Quantum Group Construction of Non-standard R-matrix for Yang-Baxter Equation

1991 Chinese Phys. Lett. 8 495

(http://iopscience.iop.org/0256-307X/8/10/001)

CHINESE PHYS. LETT. Vol.8, No.10(1991)495

Quantum Group Construction of Non-standard R-matrix for Yang-Baxter Equation*

GE Molin, SUN Changpu†, XUE Kang†

Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071

† *also Department of Physics, Northeast Normal University, Changchun 130024*

(Received 25 March 1991)

By taking the concept of weight conservation into account, new solutions of Yang-Baxter equation without spectral parameter are obtained through the non-standard R-matrix on the tensorial space $V^{1/2}\otimes V^j$ $(j \neq 1/2)$ in terms of the irreducible representation of the quantum universal enveloping algebra SL_q (2). The non-standard R-matrix $R^{1/2\ 1}$ is obtained in an explicit form for $j = 1$.

PACS: 02.20.+b, 05.50.+q

At present it is recognized that the Yang-Baxter equation(YBE) plays a crucial role in non-linear physics, such as exactly-solvable models of statistical mechanics, quantum inverse scattering method and so on.[1] According to Drinfeld[2] and Jimbo,[3] a typical scheme to obtain the solution of YBE is figured through an example of SU(2) as follows.

Consider the quantum universal enveloping algebra (QUEA) $SL_q(2)$, which is an associative algebra over the complex number field **C** and generated by J_+, J_- and J_3, satisfying

$$[J_+, J_-] = [J_3], \quad [J_3, J_\pm] = \pm 2J_\pm \ . \tag{1}$$

The universal R-matrix $\mathcal{R}(\in SL_q(2)\otimes SL_q(2))$ can be constructed in terms of $SL_q(2)$

$$\mathcal{R} = q^{J_3\otimes J_3/2}\sum_{n=0}^{\infty}\frac{(1-q^{-2})^n}{[n]!}\cdot q^{\frac{1}{2}n(n-1)}(q^{\frac{J_3}{2}}J_+\otimes q^{\frac{-J_3}{2}}J_-)^n \ , \tag{2}$$

where we have defined that $[f] = \dfrac{g^f - g^{-f}}{q-q^{-1}}$ and $[n]! = [n][n-1]\ \dots\ [1]$. For given irreducible representations $\rho^{[j]}$ of $SL_q(2)$ on the spaces $V^{[j]}$, the j-j' R-matrix $R^{j\ j'}\in \mathrm{End}\,(V^{[j]}\otimes V^{[j']})$ is given from Eq. (2) as

$$\overline{R}^{j\ j'} = \rho^{[j]}\otimes\rho^{[j']}(\mathcal{R}) \ . \tag{3}$$

The Hopf algebraic structure of $SL_q(2)$ ensures $R^{j\ j'}$ to satisfy the YBE without spectral parameter:[4]

$$R^{j_1\ j_2}_{1\ \ 2}\ R^{j_1\ j_3}_{1\ \ 3}\ R^{j_2\ j_3}_{2\ \ 3} = R^{j_2\ j_3}_{2\ \ 3}\ R^{j_1\ j_3}_{1\ \ 3}\ R^{j_1\ j_2}_{1\ \ 2} \tag{4}$$

and the condition of "CP-invariance":

$$\left(R^{j\ j'}\right)^{m_1'\ m_2'}_{m_1\ m_2} = \left(R^{j'\ j}\right)^{-m_2'\ -m_1'}_{-m_2\ -m_1} \ , \tag{5}$$

*Supported in part by the National Natural Science Foundation of China.

where $R^{j\,j'}_{i\;i'}$ is defined on $V^{[j_1]} \otimes V^{[j_2]} \otimes V^{[j_3]}$ as

$$R^{j_1\;j_2}_{1\;\;2} = \sum \left(R^{j_1\;j_2}\right)^{m'_1\;m'_2}_{m_1\;m_2} E_{m_1 m'_1} \otimes E_{m_2 m'_2} \otimes I_3 \ldots\ldots \tag{6}$$

The j-j' R-matrix constructed from Eqs. (2) and (3) in terms of a QUEA is called standard j-j' R-matrix. Through so-called Yang-Baxterization,[5,6] one obtain the solution of YBE

$$R^{j_1\;j_2}_{1\;\;2}(u)\; R^{j_1\;j_3}_{1\;\;3}(u+v)\; R^{j_2\;j_3}_{2\;\;3}(v) = R^{j_2\;j_3}_{2\;\;3}(v)\; R^{j_1\;j_3}_{1\;\;3}(u+v)\; R^{j_1\;j_2}_{1\;\;2}(u)\,. \tag{7}$$

