that i) the fluctuations of atom number and phase difference obtained by the above-mentioned two approaches seem to show large difference;[8–11] ii) there exists no bridge to connect the strong and weak tunnelling regimes yet;[11] iii) how to relate the essentially classical models[3,4] and the corresponding quantum-mechanical description in dealing with the fluctuations. In this paper, we shall solve these three important issues by providing a united approach. What is more important, we shall show that the previous studies have missed the phenomena of the multistability, phase transition and associated critical fluctuations in the macroscopic tunnelling processes between the halves of a double-well trap containing a BEC. The critical fluctuation phenomenon corresponds to the squeezed states for the phase difference between the two wells with extremely large fluctuation of atom numbers.

We consider a model system of many-atom ground state of Bose–Einstein condensate in a double-well potential.[1–7,12–17] In this system, N bosonic atoms are confined by an infinite harmonic potential that is divided into left and right wells by a barrier that can be raised and lowered arbitrarily. Making a simple two-mode approximation, considering only the lowest energy states, the creation and annihilation operators ($\hat{a}^\dagger_{L,R}$ and $\hat{a}_{L,R}$ respectively) for atoms localized in the ground state of either the left or the right potential well can be constructed. Neglecting terms that depend only on the total conserved particle number N, the Hamiltonian for the system can be written[1] as

$$\hat{H} = \frac{\Delta}{2}(\hat{n}_L - \hat{n}_R) + \frac{g\beta}{2}(\hat{n}_L^2 + \hat{n}_R^2) + \gamma(\hat{a}^\dagger_L \hat{a}_R + \hat{a}^\dagger_R \hat{a}_L)\,, \quad (1)$$

where $\hat{n}_{L,R} \equiv \hat{a}^\dagger_{L,R}\hat{a}_{L,R}$, the total conserved particle number operator $\hat{N} = \hat{n}_L + \hat{n}_R$, $g = 4\pi a_{\rm sc}\hbar^2/m$ is the mean-field energy constant ($a_{\rm sc}$ is the s-wave scattering length). The term in γ describes tunnelling between wells, whereas the term in $g\beta$, which depends on the number of atoms within each wells, describes the mean-field energy due to interactions between atoms in the same well. The term in $\Delta = (E_L - E_R)/\hbar$ describes the energy difference of the ground states in the left and right wells. The coefficients

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γ and β are determined from integrals over single-particle wave functions.[1,5]

Introducing phase operators $\hat{\phi}_{L,R}$ by the relation[19−22] $\exp(\mathrm{i}\hat{\phi}_{L,R}) = (\hat{n}_{L,R}+1)^{-1/2}\hat{a}_{L,R}$ or

$$\hat{a}_{L,R} = \exp(\mathrm{i}\hat{\phi}_{L,R})\sqrt{\hat{n}_{L,R}}\,, \tag{2}$$

where we have made use of the fact that $\hat{a}_{L,R}F(\hat{n}_{L,R}) = F(\hat{n}_{L,R}+1)\hat{a}_{L,R}$. The phase and atom number operators satisfy the commutative relations

$$[\hat{n}_{L,R},\ \hat{\phi}_{L,R}] = \mathrm{i}\,. \tag{3}$$

But phase operators thus introduced suffer from the well-known non-Hermitian problem[19−22] that

$$(\exp(\mathrm{i}\hat{\phi}_{L,R}))^\dagger \exp(\mathrm{i}\hat{\phi}_{L,R}) = 1 - |0\rangle\langle 0| \neq 1$$

