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Evolution of the Chern-Simons vortices

Li-Bin Fu*
Institute of Applied Physics and Computational Mathematics, P.O. Box 8009 (26), Beijing 100088, People's Republic of China

Yi-Shi Duan and Hong Zhang
Physics Department, Lanzhou University, Lanzhou 730000, People's Republic of China

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Based on the gauge potential decomposition theory and the ϕ-mapping theory, the topological inner structure of the Chern-Simons-Higgs (CSH) vortex is discussed in detail. The evolution of CSH vortices is also studied from the topological properties of the Higgs scalar field. The vortices are found generating or annihilating at the limit points and encountering, splitting, or merging at the bifurcation points of the scalar field ϕ.

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I. INTRODUCTION

In recent years, a great deal of work on the $(2+1)$-dimensional Abelian Chern-Simons-Higgs (CSH) model has been done by many physicists [1–4]. This model has been widely used in many fields of physics, such as the fractional spin in quantum field theory [3,5], and the quantum Hall effect in condensed matter physics [6,7].

We know that the $(2+1)$-dimensional Abelian CSH Lagrangian density is often expressed as

$$L_{CSH}(\phi, A)=\frac{1}{4}\alpha\in^{\mu\nu\lambda}A_\mu F_{\nu\lambda}+\frac{1}{2}D\phi(D\phi)^*+V(\phi), \tag{1}$$

where ϕ is the designated charged Higgs scalar field and $\frac{1}{4}\alpha\in^{\mu\nu\lambda}A_\mu F_{\nu\lambda}$ is the so-called Chern-Simons term. As pointed out by many physicists [1,3], the magnetic flux of the vortex is

$$\Phi=\oint A_i dx^i=\int\frac{1}{2}\in^{ij}\partial_i A_j dx^2=\frac{2\pi\hbar c}{e}n, \tag{2}$$

where n is a topological index, characterizing the vortex configuration. This is common to include the topological properties of the CSH vortex, but the inner structure of the vortex and its evolution have not been studied.

In this paper, based on the ϕ-mapping theory [8–11], we study the inner structure of this vortex system in detail. One shows that the vortex configuration given in Eq. (2) is a multivortex solution and the charge of the vortex is labeled only by the topological indices of the zero points of the scalar field ϕ. Furthermore, the evolution of the vortex is also investigated. And one sees that the vortices generate or annihilate at the limit points and encounter, split, or merge at the bifurcation points of the Higgs scalar fields.

II. THE TOPOLOGICAL INNER STRUCTURE OF CSH VORTEX

We know that the scalar field ϕ can be regarded as the complex representation of a two-dimensional vector field $\vec{\phi}=(\phi^1,\phi^2)$ over the base space, and

$$\phi=\phi^1+i\phi^2, \tag{3}$$

where $\phi^a(a=1,2)$ are real function. Let us define the unit vector

$$n^a=\frac{\phi^a}{||\phi||},\quad ||\phi||=(\phi\phi^*)^{1/2}. \tag{4}$$

Considering the covariant derivative

$$D\phi=d\phi-i\frac{e}{\hbar c}A\phi. \tag{5}$$

As one has showed in [12], the $U(1)$ gauge potential can be decomposed by the Higgs complex scalar field ϕ as

$$A_\mu=\frac{\hbar c}{e}\in^{ab}\partial_\mu n^a n^b+\partial_\mu\lambda, \tag{6}$$

in which λ is only a phase factor. We can introduce a topological current

$$J^\mu=\frac{1}{2}\in^{\mu\nu\lambda}\partial_\nu A_\lambda=\frac{\hbar c}{2e}\in^{\mu\nu\lambda}\in_{ab}\partial_\nu n^a\partial_\lambda n^b. \tag{7}$$

Obviously, the current (7) is conserved. Following the ϕ-mapping theory, it can be rigorously proved that

$$J^\mu=\frac{2\pi\hbar c}{e}\delta^2(\vec{\phi})D^\mu\left(\frac{\phi}{x}\right),$$

where the Jacobian $D^\mu(\phi/x)$ is defined as

$$\in^{ab}D^\mu\left(\frac{\phi}{x}\right)=\in^{\mu\nu\lambda}\partial_\nu n^a\partial_\lambda n^b.$$

From this expression, we find that J^μ does not vanish only when $\vec{\phi}=0$, i.e.,

