Berry phase in nonlinear systems

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The Berry phase acquired by an eigenstate that experienced a nonlinear adiabatic evolution is investigated thoroughly. The circuit integral of the Berry connection of the instantaneous eigenstate cannot account for the adiabatic geometric phase, while the Bogoliubov excitations around the eigenstates are found to be accumulated during the nonlinear adiabatic evolution and contribute a finite phase of geometric nature. A two-mode model is used to illustrate our theory. Our theory is applicable to Bose-Einstein condensate, nonlinear light propagation, and Ginzburg-Landau equations for complex order parameters in condensed-matter physics.

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I. INTRODUCTION

Adiabatic theory, as a fundamental issue of quantum mechanics, involves two aspects: (i) when the Hamiltonian changes slowly compared to the level spacings, an initial nondegenerate eigenstate remains to be an instantaneous eigenstate [1]; (ii) the phase acquired by the eigenstate is the sum of the time integral of the eigenenergy (dynamical phase) and a quantity independent of the time duration and related to the geometric property of the closed path in parameter space (adiabatic geometric phase or Berry phase) [2,3]. Adiabatic theory has played a crucial role in the preparation and control of quantum states [4]. Recently, the Berry phase and related geometric phases [5,6] have received renewed interest due to their important use in the implementation of quantum-computing gates [7] and applications in condensed-matter physics [8].

For a nonlinear quantum system, such as that described by the nonlinear Schrödinger equations, how does the adiabatic theory get modified? Nonlinear quantum systems have become increasingly important in physics. They often arise in the mean-field treatment of many-body quantum systems, such as Bose-Einstein condensates (BECs) of dilute atomic gases [9]. Other applications include nonlinear light propagation [10]. Recently, extending the first aspect of adiabatic theory to the nonlinear systems, i.e., investigating the adiabatic condition and adiabaticity for the nonlinear quantum evolution, has been done [11,12]. Interestingly, it was found that the adiabaticity of an eigenstate only requires that the control parameters vary slowly with respect to the Bogoliubov excitation frequencies that in general are not equivalent to the level spacings. Nevertheless, the Berry phase issue in such nonlinear systems is far from understood.

On the other hand, it should be mentioned that the geometric phase for the cyclic evolution with a finite time duration T (i.e., nonadiabatical motion) in nonlinear systems was studied many years ago [13]. It was found that the nonadiabatic geometric phase takes the form of $-\int_0^T i\langle\phi|\frac{\partial}{\partial t}|\phi\rangle dt$, analogous to its linear counterpart [6]. Here, ϕ is the wave function in the projective Hilbert space satisfying the cyclic requirement $\phi(t=0)=\phi(t=T)$. A similar formula was deduced recently in investigating the geometric phase in a Bose-Einstein-Josephson junction [14]. However, the Berry phase associated with the adiabatic evolution of an eigenstate in nonlinear systems is still unresolved. Along the considerations of [13], one might imagine that when the parameter $\mathbf{R}$ vector moves in a circuit adiabatically, the adiabatic geometric phase acquired by an eigenstate would take the usual form of $-\oint i\langle\phi(\mathbf{R})|\frac{\partial}{\partial\mathbf{R}}|\phi(\mathbf{R})\rangle d\mathbf{R}$ [15]. Here, $\phi(\mathbf{R})$ denotes one eigenstate of the nonlinear system and $\langle\phi(\mathbf{R})|\frac{\partial}{\partial\mathbf{R}}|\phi(\mathbf{R})\rangle$ is called the Berry connection. The expression indicates that the Berry phase still equals the circuit integral of the Berry connection. In a recent study of the Berry phase for a specific BEC system described by a nonlocal Gross-Pitaevskii (nonlinear) equation with a quadratic potential, the exact solutions were constructed and the Berry phase was calculated in explicit form [16]. Their obtained adiabatic geometric phase, however, does not equal the above expression; an additional term emerges that is directly provoked by the presence of nonlinearity. The reason was not determined [16]. This controversy is not properly resolved yet, calling for further investigation; it may indicate that some subtle and important aspects were missed in previous theoretical considerations.

