Vol. 37 No. 5 CHINESE SCIENCE BULLETIN March 1992

THE DIFFERENTIAL REALIZATION OF NEW SOLUTIONS FOR YANG-BAXTER EQUATION*

SUN CHANG-PU (孙昌璞)

(*Department of Physics, Northeast Normal University, Changchun* 130024, *PRC*; *Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin* 300071, *PRC***)

Received December 27, 1990.

Keywords: Yang-Baxter equation, quantum algebra, indecomposable representation.

I. INTRODUCTION

Yang-Baxter equation plays a crucial role in the nonlinear integrable system[1]. Its standard solutions can be constructed in terms of the universal R-matrix and representations of the corresponding quantum universal enveloping algebra (or called quantum algebra for short)[2], while its non-standard solutions can be obtained by using the extended Kauffman's diagram technique (EKDT)[3]. The relation between the two kinds of solutions has been discussed in Ref. [4].

In this note, using the q-differential realization

$$J_+=(1/[Z]_q)Z^2,\ J_-=(-1/[Z]_q)\frac{1}{Z^2}[Z\frac{d}{dZ}]_q,\ J_3=\frac{1}{2}+Z\frac{d}{dZ} \quad (1)$$

of the quantum algebra $sl_q(2)$ on the Bargmann space[5], which is obtained in comparison to its q-boson (q-oscillator) realization[6-9], we construct the indecomposable representations of $sl_q(2)$ for q is a root of unity, i. e. $q^p=1$ and $p=3, 4, 5\cdots$. Here, we define $[f]_q=(q^f-q^{-f})/(q-q^{-1})$ for any number or operator f. Then, we obtain some new solutions for the Yang-Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} \quad (2)$$

from these new representations. These new solutions are different from both the standard solutions and the non-standard ones.

II. NEW REPRESENTATIONS OF $sl_q(2)$

On an $sl_q(2)$-invariant subspace B^+ of the Bargmann space,

$$\{\ f(m)=Z^{2m}|m=0,1,2,\cdots,\ Z\in\mathbb{C}\}\ ,$$

the realization (1) defines an infinite dimensional representation of $sl_q(2)$

---

* Project supported by the National Natural Science Foundation for Young Scientific Workers of China.

** Mailing address.

$$\begin{cases} \rho(J_+)f(m)=\dfrac{1}{[2]_q}f(m+1), \\ \rho(J_-)f(m)=\dfrac{-1}{[2]_q}[2m]_q[2m-1]_q f(m-1), \\ \rho(J_q)f(m)=(2m+\dfrac{1}{2})f(m). \end{cases} \tag{3}$$

When q is not a root of unity, this representation is irreducible.

However, for the case that q is a root of unity, $q^p=1$, $[\alpha p]=0(\alpha=0,\ \pm1,\pm2,\cdots)$, there exists an extreme vector $f_\alpha=f(\frac{1}{2}(\alpha p+\sigma(\alpha p)))$, such that $J_- f_\alpha=0$. This extreme vector defines an invariant subspace $V_{\alpha p}$:

$$\{ f(m) \mid m\geqslant \frac{1}{2}(\alpha p+\sigma(\alpha p)) \},$$

where $\sigma(x)=\frac{1}{2}(1-(-1)^x)$.

For $J=\frac{1}{4}(\alpha p+\sigma(\alpha p)-2)$, the quotient space $Q_{\alpha\cdot p}=B^+/V_{\alpha p}$:

$$\{ \mid J, M\rangle=f(J+M) \bmod V_{\alpha p} \mid M=J, J-1, \cdots, -J \}$$

is $\frac{1}{2}(\alpha p+\sigma(\alpha p)=2J+1$ dimensional. On $Q_{\alpha,p}$, representation (3) induces a finite dimensional representation Γ:

$$\begin{cases} \Gamma(J_+)|J, M\rangle=\dfrac{1}{[2]_q}|J, M+1\rangle, \quad M\leqslant J-1; \\ \Gamma(J_+)|J, J\rangle=0, \\ \Gamma(J_-)|J, M\rangle=-[2]_q^{-1}[2(J+M)]_q[2(J+M)-1]_q|J, M-1\rangle, \\ \Gamma(J_3)|J, M\rangle=(2(J+M)+\dfrac{I}{2})|J, M\rangle. \end{cases} \tag{4}$$

It is easy to prove that representation (4) is indecomposable (there exists the similar circumstance for the Lie algebra[10]) when $\alpha\geqslant2$.

III. New R-matrix for Yang-Baxter Equation

In terms of the differential realization (1) of $sl_q(2)$, the universal R-matrix for $sl_q(2)$

$$\mathscr{R}=q^{J_3\otimes J_3}\sum_{n=0}^{\infty}\frac{(1-q^4)^n}{[n]_{q^2}}(q^{J_3}\cdot J_+\otimes q^{-J_3}\cdot J_-)^n \tag{5}$$

can be expressed in the differential operator form

$$\widetilde{\mathscr{R}}=\exp\{\ln q\cdot(Z_1\frac{d}{dZ_1}+\frac{1}{2})(Z_2\frac{d}{dZ_2}+\frac{1}{2})\}\cdot\sum_{n=0}^{\infty}(\frac{q^{-1}-q}{q+q^{-1}})^n\cdot$$

$$q^{n(3n-1)}([n]_{q^2}!)^{-1}(Z_1/Z_2)^{2n}\cdot q^{nZ_1\frac{d}{dZ_1}}[Z_2\frac{d}{dZ_2}]_q^{2n}\cdot q^{nZ_2\frac{d}{dZ_2}}\cdot \tag{6}$$