although $\exp(\mathrm{i}\hat{\phi}_{L,R})(\exp(\mathrm{i}\hat{\phi}_{L,R}))^\dagger = 1$. However, closing inspection of this problem, we realize that this problem is in fact avoidable if we focus on system's states with $n_{L,R} \neq 0$, which will be assumed to be so hereafter. Let $\hat{n} = \hat{n}_L$ and $\hat{\phi} = \hat{\phi}_L - \hat{\phi}_R$ denote the left well's atom number operator and the phase difference between the left and right wells respectively. Utilizing $\hat{n}_R = \hat{N} - \hat{n}$, and the total atom number $\hat{N} = N$ (the conserved operator $\hat{N}$ only takes a unique eigenvalue N and hence we need not consider its operator characteristic), we can, after omitting the unimportant conserved quantity $N(\Delta + g\beta N)/2$, rewrite the Hamiltonian (1) as follows:

$$\hat{H} = \Delta\hat{n} - g\beta\hat{n}(N-\hat{n}) + \gamma(\sqrt{\hat{n}(N-\hat{n})}\exp(\mathrm{i}\hat{\phi}) + \mathrm{h.c.})\,, \tag{4}$$

where the left well's atom number operator $\hat{n}$ and the phase-difference operator $\hat{\phi}$ satisfy the commutative relation

$$[\hat{n},\ \hat{\phi}] = \mathrm{i}\,. \tag{5}$$

Equations (4) and (5) are the fully quantum-mechanical model giving the united description for investigating the various problems in a Bose–Einstein condensate in a double-well potential, particularly those relevant to the phase coherence. In particular, we shall demonstrate that all the previous models, either essentially classical or the fully quantum-mechanical ones, dealing with this systems can be derived from these equations under some approximations.

Suppose that the system can be described by a two-mode coherent state $|\Psi\rangle = |\alpha_L, \alpha_R\rangle_c$ characterized by two complex parameters $\alpha_L = \sqrt{n}\exp(\mathrm{i}\phi_L)$ and $\alpha_R = \sqrt{N-n}\exp(\mathrm{i}\phi_R)$, we can derive from Hamiltonian formalism (4) for the equations of motion for the mean atom number $n = \langle\Psi|\hat{n}|\Psi\rangle$ in the left well (as well as the mean atom number N-n in the right well) and phase difference $\phi = \langle\Psi|\hat{\phi}|\Psi\rangle$ between the two wells as follows (obtained by taking average operations to the Heisenberg equation of motion $\mathrm{d}\hat{A}/\mathrm{d}t = \mathrm{i}[\hat{H},\hat{A}]$ for $\hat{A} = \hat{n}$ and $\hat{\phi}$ respectively),

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\phi}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial n}\,, \tag{6}$$

$$\mathcal{H} = \mathcal{H}_0 + \Delta n - g\beta n(N-n) + 2\gamma\sqrt{n(N-n)}\cos\phi\,, \tag{7}$$

where $\mathcal{H} \equiv \langle\Psi|\hat{H}|\Psi\rangle$, and $\mathcal{H}_0 = N(\Delta + g\beta N)/2$ is conserved quantity and can be omitted without loss of generality. The explicit form of equation (6) reads

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 2\gamma\sqrt{n(N-n)}\,\sin\phi\,, \tag{8a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Delta - g\beta(N-2n) + \gamma\frac{N-2n}{\sqrt{n(N-n)}}\cos\phi\,. \tag{8b}$$

These are nonlinear versions of the usual Josephson-junction equations[2] and are nearly identical with those describing the double-well tunnelling problem in the same system.[3,4] Raghavan *et al.* have investigated Josephson effects, π oscillations and macroscopic quantum self-trapping for similar system.[3] However, no one, to the best of our knowledge, seems to have so far noticed the important phenomena of multistability and phase transition in the macroscopic tunnelling process between the halves of a double-well trap containing a Bose–Einstein condensate, which we now turn to investigate.