In the present paper, we have made a thorough analysis of the Berry phase issue for nonlinear systems. Our analytic deduction clearly indicates that the Berry phase is dramatically modified by the nonlinearity. The underlying mechanism has been revealed: for a nonlinear system because the Hamiltonian is a functional of the instantaneous wave functions, the Bogoliubov excitations induced by the slow change of the system are allowed to feed back to the Hamiltonian. They are accumulated during an adiabatic evolution and eventually contribute a finite phase of geometric nature.

Our paper is organized as follows. Sec.II is our general formalism. In Sec. III a two-mode BEC model is used to illustrate our theory and the geometric meaning of the new phase is discussed accordingly. Section IV presents our discussion and conclusion.

II. GENERAL FORMALISM

Without losing generality, our discussion is follows the Schrödinger equation with a quadratic nonlinear term, and our deduction is readily extended to other forms of nonlinearity with the invariance of gauge transformation,

$$i\frac{\partial\psi}{\partial t} = H_0\psi + g|\psi|^2\psi, \tag{1}$$

where $H_0 = -\frac{1}{2}\nabla^2 + V(\mathbf{R};r)$. Physically, the above system can be considered as a BEC trapped in a potential $V(\mathbf{R};r)$, with $\hbar = m = 1$ [17], where $\mathbf{R}$ is the parameter vector that varies in time slowly, and g is the nonlinear parameter representing the interaction between the coherent atoms. The total energy of the system $E_T = \int dr E(\psi^*,\psi)$, where the energy density $E(\psi,\psi^*) = \psi^* H_0 \psi + \frac{1}{2} g|\psi|^4$. The above system is invariant under gauge transformations of the first kind, $\psi(r,t) \to \exp(i\eta)\psi(r,t)$ with constant η. The gauge symmetry implies that the total atom number is conserved, i.e., $\int dr|\psi|^2 = 1$.

Let λ be the overall phase of the wave function, we may take it to be the phase of the wave function at a fixed position r_0, for example, $\lambda = -\arg(\psi(r_0,t))$. We split off this overall phase by writing $\psi = e^{-i\lambda}\phi$, and ϕ belongs to the so-called projective Hilbert space. From (1) we obtain

$$\frac{d\lambda}{dt} = -i\langle\phi|\frac{\partial}{\partial t}|\phi\rangle + \int dr E(\phi^*,\phi) + \frac{g}{2}\langle\phi|\phi^*\phi|\phi\rangle. \quad (2)$$

The eigenequation of the system is

$$H_0\overline{\psi} + g|\overline{\psi}|^2\overline{\psi} = \mu\overline{\psi}, \quad (3)$$

where $\overline{\psi}$ is the eigenfunction and μ is the eigenvalue (or chemical potential).

Now we assume the parameter vector $\mathbf{R}$ varies slowly in time and introduce the dimensionless adiabatic parameter of $\varepsilon \sim |\frac{d\mathbf{R}}{dt}| \sim \frac{1}{T}$ as the measure of how slow the parameters change. The adiabatic parameter tends to zero, i.e., $\varepsilon \to 0$, indicating the adiabatic limit. T is the time duration.

Consequently, the expression of the total phase can be expanded in a perturbation series in the adiabatic parameter, i.e.,

$$\frac{d\lambda}{dt} = \alpha_0(\varepsilon^0) + \alpha_1(\varepsilon^1) + o(\varepsilon^2). \quad (4)$$

When the parameters move in a circuit, the eigenstate evolves for an infinitely long time duration in the adiabatic limit. The time integral of the zero-order term gives the so-called dynamic phase because it is closely related to the temporal process of the evolution. The time integral of the first-order term makes an additional contribution to the overall phase, which will be shown later to be of a geometric nature, that is, it only depends on the geometry of the closed path in the parameter space. The contribution of the higher-order term vanishes in the adiabatic limit.

