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Controllable band loops of ultracold atoms in a cavity

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Abstract. We theoretically investigate the atomic energy band of ultracold atoms inside a laser-driven optical cavity with Kerr medium. This cavity-atom hybrid system has two kinds of interactions: photon-atom interaction and photon-photon one. We find the loops of the atomic energy band induced by photon-atom interaction disappear when the Kerr interaction between photons exceeds a critical value. The reason of the controllably atomic energy band loops by Kerr interaction is discussed.

1 Introduction

The ultracold atoms or Bose-Einstein condensate (BEC) in a driven ultrahigh-finesse optical cavity is a nice example of nonlinear system. One kind of nonlinearity arises from the dispersive atom-light interaction, when the cavity resonance is far detuned from the atomic resonance. This atom-photon interaction imprints a position-dependent phase shift on the cavity field. In turn, this shift can affect the mechanical motion of atoms. This nonlocal interaction is quite different from the usual local atom-atom interaction. A lot of interesting results such as self-organization of atoms [1–3], quantum phase transition of Dicke model [4–7] and Bose-Hubbard model [8], adiabatic geometric phase [9], cavity-enhanced superradiant Rayleigh scattering [10], and optical bistability [11–17] and loops of (atomic) band structure have been reported [18].

The energy band of Bose-Einstein condensation in a optical lattices displays the swallowtail loop [19], when the strength of the atomic interactions is above a critical value. The loop implies multiple solutions of the atomic wave function within a single band. They can occur either near the center or the boundaries of the Brillouin zone [20,21]. For ultracold atoms in an optical cavity, the loops and bistability are two sides of the same coin, one being for the atoms and the other for the light. As the loops manifest themselves physically via a dynamical instability which destroys the superflow [22], it is an interesting question whether one can achieve the controllable loops by making use of the photon-photon interactions in the cavity. As we shall see, the answer is affirmative.

In this paper, we consider ultracold atoms in a pumped optical cavity and the cavity are filled with the additional Kerr medium, which gives rise to a strong nonlinear interaction between photons. With Kerr interaction, the photon in a cavity can block the injection of the other photon due to photon blockade effect [23]. We show here the loops of atomic energy band can be controlled by Kerr interaction. With weak Kerr interaction, the energy band of BEC shows clear loops and the intracavity photon number shows bistability. Above a critical Kerr interaction, the loops and bistablity behaviors disappear. Therefore, one can realize the controllable loops of energy band conditioned on the Kerr interaction.

2 The model of system

We consider N two-level ultracold atoms with mass m inside a high-Q optical cavity. The atoms is tightly confined in the transverse direction, such that the transverse size of the condensate is smaller than the waist of the cavity field. Thus, we only consider the dynamics of the hybrid atom-photon system along the cavity x axis. In the large detuning limit and in the rotating frame with the pump frequency, the Hamiltonian of the total system is ($\hbar = 1$) [24]

$$H = H_a + H_c + H_d. \quad (1)$$

The corresponding Hamiltonian for the condensate atoms and atom-cavity interaction can be written as

$$H_a = \int dx\ \phi^\dagger(x) \left[-\frac{1}{2m}\frac{d^2}{dx^2} + V_{ext} + U_0 \cos^2(kx) a^\dagger a \right] \phi(x), \quad (2)$$

that for the cavity field and its damping are

$$H_c = \varDelta_c a^\dagger a + i\eta(a^\dagger - a) + \frac{1}{2}\chi a^{\dagger 2} a^2, H_d = -i\kappa a^\dagger a. \quad (3)$$

In the case of weak atom-atom interactions and a shallow external trapping potential V_{ext}, we have set $V_{ext} = 0$

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and neglected the atom-atom interactions [25]. Here $\phi^\dagger(x)$ is the creation operator of atoms, and $a^\dagger$ is that of cavity photons with frequency ω_c, wave vector $k = \frac{2\pi}{\lambda}$ and mode function $\cos(kx)$. The maximum light shift which an atom experiences in the cavity mode is given by $U_0 = g_0^2/\Delta_a$ with the single atom-photon coupling constant g_0. The pump laser with strength η and frequency ω_p is detuned from the empty cavity resonance frequency ω_c, namely, $\Delta_c = \omega_c - \omega_p$. κ denotes the cavity decay. Giant optical Kerr nonlinearities are obtained by placing a $\xi^{(3)}$ medium inside a cavity [26], with $\chi = 3\omega_c^2 \mathrm{Re}[\xi^{(3)}]/2\epsilon_0 V_c$, ϵ_0 being the dielectric constant of the medium, V_c being the volume of the cavity, and $\xi^{(3)}$ being the third-order nonlinear susceptibility.