Through representation (4), the matrix representation of $\widetilde{\mathscr{R}}$ on the product space $Q_{\alpha p}\otimes Q_{\alpha p}$

is defined by

$$(R^J)_{M_1 M_2}^{M_1' M_2'} = q^{\{2(M_1+J)+\frac{1}{2}\}\{2(M_2+J)+\frac{1}{2}\}} \cdot \delta_{M_1}^{M_1'} \delta_{M_2}^{M_2'}$$

$$+ \sum_{n=1}^{2J} \{ (\frac{q^{-1}-q}{q^{-1}+q})^n \frac{1}{[n]_{q^2}!} \cdot q^{2\{\sum_{i=1}^{n}(M_1-M_2+Zi)-n\}} \prod_{l=1}^{n} [2(M_2+J)$$

$$-l+1]_q [2(M_2+J-l)+1]_q \cdot \delta_{M_1-n}^{M_1'} \delta_{M_2-n}^{M_2'} \}. \quad (7)$$

The general theory of quantum group ensures that the R-matrix R^J given by (7) satisfies the Yang-Baxter equation (1). In fact, we can use the EKDT to check the conclusion for each representation. For example, when $\alpha=2$ and $p=3$, the 3D representation of $sl_q(2)$ gives a 9×9 R-matrix

$$R^1 = q^{5/4} \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}, \ (q^3=1)$$

on $Q_{23} \otimes Q_{23}$, where

$$A_1 = \begin{bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{bmatrix}, \quad A_2 = \begin{bmatrix} q & q^{-1}-q & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & q \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 1 & q^{-1}-q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{bmatrix}.$$

IV. Discussion

1. The Yang-Baxter equation discussed in this note has not the spectral parameter and is usually called braid group relation. The solutions for the Yang-Baxter equation with spectral parameter that is useful for physical models can be obtained from the R-matrices achieved in this note by the Yang-Baxterization scheme built by M. L. Ge et al. [3]

2. The R-matrices in this note are non-generic because they cannot be obtained either the standard solutions or the non-standard ones by letting $q^p=1$.

3. Other new solutions on $Q_{\alpha,p} \otimes Q_{\alpha',p}$ with different $Q_{\alpha,p}$ and $Q_{\alpha',p}$ can also be given by the differential realization (7) of the R-matrix.

4. In fact, if we substitute the indecomposable representation

$$\rho = \left[\begin{array}{c|c} \rho^{[\lambda]} & A \\ \hline & B \end{array} \right] \qquad A, B \neq \mathbb{O}$$

of the quantum algebra $G_q: \{e_a, e^a\}$ into the universal R-matrix $\mathscr{R} = \sum_a e_a \otimes e^a$, the obtained R-matrix will possess the following embedding structure

$$R = \rho \otimes \rho(\mathscr{R}) = \sum_a \rho(e_a) \otimes \rho(e^a) = \left[\begin{array}{c|c} R^\lambda & C \\ \hline O & D \end{array} \right], \quad C, D \neq \mathbb{O},$$

where R^λ is an R-matrix defined by the subrepresentation $\rho^{[\lambda]}$ and still satisfies a Yang-Baxter equation. Such a kind of solutions are indeed new.

The author thanks Profs. M. L. Ge, Z. Y. Wu and Drs. K. Xue and X. F. Liu for the useful discussion.

REFERENCES

[1] Yang, C. N., *Phys. Rev. Lett.*, **19**(1967), 1312;
Baxter, R. J., *Exactly Sovable Model in Statistical Mechanics*, Academic Press, 1982;
Yang, C. N. & Ge, M. L. (Eds.), *Braid Group, Knot Theory and Statisticval Mechanics*, World Scientific, 1989.

[2] Drinfeld, V. G., *Proc. IMC*, Berkely, 1986, p. 789;
Jimbo, M., *Lett. Math. Phys.*, **10**(1985), 63; **11**(1986), 247.
Reshetikhin, N. Y., *LOMI*, preprint, E4 & E8, 1987.
Majid, S., *Inter. J. Mod. Phys.*, **A5**(1990), 1.

[3] Kauffman, L. H., *Ann. Math. Studies*, **115**(1988), 1;
Ge, M. L. et al., *Inter. J. Mod. Phys.*, **A4**(1989), 3551; *J. Phys.*, *A*, **23**(1990), 605 & 619.

[4] Sun, C. P. et al., *J. Phys. A*, **23**(1990), L645.

[5] Bargmann, V., *Commun, Appl. Phys.*, **14**(1961), 187.

[6] Sun, C. P. & Fu, H. C., *J. Phys. A*, **22**(1989), L 983.
Biedenharn, L. C., *ibid.*, **22**(1989), L873;
Macfarlane, A. J., *ibid.*, **22**(1989), 4551.

[7] Sun, C. P. & Fu, H. C., *Commun. Theor. Phys.*, **13**(1990), 217;
Sun, C. P. & Ge, M. L., *J. Math. Phys.*, **33**(1991), 595.

[8] Kulish, P & Damaskvski, E, *J. Phys. A*, **23**(1990), L415.

[9] Yan, H., *ibid.*, **23**(1990), L1155.

[10] Sun, C. P., *ibid.*, **20**(1987), 4551; 5823; L1157.
Sun, C. P., & Fu, H. C., *IL NUOVO CIMENTO*, **115B**(1990), 1.