It is worth notice that the Kauffman's diagram technique(KDT)[7] to directly obtain j-R matrix $R^j \equiv R^{j\,j'}(j'=j)$ has been extended to the case of arbitrary classical Lie algebra and non-standard j-R matrix or non-standard braid group representations obtained by this extended KDT can not be covered by standard ones.[8]

Now, a question rises naturally: Does there exist a non-standard $R^{j\,j'}$ matrix when $j \neq j'$? We recall that the concept of weight conservation is a key point for calculation of our extended KDT and proved to be satisfied by the quantum group constructed $R^{j\,j}$ matrix.[9] Non-standard $R^{j\,j}$ matrices besides standard ones have been constructed in terms of quantum group or QUED in Ref. 9. Since $R^{j\,j}$ matrix is a special case of $R^{j\,j'}$ matrix when $j = j'$, we naturally hope to construct $R^{j\,j'}$ matrix $(j \neq j')$ through quantum group or QUED.

By a proof similar to that for $R^{j\,j}$ matrix, we easily observe that the standard $R^{j\,j'}$ matrix $(j \neq j')$ still satisfies weight conservation. Now, we consider the case that j or $j' = 1/2$. It follows from Eq. (4) that the $R^{j\;1/2}$ matrix and the $R^{1/2\;j}$ matrix satisfy

$$\left\{\begin{array}{lll} R^{1/2\;1/2}_{1\;\;\;2}\, R^{1/2\;j}_{1\;\;\;3}\, R^{1/2\;j}_{2\;\;\;3} & = & R^{1/2\;j}_{2\;\;\;3}\, R^{1/2\;j}_{1\;\;\;3}\, R^{1/2\;1/2}_{1\;\;\;2}\,, \quad (8-\mathrm{a}) \\ R^{1/2\;j}_{1\;\;\;2}\, R^{1/2\;1/2}_{1\;\;\;3}\, R^{j\;1/2}_{2\;3} & = & R^{j\;1/2}_{2\;3}\, R^{1/2\;1/2}_{1\;\;\;3}\, R^{1/2\;j}_{1\;\;\;2}\,, \quad (8-\mathrm{b}) \\ R^{j\;1/2}_{1\;2}\, R^{j\;1/2}_{1\;3}\, R^{1/2\;1/2}_{2\;\;\;3} & = & R^{1/2\;1/2}_{2\;\;\;3}\, R^{j\;1/2}_{1\;3}\, R^{j\;1/2}_{1\;2}\,. \quad (8-\mathrm{c}) \end{array}\right.$$

The "CP-invariance" requires that Eqs. (8-a) and (8-c) are equivalent. Let

$$R^{1/2\;j} = \overline{R}^{1/2\;j} + \Delta^{1/2\;j}\,, \quad R^{j\;1/2} = \overline{R}^{j\;1/2} + \Delta^{j\;1/2}\,, \quad R^{1/2\;1/2} = \overline{R}^{1/2\;1/2}$$

be a solution of Eq. (4) where $\overline{R}^{1/2\;j}$, $\overline{R}^{j\;1/2}$ and $\overline{R}^{1/2\;1/2}$ are standard. Then,

$$\left\{\begin{array}{l} \overline{R}^{1/2\;1/2}_{1\;\;\;2}\left(\Delta^{1/2\;j}_{1\;\;\;3}\,\overline{R}^{1/2\;j}_{2\;\;\;3} + \overline{R}^{1/2\;j}_{1\;\;\;3}\,\Delta^{1/2\;j}_{2\;\;\;3} + \Delta^{1/2\;j}_{1\;\;\;3}\,\Delta^{1/2\;j}_{2\;\;\;3}\right) = \\ \quad\left(\Delta^{1/2\;j}_{2\;\;\;3}\,\Delta^{1/2\;j}_{1\;\;\;3} + \Delta^{1/2\;j}_{2\;\;\;3}\,\overline{R}^{1/2\;j}_{1\;\;\;3} + \overline{R}^{1/2\;j}_{2\;\;\;3}\,\Delta^{1/2\;j}_{1\;\;\;3}\right)\overline{R}^{1/2\;1/2}_{1\;\;\;2}\,, \\ \\ \Delta^{1/2\;j}_{1\;\;\;2}\,\overline{R}^{1/2\;1/2}_{1\;\;\;3}\,\overline{R}^{j\;1/2}_{2\;3} + \overline{R}^{1/2\;j}_{1\;\;\;2}\,\overline{R}^{1/2\;1/2}_{1\;\;\;3}\,\Delta^{j\;1/2}_{2\;3} + \Delta^{1/2\;j}_{1\;\;\;2}\,\overline{R}^{1/2\;1/2}_{1\;\;\;3}\,\Delta^{j\;1/2}_{2\;3} = \\ \quad\Delta^{j\;1/2}_{2\;3}\,\overline{R}^{1/2\;1/2}_{1\;\;\;3}\,\overline{R}^{1/2\;j}_{1\;\;\;2} + \overline{R}^{j\;1/2}_{2\;3}\,\overline{R}^{1/2\;1/2}_{1\;\;\;3}\,\Delta^{1/2\;j}_{1\;\;\;2} + \Delta^{j\;1/2}_{2\;3}\,\overline{R}^{1/2\;1/2}_{1\;\;\;3}\,\Delta^{1/2\;j}_{1\;\;\;2}\,. \end{array}\right. \tag{9}$$