In the quantum evolution with slowly changing parameters, we assume $\phi = \overline{\phi}(\mathbf{R}) + \delta\phi(\mathbf{R})$, where $\overline{\phi}(\mathbf{R})$ is the wave function of the instantaneous eigenstate corresponding to the local minimum energy. $\delta\phi$ denotes the secular part of the Bogoliubov excitations induced by the system's slow change, while the rapid oscillations in the excitations are ignored because they vanish after a long-term average. $\delta\phi$ depends on the adiabatic parameter and is of order ε, then from Eq. (2) and with the help of relation Eq. (3), we have the explicit expressions as follows:

$$\alpha_0(\varepsilon^0) = \mu(\mathbf{R}), \quad (5)$$

$$\alpha_1(\varepsilon^1) = -i\int dr\left(\overline{\phi}^*\frac{\partial}{\partial t}\overline{\phi}\right) + g\int dr(\overline{\phi}^2\overline{\phi}^*\delta\phi^* + \overline{\phi}^{*2}\overline{\phi}\delta\phi). \quad (6)$$

From the above expressions, we see that the dynamical phase has been modified to be the time integral of the chemical potential rather than the energy. This is because the instantaneous eigenstates are feedback to the Hamiltonian. More interestingly, the first-order term, i.e., the Berry phase term has been modified due to the feedback of the Bogoliubov excitations to the Hamiltonian. To evaluate it qualitatively and express the modified geometric phase explicitly, let us introduce a set of orthogonal basis $|k\rangle$ ($k = 1,2,\ldots,N$) and the variable ψ_j, which is the jth component, i.e., $\psi_j = \langle j|\psi\rangle$. Without losing generality, the projective Hilbert space is set to be of a specific gauge such that the phase of the Nth component is zero. In the projective Hilbert space, the new variables (n_j,θ_j) are introduced through $\phi_j = \sqrt{n_j}e^{i\theta_j}$. Substituting the expression of $\psi_j = \sqrt{n_j}e^{i\theta_j}e^{-i\int^t \beta dt}$ into the nonlinear equation, and separating real and imaginary parts, we have the following differential equations for the density n_j and phase θ_j, respectively:

$$\frac{dn_j}{dt} = f_j, \quad \frac{d\theta_j}{dt} = h_j, \quad j = 1,2,\ldots,N-1, \quad (7)$$

where f_j and h_j are functions of the amplitudes, relative phases, matrix elements $C_{jk}(\mathbf{R}) = \langle j|H_0(\mathbf{R})|k\rangle$, and the overlap integral $D_{j,k,l,m} = \langle j|\langle k|l\rangle|m\rangle$. Their explicit expressions can be readily deduced but are not given here.

The norm conservation condition $n_N = 1 - \sum_{k=1}^{N-1} n_k$ has been used to remove the variable n_N in the above equations. In the representation of new variables, $(\overline{n}_j,\overline{\theta}_j)$ satisfy equations of the equilibrium state, i.e.,

$$\left(\frac{dn_j}{dt},\frac{d\theta_j}{dt}\right)\Bigg|_{(\overline{n}_j,\overline{\theta}_j)} = 0. \quad (8)$$

$(\overline{n}_j,\overline{\theta}_j)$ are functions of the parameter $\mathbf{R}$ corresponding to the eigenstates of the system. Let us make a perturbation expansion around the eigenstate with

$$n_j = \overline{n}_j(\mathbf{R}) + \delta n_j(\mathbf{R}), \quad \theta_j = \overline{\theta}_j(\mathbf{R}) + \delta\theta_j(\mathbf{R}). \quad (9)$$

