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To cite this article: Duan YiShi *et al* 2000 *Commun. Theor. Phys.* **33** 693

Commun. Theor. Phys. (Beijing, China) **33** (2000) pp 693–696
Vol. 33, No. 4, June 15, 2000

Topological Structure of Chern–Simons Vortex*

DUAN YiShi, FU LiBin and ZHANG Hong

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, China

(Received May 27, 1998; Revised October 5, 1998)

Abstract *Based on the gauge potential decomposition theory and the ϕ-mapping method, the topological inner structure of the Chern–Simons–Higgs vortex has been studied strictly. It is shown that there exits a multi-charged vortex at every zero point of the Higgs scalar field ϕ. The multivortex solutions in the Chern–Simons–Higgs model are obtained strictly.*

PACS numbers: 02.40.Pc, 02.40.-k, 11.15.-q
Key words: topology structure, Chern–Simons vortex

In recent years, a great deal of work on the Abelian Chern–Simons–Higgs (CSH) model in 2+1 dimensions has been done by many physicists.[1–3] This model has been widely used in many fields in physics, such as the fractional spin in quantum field theory,[3,4] and the quantum Hall effect in condensed matter physics.[5,6] Though it is common to include the topological properties of the Abelian CSH vortex, the topological structure of this vortex has not been studied strictly. In this paper, based on the gauge potential decomposition theory and the ϕ-mapping theory, the inner structure of the CSH vortex will be discussed, and the multivortex will be shown naturally. The ϕ-mapping topological current theory and the composed theory of gauge potential [7–14] are important to study the topological invariant and topological structure of physics system. These theories have been used to study the topological current of monopoles,[8] topological string theory,[9] topological characteristics of dislocations and disclinations continuum,[10,11] topological structure of the defects of space-time in early universe as well as its topological bifurcation,[12,13] the topological structure of Gauss–Bonnet–Chern theorem,[14] and the topological structure of London equation in superconductor.[15]

It is convenient to introduce the notions required in a mathematically precise manner.[16] As is known, the U(1) gauge potential $A = A_\mu \mathrm{d}x^\mu$ is a U(1)-connection on a complex line bundle over three-dimensional Minkowski space, and the Higgs complex scalar field ϕ denotes a section of this bundle. Then the covariant derivative is given by

$$D\phi = \mathrm{d}\phi - \mathrm{i}\frac{e}{\hbar c}A\phi\,. \tag{1}$$

The complex field ϕ can be expressed as

$$\phi = \phi^1 + \mathrm{i}\phi^2\,,$$

one can regard the scalar field ϕ as the complex representation of a two-dimensional vector field over the base space

$$\vec{\phi} = (\phi^1, \phi^2)\,,$$

where ϕ^a $(a = 1, 2)$ are real functions.[15] Now, let us define the unit vector

$$n^a = \frac{\phi^a}{||\phi||}, \qquad ||\phi|| = (\phi\phi^*)^{1/2} \tag{2}$$

satisfying

$$n^a n^a = 1 \qquad (a = 1, 2)\,.$$

*The project supported by National Natural Science Foundation of China

From a local point of view, all of the interesting behaviors of $\vec{\phi}$ occur around its zeros. If $\vec{\phi}^a(x) = 0$, the direction of $\vec{\phi}(x)$ changes radically in a small neighborhood of x. The field may circulate around x. Therefore, the direction of $\vec{\phi}$ or the unit vector $\vec{n}$ carries important topological information of the physical system.

Now let us study the CSH model. We know that the Abelian CSH Lagrangian density in 2+1 dimensions is often expressed as

$$L_{\rm CSH}(\phi,\, A) = \frac{1}{4}\alpha\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} + \frac{1}{2}D\phi(D\phi)^* + V(\phi)\,, \tag{3}$$

where $\frac{1}{4}\alpha\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}$ is the so-called Chern–Simons term having α as the CS factor. It may be mentioned that the Maxwell term $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is not considered. This does not affect our analysis to the extent of discussing the topological structure of the CSH vortex. In fact, quite a few authors[1,3] have discussed some important physical aspects related with the CSH solutions without considering the Maxwell term. From the above Lagrangian density, the gauge field equation can be obtained

$$\frac{1}{2}\alpha\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = J^\mu\,. \tag{4}$$

It is easy to know that the current J^μ is conserved.