The multistability and phase transition in the macroscopic tunnelling process are clearly seen from Fig. 1. Let us describe their main features. First of all, there are two kinds of evolution pattern for atom number n $(N-n)$ in the left (right) well and phase difference between the two wells. They are i) stable and unstable steady states (fixed points in the phase diagrams) denoting no tunnelling at all although tunnelling rate is non-zero; ii) periodic tunnelling processes where the tunnelling amplitude can be very large, i.e., macroscopic quantum tunnelling, even for small tunnelling rate γ. Secondly, the phase transition takes place when an "order" parameter $|\xi| = |g\beta N/(2\gamma)|$ characterizes the relative magnitude of the mean-field energy per particle and the tunnelling energy goes across the critical parameter ξ_c of order one. In other words, one of the three stable fixed point remains while the unique unstable fixed point and the two of the three stable fixed points for $|\xi| > \xi_c$ merge into one stable fixed point when $|\xi|$ goes from below to above the critical value ξ_c. Therefore the phase transition corresponds to the sudden structural change in $n-\phi$ phase diagrams when the "order" parameter $|\xi|$ goes across its critical value ξ_c. Thirdly, there exist three stable and one unstable fixed points when $|\xi| > \xi_c$,whereas there exist two stable fixed points and no unstable one otherwise. The concrete value of the critical parameter depends on the parameter $\delta = \Delta/(2\gamma)$ corresponding to the ratio of the ground energy difference of the two wells to the tunnelling rate. When $|\xi| > \xi_c$, the two of the three stable fixed points (i.e. the two stable fixed points at $\phi = 0$ in the lower phase diagrams of Fig. 1) are symmetric about $n = N/2$ if the right and left wells are identical with each other ($\delta = 0$) and they are asymmetric if the two wells are different from each other ($\delta \neq 0$).

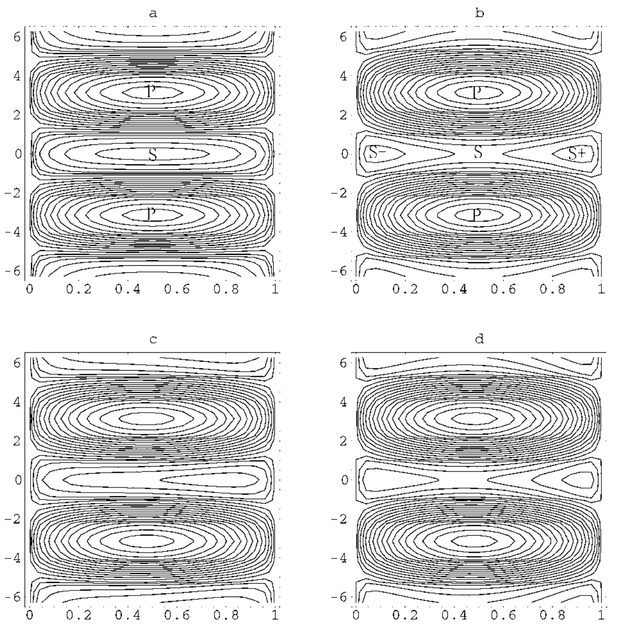

Fig. 1 The $\mathcal{H}/(2\gamma)$ contours for several values of parameters $\xi = g\beta N/(2\gamma)$ and $\delta = \Delta/(2\gamma)$. The horizontal and vertical axes denote reduced atom number $x = n/N$ and the phase difference ϕ respectively. The four diagrams correspond to the parameter choices: a) ($\xi = 0.6, \delta = 0$), b) ($\xi = 1.8, \delta = 0$), c) ($\xi = 1.1, \delta = 0.1$) and d) ($\xi = 1.8, \delta = 0.1$). We have explicitly designated the fixed points S, $S_\pm$ and P in a) and b).