Here, $\overline{\phi}_j(\mathbf{R}) = \sqrt{\overline{n}_j(\mathbf{R})}e^{i\overline{\theta}_j(\mathbf{R})}$, and $\delta n_j(\mathbf{R}),\delta\theta_j(\mathbf{R})$ are the excitations of order ε. Then, inserting the above expansion into equations (7) and, ignoring the higher-order terms such as $\frac{\partial\delta n_j}{\partial t},\frac{\partial\delta\theta_j}{\partial t}$, and denoting $\upsilon = (n_1,\theta_1;\ldots;n_{N-1},\theta_{N-1})$, we obtain

$$\frac{d\overline{\upsilon}}{d\mathbf{R}}\frac{d\mathbf{R}}{dt} = \mathcal{L}\delta\upsilon, \quad (10)$$

where the matrix takes the form

$$\mathcal{L} = \{\mathcal{L}_{jk}\}_{(N-1,N-1)}, \quad \mathcal{L}_{jk} = \begin{pmatrix} \frac{\partial f_j}{\partial n_k} & \frac{\partial f_j}{\partial\theta_k} \\ \frac{\partial h_j}{\partial n_k} & \frac{\partial h_j}{\partial\theta_k} \end{pmatrix}_{\upsilon=\overline{\upsilon}}. \quad (11)$$

Then, inversely, we have

$$\delta\upsilon = \mathcal{L}^{-1}\frac{d\overline{\upsilon}}{d\mathbf{R}}\frac{d\mathbf{R}}{dt}. \quad (12)$$

The differential relation between the new variables and old ones takes the form

$$\begin{pmatrix}\delta\phi_j \\ \delta\phi_j^*\end{pmatrix} = \Pi_j\begin{pmatrix}\delta n_j \\ \delta\theta_j\end{pmatrix}, \quad (13)$$

in which

$$\Pi_j = \begin{pmatrix} \frac{1}{2}\overline{n}_j^{-1/2} e^{i\overline{\theta}_j} & i\sqrt{\overline{n}_j} e^{i\overline{\theta}_j} \\ \frac{1}{2}\overline{n}_j^{-1/2} e^{-i\overline{\theta}_j} & -i\sqrt{\overline{n}_j} e^{-i\overline{\theta}_j} \end{pmatrix}. \tag{14}$$

Substituting (12) and (13) into (6), we finally obtain the explicit expression of the adiabatic geometric phase that contains two terms,

$$\gamma_g = \gamma_B + \gamma_{NL}, \tag{15}$$

where the first term is the usual Berry phase formula,

$$\gamma_B = -i \oint \langle \overline{\phi} | \nabla_{\mathbf{R}} | \overline{\phi} \rangle \, d\mathbf{R} = \oint \sum_{j=1}^{N-1} \overline{n}_j \frac{\partial \overline{\theta}_j}{\partial \mathbf{R}} \, d\mathbf{R}, \tag{16}$$

and the additional term is from the nonlinearity, taking the form

$$\gamma_{NL} = g \oint \left\langle \Lambda | \Pi \circ \mathcal{L}^{-1} \left| \frac{d\overline{v}}{d\mathbf{R}} \right. \right\rangle d\mathbf{R}. \tag{17}$$

Here,

$$\Lambda = \left(\left(\overline{n}_1 + \sum_{j=1}^{N-1} \overline{n}_j - 1 \right) \overline{n}_1^{1/2} e^{i\overline{\theta}_1}, \right.$$
$$\left. \left(\overline{n}_1 + \sum_{j=1}^{N-1} \overline{n}_j - 1 \right) \overline{n}_1^{1/2} e^{-i\overline{\theta}_1}, \ldots \right),$$
$$\frac{d\overline{v}}{d\mathbf{R}} = \left(\frac{d\overline{n}_1}{d\mathbf{R}}, \frac{d\overline{\theta}_1}{d\mathbf{R}}, \ldots, \frac{d\overline{n}_{N-1}}{d\mathbf{R}}, \frac{d\overline{\theta}_{N-1}}{d\mathbf{R}} \right)^T,$$

and diagonal matrix $\Pi = \mathrm{diag}(\Pi_1, \Pi_2, \ldots, \Pi_{N-1})$. Notice that to simplify the expression of Λ, we use the approximation that the overlap intergral $D_{j,k,l,m} \simeq 0$ when the subscripts are not all identical.

Both γ_B and γ_{NL} have the geometric property of parameter space. The novel second term indicates that the Bogoliubov excitations induced by the slow change of the system, which is negligible in the linear case, could be accumulated in the nonlinear adiabatic evolution and could contribute to the finite phase of a geometric nature.