As in the case of decomposition of the U(1) and SO(2) gauge potentials,[13,18] we can prove

$$A = k\epsilon^{ab}{\rm d}n^a n^b + {\rm d}\lambda\,,$$

where λ is only a phase factor, and $k = \hbar c/e$. From the above formula and using ϕ-mapping method,[13] we obtain

$$F = 2\pi k\epsilon^{ab}\delta^2(\vec{\phi}){\rm d}\phi^a \wedge {\rm d}\phi^b\,, \tag{5}$$

where we have used the formula

$$\frac{\partial^2}{\partial\phi^a\partial\phi^a}\Big(\ln\frac{1}{||\phi||}\Big) = -2\pi\delta^2(\vec{\phi})\,.$$

Substituting Eq. (5) into Eq. (4), we can obtain

$$J^\mu = 2\pi k\alpha\delta^2(\vec{\phi})D^\mu\Big(\frac{\vec{\phi}}{x}\Big)\,,$$

and

$$D^\mu\Big(\frac{\vec{\phi}}{x}\Big) = \frac{1}{2}\epsilon^{\mu\nu\lambda}\epsilon_{ab}\frac{\partial\phi^a}{\partial x^\nu}\frac{\partial\phi^b}{\partial x^\lambda}\,,$$

which is the same as the topological current of two-dimensional vector field in mathematical expression.[14] It is surprised that the current given in Eq. (4) is strictly a topological current. So, it is convenient to name it as CSH current. From this expression, we find that J^μ does not vanish only when $\vec{\phi} = 0$, i.e.,

$$\phi^1(x^0, x^1, x^2) = 0\,, \qquad \phi^2(x^0, x^1, x^2) = 0\,. \tag{6}$$

Suppose that the vector field $\vec{\phi}(\phi^1, \phi^2)$ possesses l zeros denoted as z_i $(i = 1, \cdots, l)$. According to the implicit function theorem,[19] when the zero points $\vec{z}_i$ are the regular points of $\vec{\phi}$ which require the Jacobian determinant

$$D\Big(\frac{\vec{\phi}}{x}\Big)\Big|_{z_i} = D^0\Big(\frac{\vec{\phi}}{x}\Big)\Big|_{z_i} \neq 0\,,$$

the solutions of Eq. (6) can be generally obtained

$$\vec{x} = \vec{z}_i(t)\,, \qquad i = 1, 2, \cdots, l\,, \qquad x^0 = t\,. \tag{7}$$

Then, from Eq. (6), it is easy to prove that[7]

$$D^\mu\Big(\frac{\vec{\phi}}{x}\Big)\Big|_{z_i} = D\Big(\frac{\vec{\phi}}{x}\Big)\Big|_{z_i}\frac{\mathrm{d}x^\mu}{\mathrm{d}t},$$

so the CSH current (4) can be rigorously expressed as

$$J^\mu = 2\pi k\alpha\delta^2(\vec{\phi})D\Big(\frac{\vec{\phi}}{x}\Big)\frac{\mathrm{d}x^\mu}{\mathrm{d}t}. \tag{8}$$