The fixed points and the critical parameter ξ_c are easily shown to be determined by the equations $\sin\phi_0 = 0$ or $\phi_0 = 0, \pi$ and

$$(1-2x_0)(\cos\phi_0 - 2\xi\sqrt{x_0(1-x_0)}) = 2\delta\sqrt{x_0(1-x_0)}\,, \quad (9)$$

where $\xi = g\beta N/(2\gamma)$, $\delta = \Delta/(2\gamma)$, and $x_0 = n_0/N$. We find from this equation that the critical parameter $\xi_c = 1$ for $\delta = 0$, and $\xi_c > 1$ for nonzero δ. For instance, $\xi_c \approx 1.37$ for $\delta = 0.1$. In the case $\delta = 0$ denoting identical ground energy for two wells, we can easily obtain the explicit expressions for the fixed points. They are the fixed point P whose $n_0 = N/2$ and $\phi_0 = 0$ for $\xi > 0$ and $\phi_0 = \pi$ for $\xi < 0$, the fixed points $S_\pm$ whose $n_0 = N(1 \pm \sqrt{1-\xi^{-2}})/2$ and $\phi_0 = 0$ for $\xi > 0$ and $\phi_0 = \pi$ for $\xi < 0$, the fixed point S whose $n_0 = N/2$ and $\phi_0 = \pi$ for $\xi > 0$ and $\phi_0 = 0$ for $\xi < 0$. The fixed point P is unstable for $|\xi| > 1$ and stable for $|\xi| \leq 1$, S is stable and exists for any ξ, and $S_\pm$ are stable but they exist only when $|\xi| \geq 1$.

Now let us illustrate that equations (4) and (5) provide a natural basis for describing quantum-mechanically the atom number and phase statistics. For this goal, we expand the left well's atom number operator $\hat{n}$ and the phase-difference operator $\hat{\phi}$ in the Hamiltonian (4) around one of the stable steady states discussed in the paragraph where equation (9) locates, i.e., $\hat{n} = n_0 + \hat{\eta}$ and $\hat{\phi} = \phi_0 + \hat{\psi}$ with (n_0, ϕ_0) denoting one of the stable steady states (note $\sin\phi_0 = 0$) and $[\hat{\eta}, \hat{\psi}] = \mathrm{i}$. Then after neglecting the terms equal to or higher than the order of $\mathcal{O}(\hat{\eta}^3, \hat{\eta}\hat{\psi}^2, \hat{\eta}^2\hat{\psi}, \hat{\psi}^3)$ and taking $\hat{\psi} = -\mathrm{i}\partial/\partial\eta$ and $\hat{\eta} = \eta$ in order to satisfy $[\hat{\eta}, \hat{\psi}] = \mathrm{i}$, we can write Eq. (4) as follows:

$$\hat{H} \approx H_0 + E_1\frac{\partial}{\partial\eta} + E_2\eta\frac{\partial}{\partial\eta} - \frac{E_J}{2}\frac{\partial^2}{\partial\eta^2} + \frac{E_C}{2}\eta^2\,, \quad (10)$$

where H_0 is a constant equal to $(\mathcal{H} - \mathcal{H}_0)$ in Eq. (7) evaluated at the fixed point considered, and the coefficients in Eq. (10) are given in Eq. (11) for the fixed point S and in Eq. (13) for the fixed points $S_\pm$ respectively. Following the same argument as the one in Ref. [5] given in the paragraph immediately after its equation (16), we can neglect $E_{1,2}$ terms in Eq. (10) in evaluating the atom number and phase statistics. After this approximation, equation (10) is nothing but a harmonic oscillator model and fluctuations of the atom number in the left well and the phase difference are easily shown to be $\Delta n = (\Delta\phi)^{-1} = (E_J/E_C)^{1/4}$ when the harmonic oscillator is in its ground state.[18]

In the case where $\delta = 0$ denoting identical ground energy for the two wells, if one considers the stable fixed point S whose concrete expression is given after Eq. (9), the coefficients in Eq. (10) can be calculated to be

$$E_1 = 0, \quad E_2 = \frac{g\beta}{|\xi|}, \quad E_J = \frac{g\beta N^2}{2|\xi|}, \quad E_C = \frac{2g\beta}{1+|\xi|}\,. \quad (11)$$