Applying the mean-field approximation, the equations of motion for atoms with $\psi(x) = \langle\phi(x)\rangle/\sqrt{N}$ derived from Heisenberg operator equations reduce to

$$i\frac{d}{dt}\psi(x) = \left[-\frac{1}{2m}\frac{d^2}{d^2x} + U_0\cos^2(x)\alpha^\dagger\alpha\right]\psi(x), \quad (4)$$

and that for optical field $\alpha = \langle a\rangle$ is

$$i\frac{d}{dt}\alpha = \Delta_c\alpha + i\eta + \chi\alpha^\dagger\alpha^2 - i\kappa\alpha + NU_0\langle\cos^2(x)\rangle\alpha, \quad (5)$$

where the length is scaled by k, the energy by $E_R = k^2/2M$, the time by $1/E_R$, and $\langle\cos^2(x)\rangle \equiv \int dx\, \psi^*(x)\cos^2(x)\psi(x)$.

In the experiment of Brennecke et al. [24], $\kappa \sim$ MHz and $\omega_r \sim$ kHz, that the cavity decay is almost three orders of magnitudes faster than the motion of the condensate. So it is reasonable to assume that the cavity field follows the condensate adiabatically and the cavity field is solved as

$$n_{ph} = \alpha^\dagger\alpha = \frac{\eta^2}{\kappa^2 + (\Delta_c' + NU_0\langle\cos^2(x)\rangle)^2}, \quad (6)$$

with $\Delta_c' = \Delta_c + \chi n_{ph}$. Substituting this back into equation (4) gives us a nonlinear Schrödinger equation for the atomic motion:

$$i\frac{d}{dt}\psi(x) = \Big[-\frac{1}{2m}\frac{d^2}{dx^2} + \frac{U_0\eta^2}{\kappa^2 + (\Delta_c' + NU_0\langle\cos^2(x)\rangle)^2} \times \cos^2(x)\Big]\psi(x). \quad (7)$$

Notice the photon number in this equation is not a external parameter but must be determined self-consistently with equation (6). The physical reason is the interdependence between the atoms and the photon, deriving from the atom-light coupling. More specifically, the spatial distribution of atoms controls the additional phase shift of the light, and the depth of the optical lattice influencing the atomic wave function is determined by the amplitude of the light. Therefore, equations (6) and (7) must be solved self-consistently.

Formally, the equations of motion are very similar to the counterparts in reference [18], except for the additional Kerr term in the steady photon number in equation (6). Actually, they are very different as the photon number is partly determined by the Kerr term in equation (6). Taking the interaction between the photon and atoms into consideration, the Kerr interaction of photon will play a non-trivial role in the properties of the system.

Moreover, on the one hand, the nonlinear Schrödinger equation (7) could be derived as an equation of motion using Hamilton's (classical) equation,

$$i\frac{d}{dt}\psi(x) = \frac{\delta E[\psi(x), \psi(x)^*]}{\delta\psi(x)^*}, \quad (8)$$

with energy functional

$$E[\psi(x)] = \frac{N}{\pi}\int dx\, \left|\frac{d\psi(x)}{dx}\right|^2 - \frac{\eta^2}{\kappa}\arctan\left(\frac{\Delta_c' + NU_0\langle\cos^2(x)\rangle)^2}{\kappa}\right), \quad (9)$$

where the first term represents the kinetic energy. The second one denotes an atom-light coupling, that can be interpreted as the product of the phase of the steady-state cavity field and the incident photon current from the pump laser.

On the other hand, according to the Bloch theorem, the eigenfunction of equation (6) with the periodic potential $\cos^2(x)$ can be written as

$$\psi(x) = \exp(iqx)U_q(x), \quad (10)$$

with the quasimomentum q cofined in the first Brillouin zone $q \in [-1, 1)$ as the potential being periodic with period π, and $U_q(x + \pi) = U_q(x)$. Expanding the periodic function $U_q(x)$ to a Fourier series, the Bloch wavefunction can therefore be written as

$$\psi(x) = \exp(iqx)\sum_n a_{q,n}\exp(i2nx). \quad (11)$$

Substituting this Bloch wavefunction into equation (9), the resulting function are numerically extremized with respect to $a_{q,n}$. The normalization of $\psi_q(x)$ is maintained throughout. The parameters $a_{q,n}$ are taken to be real, and the Fourier series is terminated at $n = -M, \ldots, M$, determined by the convergence of the energy functional [20,21].