As we proved in Refs [13] and [20], the δ-function $\delta^2(\vec{\phi})$ can be expanded by

$$\delta^2(\vec{\phi}) = \sum_{i=1}^{l}\frac{\beta_i}{|D(\vec{\phi}/\vec{x})|_{z_i}}\delta^2(\vec{x}-\vec{z}_i),$$

where the positive integer β_i is the Hopf index.[17,21,22] Substituting this expansion of $\delta^2(\vec{\phi})$ into Eq. (8), we have

$$J^\mu = 2\pi k\alpha\sum_{i=1}^{l}\beta_i\eta_i\delta^2(\vec{x}-\vec{z}_i)\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\Big|_{z_i}, \tag{9}$$

where $\eta_i = \mathrm{sign}(D(\vec{\phi}/\vec{x})_{z_i}) = \pm 1$ is Brouwer degree.[8,10] The obvious meaning of β_i is that when the point $\vec{x}$ covers the neighborhood of zero point $\vec{z}_i$ once, the vector field $\vec{\phi}$ covers the corresponding region β_i times. On the other hand, the direction of this vector field will change $2\pi\beta_i$ in the small neighborhood of $\vec{\phi}(x) = 0$. In fact,[8] the integer $\beta_i\eta_i$ is the winding number of the point $\vec{z}_i$.

As is known, upon integration over the entire plane, this has the important consequence that any excitation with charge $Q = \int \mathrm{d}^2r\rho$ also carries magnetic flux $\Phi = \oint A_i\mathrm{d}x^i$ given by

$$Q = \alpha\Phi. \tag{10}$$

This charged system also carries a vortex-like magnetic field.[23] The magnetic flux of the vortex is

$$\Phi = \oint A_i\mathrm{d}x^i = \frac{1}{2}\int\epsilon^{ij}F_{ij}\mathrm{d}^2x.$$

From Eq. (4), the charge density is given by

$$\rho = \frac{1}{2}\alpha\epsilon^{0\alpha\beta}F_{\alpha\beta}, \tag{11}$$

then, according to Eq. (9), the density of topological charge can be rewritten as

$$\rho = J^0 = 2\pi k\alpha\sum_{i=1}^{l}\beta_i\eta_i\delta^2(\vec{x}-\vec{z}_i). \tag{12}$$

From the above discussions, we see that the density $\rho(x)$ is similar to a system of l classical point-like particles with topological $g_i = \alpha\beta_i\eta_i\Phi_0$ moving in the (2+1)-dimensional spacetime. And, the solution (7) can be regarded as the trajectory of the i-th vortex. From Eq. (12), we obtain that the total charge of the system is

$$Q = \int\rho(x)\mathrm{d}^2x = 2\pi k\alpha\sum_{i=1}^{l}\beta_i\eta_i,$$

it carries magnetic flux

$$\Phi = 2\pi k\sum_{i=1}^{l}\beta_i\eta_i = N\Phi_0$$

and

$$N = \sum_{i=1}^{l}\beta_i\eta_i, \tag{13}$$

where $\Phi_0 = 2\pi k = hc/e$ is a unit magnetic flux. It is obvious to see that there exit l isolated vortices of which the i-th vortex possesses charge $\beta_i\eta_i\alpha\Phi_0$ and carries magnetic flux $\beta_i\eta_i\Phi_0$. On the other hand, we can say that there exits a multi-charged vortex at every zero point of the scalar field ϕ, or at the singular point of the unit vector $\vec{n}$. And, vortices correspond to $\eta_i = +1$, while antivortices correspond to $\eta_i = -1$.[15] So, it is naturally obtained that the CSH current is a vortex current.

Now, let us define a general velocity

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}, \qquad u^\mu = 1,$$

and taking account of Eq. (12), the CSH current can be written in a simple and compact form $J^\mu = \rho u^\mu$. It is surprised that the topological current density just takes the same form as the current density in classical electrodynamics or hydrodynamics. So, we can obtain a moving system of the CSH vortices, which is a generalization of 't Hooft theory. In fact, the integer N in Eq. (13) is equal to the degree of ϕ-mapping deg ϕ. This is closely related to the Poincare–Hopf theorem. Thus we find that the unit vector $\vec{n}$ carries the topological information of the topological current, and the expressions (9) and (12) give the topological structure of the CSH current and density.

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