Consequently, fluctuations of the atom number in the left well and the phase difference in this case are given by

$$\Delta n = \frac{1}{\Delta\phi} = \frac{\sqrt{N}}{\sqrt{2}(1+|\xi|)^{1/4}}\,, \quad (12)$$

where $\xi = g\beta N/(2\gamma)$. The equation (16) of Ref. [5] deals with the same system as ours, and has nearly identical form as our equations (10) and (11) but with two slightly different coefficients $E_C = 2g\beta(1 + 0.5|\xi|^{-1})$ and $E_2 = 0.5g\beta/|\xi|$ in our notation. It is instructive to note that the atom number fluctuation $\Delta n \propto N^{1/4}$ in the weak tunnelling regime ($|\xi| \gg 1$) just as is given by Leggett and Sols,[8] whereas in the strong tunnelling regime ($|\xi| \ll 1$), the atom number fluctuation $\Delta n \propto \sqrt{N}$ just as is obtained by Javanainen and Wilkins.[7,11] Therefore the problem quarrelled by them[8,11] is naturally settled.

The previous studies on the atom number statistics for the double-well system have failed to notice the multistability and the phase transition in this system when the parameter $|\xi|$ goes across the critical parameter ξ_c.

Consequently, no one has so far discussed the atom number and phase statistics around the stable fixed points $S_\pm$ when $|\xi| > \xi_c$, which will be the subject of the present paragraph. The critical parameter ξ_c is unity in the case where $\delta = 0$ denoting identical ground energy for the two wells. In this case, we consider fluctuations around the stable fixed points $S_\pm$ whose concrete expression is given after Eq. (9), and obtain the coefficients in Eq. (10) as follows:

$$E_1 = \pm 0.5 g\beta N\sqrt{1-\xi^{-2}}, \qquad E_2 = -g\beta\xi^2\,, \tag{13a}$$

$$E_J = -\frac{g\beta N^2}{2\xi^2}, \qquad E_C = -2g\beta(\xi^2 - 1)\,. \tag{13b}$$

Fluctuations of the atom number in the left well and the phase difference in this case are given by

$$\Delta n = \frac{1}{\Delta\phi} = \frac{\sqrt{N}}{\sqrt{2|\xi|}(\xi^2-1)^{1/4}}\,, \tag{14}$$

where $|\xi| = |g\beta N/(2\gamma)| > 1$. In the weak tunnelling regime ($|\xi| \gg 1$), the atom number fluctuation $\delta n \approx \sqrt{0.5/N}$ is very small in large-N circumstances, which demonstrates sub-Poissonian fluctuations and the atoms in any one of the wells can be thought to be approximately in a Fock state. Another interesting phenomenon is that the atom numbers in both the wells display strong fluctuations when the parameter $|\xi| \equiv |g\beta N/(2\gamma)|$ approaches one from above. The atom number fluctuations have the form $\Delta n \approx \sqrt{N/2}[2(|\xi|-1)]^{-1/4}$ as $|\xi| \to 1$. This form demonstrates the typical strong critical fluctuation phenomena in phase transitions.[23,24] The corresponding "order" parameter and critical index for atom number fluctuation in our case are $|\xi|$ and 1/4 respectively.[23,24] However, it should be emphasized that the critical fluctuation phenomenon corresponds in fact to squeezed states for the phase difference ϕ since the phase difference fluctuation $\Delta\phi \approx \sqrt{2/N}[2(|\xi|-1)]^{1/4} \to 0$ as the "order" parameter ξ approaches unity from above.

The multistability and phase transition as well as the critical fluctuation phenomenon (i.e. phase squeezed states) in the macroscopic tunnelling in a double-well trap containing a Bose–Einstein condensate (BEC) are well within the reach of nowadays BEC-related technology. As a matter of fact, the recent experiment by Kasevich's group[1] has already reached the strong tunneling regime and the parameter $\xi = g\beta N/(2\gamma)$ in that experiment can at least reach as low as 1.5 as given in the caption of its Fig. 1D.[1] We therefore believe that one should be able to discover the new phenomena investigated here with the same apparatus as in that experiment.

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