In the following, we will study the band structure of atoms, i.e., the relationship between the energy functional and quasimomentum. This requires the quasimomentum to be a good quantum number, which implies cold atoms are delocalized in many lattice sites and have long-range phase coherence. Experimentally, we can prepare these Bloch states making use of the similar method as in the free optical lattice case. A cigar-shaped BEC near the zero quasimomentum Bloch state can be created by slowly turn on an optical lattice along its longitudinal direction. Other Bloch states can be achieved by accelerating the lattice for a certain amount of time. These experimental techniques of adiabatic turning-on and accelerating optical lattices have been demonstrated successfully with either cold atoms [27,28] or BECs [29].

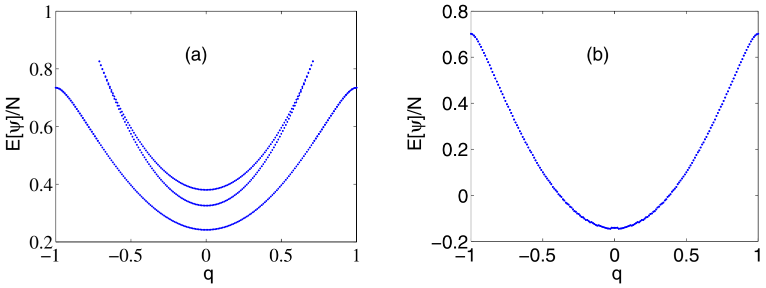

Fig. 1. (Color online) The energy functional $E[\psi(x)]$ in the first band as a function of the quasimomentum q. (a) and (b) correspond to Kerr parameter $\chi = 0$ and $\chi = 700 w_R$, respectively. Other parameters are $N = 10^4$, $U_0 = w_R$, and $\kappa = 350 w_R$, $\eta = 909.9 w_R$, $\Delta_c = 3140 w_R$ [18].

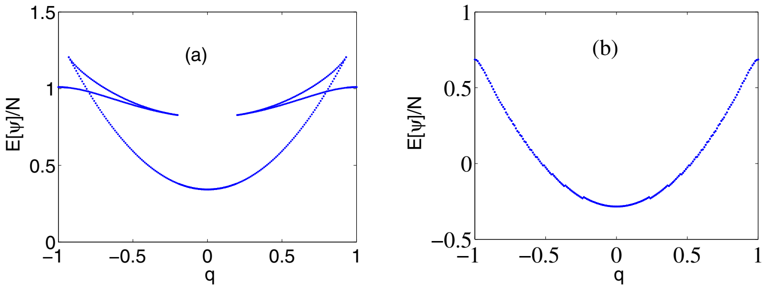

Fig. 2. (Color online) The energy functional $E[\psi(x)]$ in the first band as a function of the quasimomentum q. (a) and (b) correspond to Kerr parameter $\chi = 0$ and $\chi = 4000 w_R$, respectively. The other parameter is the same to Figure 1 except for $\eta = 2.8 \times 325 w_R$, $\Delta_c = 2100 w_R$.

3 Results and discussions

In previous section, we have discussed the model of the hybrid cavity-cold atoms system. In this section, we study the band structure of atoms numerically. The swallowtail loop of the band structure was observed in [19–21] for an interacting BEC in an free optical lattice. However, the nonlinearity induced loops in the two cases are very different. The nonlinearity arising from atom-atom interaction depends on the density of a BEC in an optical lattice, which is local usually, whereas the nonlinearity in the cavity-cold atom case is nonlocal usually. In a high-fineness cavity, the photons can bounce off so many times before it decays outside, which effectively induces the non-local interaction. In a word, the cavity is vital for this nonlocal interaction.

The loops in the atomic energy band can induce a hysteresis effect [22] when the quasimomentum is swept through the band, and a loss of adiabaticity even with infinitely slowly swept [30]. This scenario has been confirmed recently in experiment [31], and has some similarity with the hysteresis of the bistability of cavity photons [32], which has two different critical parameters corresponding to the negative or positive scanning.