III. TWO-LEVEL MODEL

As an illustration of our theoretical formalism, we consider the simple two-mode BEC model described by the nonlinear equation [18]

$$i \frac{d}{dt} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = H(\Psi_1, \Psi_2) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{18}$$

with

$$H(\Psi_1, \Psi_2) = \begin{pmatrix} Z|\Psi_1|^2 & \frac{\rho}{2} e^{-i\varphi} \\ \frac{\rho}{2} e^{i\varphi} & Z|\Psi_2|^2 \end{pmatrix}, \tag{19}$$

and the $\mathbf{R} = (X = \rho\cos\varphi,\ Y = \rho\sin\varphi, Z)$ are parameters. For simplicity, we fix ρ and Z and change the parameter φ from 0 to 2π adiabatically.

The eigenequations read

$$H \circ \begin{pmatrix} \overline{\Phi}_1 \\ \overline{\Phi}_2 \end{pmatrix} = \mu \begin{pmatrix} \overline{\Phi}_1 \\ \overline{\Phi}_2 \end{pmatrix}. \tag{20}$$

For the nonlinear system, the number of eigenstates may be larger than the dimension of the Hilbert space and the eigenstate could be unstable [11]. We have obtained four eigenstates for the case $Z > \rho$ by solving the above eigenequation. Three of them are stable and one is unstable. We choose the following stable eigenstate to illustrate our theory, i.e.,

$$\begin{aligned} \overline{\Phi}_1 &= \left[\frac{1}{2} \left(1 - \sqrt{1 - \frac{\rho^2}{Z^2}} \right) \right]^{1/2}, \\ \overline{\Phi}_2 &= \left[\frac{1}{2} \left(1 + \sqrt{1 - \frac{\rho^2}{Z^2}} \right) \right]^{1/2} e^{i\varphi}, \end{aligned} \tag{21}$$

with eigenvalue of $\mu = Z$.

The Berry term of the geometric phase is readily deduced,

$$\gamma_B = -i \int_0^{2\pi} \left(\overline{\Phi}_2 \frac{\partial}{\partial\varphi} \overline{\Phi}_2 \right) d\varphi = \pi \left(1 + \sqrt{1 - \frac{\rho^2}{Z^2}} \right). \tag{22}$$

Now we are going to derive the additional term γ_{NL} of the geometric phase. Let us introduce the new variables (n,θ) through $(\Phi_1, \Phi_2) = (\sqrt{1-n}, \sqrt{n} e^{i\theta})$. Substituting $(\Psi_1, \Psi_2) = e^{-i\int^t \beta dt}(\Phi_1, \Phi_2)$ into Eq. (18), and separating the real and imaginary parts, we have four differential equations, two of which are identical due to the norm conservation:

$$\frac{d}{dt} n = -\rho\sqrt{n - n^2} \sin(\theta - \varphi), \tag{23}$$

$$\frac{d}{dt} \theta = -\frac{\rho\sqrt{1-n}}{2\sqrt{n}} \cos(\theta - \varphi) - Zn + \beta, \tag{24}$$

$$\beta = Z(1-n) + \frac{\rho}{2} \sqrt{\frac{n}{1-n}} \cos(\theta - \varphi). \tag{25}$$

The eigenstate makes up the fixed point of Eqs. (23) and (24), i.e.,

$$\overline{n} = \frac{1}{2} \left(1 + \sqrt{1 - \frac{\rho^2}{Z^2}} \right), \quad \overline{\theta} = \varphi. \tag{26}$$