Using the explicit matrix elements of the irreducible representation $\rho^{[j]}$ of $SL_q(2)$[10]

$$\left(\rho^{[j]}(J_\pm)\right)^{m'}_{m} = \left([j \mp m][j \pm m + 1]\right)^{1/2} \delta^{m'}_{m\pm 1}\,, \quad \left(\rho^{[j]}(J_3)\right)^{m'}_{m} = 2m'\delta^{m'}_{m}\,, \tag{10}$$

the standard $R^{1/2\ j}$ matrix and $\Delta^{1/2\ j}$ are explicitly written as

$$\begin{aligned}\overline{R}^{1/2\ j} &= q^{1/2\ \rho^{[1/2]}(J_3)\otimes\rho^{[j]}(J_3)}\sum_{n=0}^{1}(1-q^{-2})^n\left\{q^{\rho^{[\frac{1}{2}]}(J_3)}\cdot\rho^{[1/2]}(J_+)\right.\\ &\left.\otimes q^{-\rho^{[1/2]}(J_3)}\cdot\rho^{[j]}(J_-)\right\}^n\cdot q^{1/2n(n-1)}\ ,\end{aligned} \tag{11}$$

and

$$\begin{aligned}\Delta^{1/2\ j} &= \sum_{k=0}^{2j}A_k\rho^{[1/2]}(J_-)\rho^{[1/2]}(J_+)\otimes\left(\rho^{[j]}(J_-)\right)^k\left(\rho^{[j]}(J_+)\right)^k\\ &+\sum_{k=1}^{2j}B_k\rho^{[1/2]}(J_+)\otimes\left(\rho^{[j]}(J_+)\right)^{k-1}\left(\rho^{[j]}(J_-)\right)^k\\ &+\sum_{k=1}^{2j}C_k\rho^{[1/2]}(J_-)\otimes\left(\rho^{[j]}(J_+)\right)^k\left(\rho^{[j]}(J_-)\right)^{k-1}\ ,\end{aligned} \tag{12}$$

respectively. Equation (9) determines the coefficients A_k, B_k and C_k $(k=1,2,...,2j)$.

For example, when $j=1$,

$$\begin{aligned}(\Delta^{1/2\ 1})^{m_1'\ m_2'}_{m_1\ m_2} &= \delta^{m_1'}_{-1/2}\delta^{m_1}_{-1/2}\left(A_0\delta^{m_2'}_1\delta^{m_2}_1+(A_0+[2]A_1)\delta^{m_2'}_0\delta^{m_2}_0+(A_0+[2]A_1\right.\\ &\left.+[2]^2A_2)\ \delta^{m_2'}_{-1}\delta^{m_2}_{-1}\right)+\delta^{m_1'}_{1/2}\delta^{m_1}_{-1/2}\left((B_1+[2]B_2)[2]^{1/2}\delta^{m_2'}_0\delta^{m_2}_1\right.\\ &\left.+[2]^{1/2}B_1\delta^{m_2'}_{-1}\delta^{m_2}_0\right)+\delta^{m_1'}_{-1/2}\delta^{m_1}_{1/2}\left((C_1+[2]C_2)[2]^{1/2}\delta^{m_2'}_1\delta^{m_2}_0\right.\\ &\left.+[2]^{1/2}C_1\delta^{m_2'}_0\delta^{m_2}_{-1}\right)\ ,\end{aligned} \tag{13}$$

where the coefficients are determined to be two sets:

$$\begin{aligned}&A_0=q^{-2}(Q-q)\,,\quad A_1=[2]^{-1}(q^{-1}-q^{-2})(Q-q)\,,\quad A_2=[2]^{-2}(1-q^{-1})(Q-q)\,,\\ &C_1=C_2=0\,,\quad B_1=[2]^{-1/2}\left(q_2-\{(1-q^{-2})(q^2-q^{-2})\}^{1/2}\right)\,,\quad B_2=q_1-q_2\,,\\ &q_1q_2=q^{-1}Q(1-q^{-2})(q^2-q^{-2})\,,\end{aligned} \tag{14}$$

and

$$\begin{aligned}&A_0=\tilde{Q}-q^{-1}\,,\quad A_1=[2]^{-1}(q-1)(\tilde{Q}-q^{-1})^1\,,\quad A_2=(1-q)(\tilde{Q}+1)\,,\\ &B_1=[2]^{-1/2}\{(1-q^{-2})(q^2-q^{-2})\}^{1/2}\,,\quad B_2=[2]^{-1}\tilde{q}_1\,,\\ &C_1=[2]^{-1/2}\tilde{q}_2\,,\quad C_2=-[2]^{3/2}\tilde{q}_2\,,\quad \tilde{q}_2=-q\tilde{q}_1\,,\end{aligned} \tag{15}$$

where $Q, q_1, \tilde{Q}$ and $\tilde{q}_1$ are arbitrary complex parameters. Equations (14) and (15) define two non-standard $R^{\frac{1}{2}\ j}$ matrices respectively

$$R^{1/2\ 1}(I)=\begin{bmatrix} q & & & & & \\ & 1 & q_1 & & & \\ & 0 & q^{-2}Q & & & \\ & & & q^{-1}, & q_2 & \\ & & & 0 & q^{-1}Q & \\ & & & & & Q \end{bmatrix}, \tag{16}$$

$$R^{1/2\ 1}(II)=\begin{bmatrix} q & & & & & \\ & 1 & \tilde{q}_1 & & & \\ & 0 & \tilde{Q} & & & \\ & & & q & 0 & \\ & & & \tilde{q}_2 & q\tilde{Q} & \\ & & & & & Q \end{bmatrix}. \tag{17}$$

When

$$Q=q,\ q_1=q_2=\left\{(1-q^{-2})(q^2-q^{-2})\right\}^{1/2}, \tag{18}$$

non-standard $R^{1/2\ 1}$ matrix $R^{1/2\ 1}(\mathrm{I})$ is reduced to the standard one $\overline{R}^{1/2\ 1}$ and thus $R^{1/2\ 1}(\mathrm{I})$ can be regarded as a three-parameter deformation of the standard R-matrix.

Finally, it is pointed out that there is no braid group representation corresponding to the $R^{j\ j'}$ matrix when $j \neq j'$ and the non-standard $R^{j\ j'}$ matrices can also be Yang-Baxterized by our general scheme.[6]

References

[1] C. N. Yang, Phys. Rev. Lett. 19 (1967) 1312; R. J. Baxter, *Exactly Solvable Models in Statistical Mechanics* (Academic Press, London, 1982); C. N. Yang and M. L. Ge, (ed.) *Braid Group, Knot Theory and Statistical Mechanics* (World Scientific, Singapore, 1989).

[2] V. G. Drinfeld, Proc. IMC., Berkely: 1986, p. 798.

[3] M. Jimbo, Lett. Math. Phys. 10 (1985) 63; 11 (1986) 247; Commun. Math. Phys. 102 (1986) 537.

[4] N. Yu. Reshetikhin, Preprint LOMI E-9-88, 1988.

[5] V. F. Jones, Inter. J. Mod. Phys. B 4 (1990) 701.

[6] M. L. Ge, L. H. Gwa, F. Piao and K. Xue, J. Phys. A 23 (1990) 2273; M. L. Ge, Y. S. Wu and K. Xue, Inter. J. Mod. Phys. 6A (1991) (in press).

[7] L. H. Kauffman, Ann. Math. Studies 115 (1988) 1.

[8] M. L. Ge, L. Y. Wang, K. Xue and Y. S. Wu, Inter. J. Mod. Phys. A 4 (1989) 3351; M. L. Ge, L. Y. Wang, Y. Q. Li and K. Xue, J. Phys. A 23 (1990) 605; M. L. Ge, Y. Q. Li and K. Xue, J. Phys. A 23 (1990) 619; M. L. Ge, L. Y. Wang and X. P. Kong, J. Phys. A 24 (1990) 569; M. L. Ge, L. H. Gwa and K. Xue, J. Phys. A 23 (1990) 2273.

[9] M. L. Ge, C. P. Sun, L. Y. Wang and K. Xue, J. Phys. A 23 (1990) L 645.

[10] C. P. Sun and H. C. Fu, J. Phys. A 22 (1989) L 983.