When there is no Kerr interaction, the energy spectrum $E[\psi]$ as a function of quasimomentum q is shown in Figure 1a with the same parameters as in reference [18] to check the correctness of our codes. This figure implies that our codes are consistent with that of reference [18]. With the sufficient large Kerr parameter χ, the loop structure of the energy band disappears as displayed in Figure 1b. The single loop of Figure 1 appears near the center of the Brillouin zone. Is it possible to control the symmetry loops near the boundaries of the Brillouin zone? Figure 2 shows this point is affirmative. However, interestingly, a much larger Kerr parameter than the former case is needed. We will come back to this point latter. These results implies we can efficiently control the loop making use of Kerr interaction between cavity photons.

The upper results can be understood as follows. If χ is small, the main interaction of the system is the dispersive atom-light interaction, which can induce the loop of atomic energy band and optical bistability of cavity field. When χ is very big, the photon blockade induced by Kerr can drastically reduces the photon number, then coupling between atoms and the cavity field. It is reasonable to expect that there is no loop in the limit of the sufficiently large χ.

Specifically, for a limited Fourier series

$$\psi_1(x) = e^{iqx}[c_0(t) + c_1(t)e^{i2x} + c_1(t)e^{-i2x}], \qquad (12)$$

it is checked numerically that we can approximately obtain the same results as the larger M. For this state, it is easy to complete the integration $\langle \cos^2(kx) \rangle$ as

$$\begin{aligned}\langle \cos^2(kx) \rangle &= \frac{1}{2} + \frac{1}{2}(\mathrm{Re}(c_0 c_2^* + \mathrm{Re}(c_0 c_1^*)) \\ &= \frac{1}{2} + \frac{1}{2}(X(t) + Y(t)). \qquad (13)\end{aligned}$$

The equations of motion with $X(t)$ and $Y(t)$ can be derived as

$$\begin{aligned}&\ddot{X} + (4q+4)^2 X + (q+1)U_0 n_{ph}|c_0|^2 = 0, \\ &\ddot{Y} + (4q-4)^2 Y - (q-1)U_0 n_{ph}|c_0|^2 = 0, \\ &i\dot{a} = \Delta_c a + i\eta + \chi a^\dagger a^2 - i\kappa a \\ &\quad + NU_0 \left[\frac{1}{2} + \frac{1}{2}(X(t) + Y(t))\right] a. \qquad (14)\end{aligned}$$

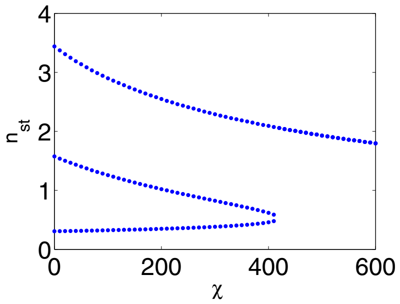

Fig. 3. (Color online) The photon number in the steady state n_{st} as a function of the Kerr parameter for the fixed quasimomentum $q = 0.5$. The other parameters is the same to Figure 1.

The steady state equation of photon with reduces to

$$n_{ph} = \langle a^\dagger a \rangle = \frac{\eta^2}{\kappa^2 + \left[\Delta_c'' + \frac{NU_0^2}{16(q^2-1)} n_{ph} + \chi n_{ph} \right]^2}, \quad (15)$$

with $\Delta_c'' = \Delta_c + \frac{NU_0}{2}$ and $|c_0|^2 \simeq 1$. This equation is a basic equation in the field of nonlinear optics [33], which can display the bistability or not depending on the parameters. And a standard linear stability analysis of equation (14) shows that only two solutions are dynamically stable.

And according to the theory of the roots of the polynomial equation, we can easily understand the following facts from equation (15): (a) a sufficient large Kerr parameter χ can suppress the loop when the equation has a single root, and (b) in order to suppress loop, the Kerr interaction in the case loop appearing near the boundaries of Brillouin zone (with finite q) is much larger than the case loop appearing at the center (with $q = 0$). As an example, Figure 3 displays the variation of the optical photon number n_{st} in the cavity with respect to the Kerr parameter χ. This figure clearly displays we can control the bistability with Kerr interaction of photon. The two effects, the bistability of photon and the loop of atomic band structure, are the different sides of one coin. So the Kerr provide an controllable knobs to the loop.

4 Conclusion

In summary, we have studied the effect of a Kerr medium on the behavior of a BEC in a pumped cavity field. We have shown that the loop structure may completely disappears when the Kerr interaction exceeds a critical strength. These effects are due to the photon blockade mechanism. All the results demonstrate that the Kerr interaction is a new handle to coherently control the dynamics of BEC in a cavity field and hence could be useful in the realization of tuneable quantum-mechanical devices in the future.

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