Let us make a perturbation expansion around the eigenstate with $n = \overline{n}(\varphi) + \delta n$ and $\theta = \overline{\theta}(\varphi) + \delta\theta$. Then inserting the above expansion into Eqs. (23) and (24), we get

$$\begin{pmatrix} \frac{\partial \overline{n}}{\partial \varphi} \\ \frac{\partial \overline{\theta}}{\partial \varphi} \end{pmatrix} \frac{d\varphi}{dt} = \mathcal{L} \begin{pmatrix} \delta n \\ \delta\theta \end{pmatrix}, \tag{27}$$

with

$$\mathcal{L} = \begin{pmatrix} 0 & -\rho\sqrt{\overline{n} - \overline{n}^2} \\ -2Z + \frac{\rho}{4(\overline{n} - \overline{n}^2)^{3/2}} & 0 \end{pmatrix}. \tag{28}$$

Then, we have

$$\gamma_{NL} = Z \int_0^{2\pi} \left\langle \Lambda | \Pi \mathcal{L}^{-1} \circ \left(\frac{\partial \overline{n}}{\partial \varphi}, \frac{\partial \overline{\theta}}{\partial \varphi} \right)^T \right\rangle d\varphi, \tag{29}$$

where $\Lambda = \big((2\overline{n} - 1)\sqrt{\overline{n}} e^{-i\overline{\theta}}, (2\overline{n} - 1)\sqrt{\overline{n}} e^{i\overline{\theta}}\big)$. Finally, we get

$$\gamma_{NL} = \frac{\pi\rho^2}{Z\sqrt{Z^2 - \rho^2}}. \tag{30}$$

Combining formulas (22) and (29), we arrive at the explicit expression of the adiabatic geometric phase:

$$\gamma_g = \pi \left(1 + \sqrt{1 - \frac{\rho^2}{Z^2}} + \frac{\rho^2}{Z\sqrt{Z^2 - \rho^2}}\right). \quad (31)$$

This new geometric phase has been verified by numerically solving the nonlinear Schödinger equation (18). This phase can also be interpreted as the flux of a virtual magnetic field M through the surface enclosed by the close circuit in parameter space. The virtual field has been deduced to take form of

$$M = \frac{\mathbf{R}}{2(Z^2 - \rho^2)^{3/2}}, \quad (32)$$

in contrast to the Dirac monopole field $\frac{\mathbf{R}}{2R^3}$ claimed by Berry for the linear case [2].

Now we discuss the geometric meaning of the new phase. The state vector ψ can be parametrized in a Bloch sphere according to

$$|\psi\rangle\langle\psi| = \tfrac{1}{2}(I + s \cdot \sigma). \quad (33)$$

Here, $\psi = (\cos\frac{\alpha}{2}e^{-i\delta/2}, \sin\frac{\alpha}{2}e^{i\delta/2})$, $s = (\sin\alpha\cos\delta, \sin\alpha \sin\delta, \cos\alpha)$, and I and σ are unit and Pauli matrix, respectively. In the new variables, the nonlinear phase can be expressed as a function of the solid angle in a Bloch sphere [i.e., $\Omega_B = 2\pi(1 - \cos\alpha)$] and the solid angle in the parameter space [i.e., $\Omega_P = 2\pi(1 - Z/\sqrt{\rho^2 + Z^2})$]. The two angles usually are not identical in the nonlinear case. To understand it, let us first consider the linear case $Z = 0$. The system can be viewed as a spin s driven by an external magnetic field $B = (\rho\cos\varphi, \rho\sin\varphi, Z)$. When the magnetic field varies in time adiabatically, from the dynamical equation

$$\frac{ds}{dt} = B \times s, \quad (34)$$

we know that the spin will stay in parallel with the field. Obviously, in this case, $\Omega_B = \Omega_P$. While in the presence of the nonlinearity, the magnetic field will be modulated self-consistently by the spin, the effective field $B^* = (\rho\cos\varphi, \rho\sin\varphi, Z\cos\alpha)$. This can be seen from Eq. (19) where parameter Z has been renormalized by $|\Psi_1|^2 - |\Psi_2|^2 = \cos\alpha$. The adiabatic evolution of the spin is expected to parallel the modulated field B^* rather than B. Thus, for a cyclic adiabatic evolution of the spin, the solid angles in the Bloch sphere and parameter space are not identical in general. On the other hand, from Eqs. (22) and (29) and using the relations $\overline{n} = \frac{\Omega_B}{4\pi}$ and $\rho/Z = \tan\arccos(1 - \Omega_P/2\pi)$, we can express our nonlinear geometric phase in terms of these solid angles, i.e.,

$$\gamma_g = \frac{\Omega_B}{2} + \frac{\pi\left(1 - \frac{\Omega_B}{2\pi}\right)\left(1 - \frac{\Omega_P}{2\pi}\right)}{\left(1 - \frac{\Omega_P}{2\pi}\right) - \sqrt{\left(2 - \frac{\Omega_P}{2\pi}\right)\left(\frac{\Omega_P}{2\pi}\right)} \Big/ \left[\left(2 - \frac{\Omega_B}{2\pi}\right)\left(\frac{\Omega_B}{2\pi}\right)\right]^{\frac{3}{2}}}. \quad (35)$$

For the linear case $Z = 0$, $\Omega_P = 2\pi$, the above expression reduces to the well-known relation $\gamma_g = \frac{\Omega_B}{2}$, i.e., the Berry phase equals half the solid angle.

IV. DISCUSSION AND CONCLUSION

Before concluding, we present some discussion. Let us first recall the nonadiabatic geometric phase for a cyclic motion. By inserting $\psi = e^{-i\lambda}\phi$ into the Schrödinger equation $i\frac{\partial\psi}{\partial t} = H\psi$, one can readily obtain that $\frac{d\lambda}{dt} = -i\langle\phi|\frac{\partial}{\partial t}|\phi\rangle + \langle\phi|H|\phi\rangle$. Here, ϕ is the wave function in the projective Hilbert space satisfying the cyclic requirement $\phi(t=0) = \phi(t=T)$. The total phase acquired during the cyclic evolution contains two parts, i.e., $-\int_0^T i\langle\phi|\frac{\partial}{\partial t}|\phi\rangle dt$ and $\int_0^T \langle\phi|H|\phi\rangle dt$. The former has no relation to Hamiltonian H and therefore can be regarded as the geometric phase for the cyclic state [13]. The above deduction is correct regardless of nonlinearity, just requiring that the system is invariant under gauge transformation (obviously the $H = H_0 + g|\psi|^2$ discussed in the present paper has this symmetry). A question arises as to why the above simple deduction does not apply to our adiabatic case. The main reason is that the adiabatic limit is a process, i.e., one can approach it but one cannot reach it. That is, for any small adiabatic parameter ϵ, i.e., $\mathbf{R}$ sweeps at any small rate, the dynamical solution obtained by solving the Schrödinger equation deviates from the adiabatic solution by a small quantity of order ϵ. This point has been clarified by the discussion in previous sections and expressed explicitly by $\phi = \overline{\phi}(\mathbf{R}) + \delta\phi(\mathbf{R})$. Moreover, as revealed by our previous discussion, in the presence of nonlinearity, this infinitesimal deviation could be accumulated to contribute a finite phase during an infinitely long time evolution $T \sim \frac{1}{\epsilon} \to \infty$. Whereas, for the cyclic evolution with a finite time duration, this kind of accumulation cannot emerge. Therefore, the adiabaticity is crucial to the emergence of our nonlinear correction.

In conclusion, we have investigated the phase acquired by an eigenstate during the nonlinear evolution in which the parameter vector moves adiabatically in a circuit. The acquired phase consists of two parts: the dynamical phase and the adiabatic geometric phase. Compared to the linear case, the dynamical phase is found to be modified as the time integral of the chemical potential rather than the energy. This is because the instantaneous eigenstates are feedback to the nonlinear Hamiltonian. More interestingly, the adiabatic geometric phase or Berry phase is found also to be modified by the nonlinearity. The underlying mechanism has been revealed. The above nonlinear corrections are of significance, for instance, they could affect the interferences of BEC matter waves, and therefore are expected to be observable in future experiments. Since some of the theories covered by our results are mean-field limits of quantum many-body theories, the possibility of generalizing these considerations to quantized field theories from the correspondence principle is of great interest for future study.

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