*Email address: lbfu@263.net

$$\phi^a(x^0,x^1,x^2)=0,\quad a=1,2. \tag{8}$$

Suppose that the vector field $\vec{\phi}(\phi^1,\phi^2)$ possesses l zeros, denoted as z_i $(i=1,\ldots,l)$. According to the implicit function theorem [15], when the zeros points $\vec{z}_i$ are the regular points of $\vec{\phi}$, that requires the Jacobians determinant

$$D\left(\frac{\vec{\phi}}{x}\right)\bigg|_{z_i}=D^0\left(\frac{\vec{\phi}}{x}\right)\bigg|_{z_i}\neq 0. \tag{9}$$

The solutions of Eq. (8) can be generally obtained:

$$\vec{x}=\vec{z}_i(t),\quad i=1,2,\ldots,l,$$
$$x^0=t. \tag{10}$$

From Eq. (8), it is easy to prove that

$$D^\mu\left(\frac{\vec{\phi}}{x}\right)\bigg|_{z_i}=D\left(\frac{\vec{\phi}}{x}\right)\bigg|_{z_i}\frac{dx^\mu}{dt}. \tag{11}$$

According to the δ-function theory [13] and the ϕ -mapping theory, one can prove that

$$J^\mu=\frac{2\pi\hbar c}{e}\sum_{i=1}^{l}\beta_i\eta_i\delta^2(\vec{x}-\vec{z}_i)\frac{dx^\mu}{dt}\bigg|_{z_i}, \tag{12}$$

in which the positive integer β_i is the Hopf index and $\eta_i=\mathrm{sgn}(D(\vec{\phi}/\vec{x})_{z_i})=\pm 1$ is the Brouwer degree [14,9]. Then the density of topological charge can be expressed as

$$\rho=J^0=\frac{2\pi\hbar c}{e}\sum_{i=1}^{l}\beta_i\eta_i\delta^2(\vec{x}-\vec{z}_i). \tag{13}$$

From Eq. (7), it is easy to see that

$$\rho(x)=\frac{1}{2}\in^{ij}\partial_iA_j, \tag{14}$$

so, the total charge of the system given in Eq. (2) can be rewritten as

$$Q=\int\rho(x)d^2x=\Phi_0\sum_{i=1}^{l}\beta_i\eta_i, \tag{15}$$

where $\Phi_0=2\pi\hbar c/e$ is the unit magnetic flux. And the topological index n in Eq. (2) has the following expression

$$n=\sum_{i=1}^{l}\beta_i\eta_i. \tag{16}$$

It is obvious to see that there exist l isolated vortices of which the ith vortex possesses charge $\beta_i\eta_i\Phi_0$. And the vortex corresponds to $\eta_i=+1$, while the antivortex corresponds to $\eta_i=-1$. One can conclude that the vortex configuration given in Eq. (2) is a multivortex solution which possesses the inner structure described by expression (15).

III. THE GENERATION AND ANNIHILATION OF VORTICES

As discussed before, the zeros of the vector field $\vec{\phi}$ play an important role in describing the topological structure of the vortices. Now we begin investigating the properties of the zero points. As we knew before, if the Jacobian

$$D^0\left(\frac{\phi}{x}\right)\neq 0, \tag{17}$$

we will have the isolated zeros of the vector field $\vec{\phi}$. However, when the condition fails, the above discussion will change in some way and lead to the branch process. We denote one of the zero points as $(t^*,\vec{z}_i)$. If the Jacobian

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{z}_i)}\neq 0, \tag{18}$$

we can use the Jacobian $D^1(\phi/x)$ instead of $D^0(\phi/x)$ for the purpose of using the implicit function theorem [15]. Then we have a unique solution of Eqs. (8) in the neighborhood of the limit point $(t^*,\vec{z}_i)$

$$t=t(x^1),\quad x^2=x^2(x^1) \tag{19}$$

with $t^*=t(z_i^1)$. We call the critical points $(t^*,\vec{z}_i)$ the limit points. In the present case, we know that

$$\frac{dx^1}{dt}\bigg|_{(t^*,\vec{z}_i)}=\frac{D^1(\phi/x)}{D(\phi/x)}\bigg|_{(t^*,\vec{z}_i)}=\infty, \tag{20}$$

i.e.,

$$\frac{dt}{dx^1}\bigg|_{(t^*,\vec{z}_i)}=0. \tag{21}$$

Then, the Taylor expansion of $t=t(x^1)$ at the limit point $(t^*,\vec{z}_i)$ is [8]

$$t-t^*=\frac{1}{2}\frac{d^2t}{(dx^1)^2}\bigg|_{(t^*,\vec{z}_i)}(x^1-x_i^1)^2, \tag{22}$$

which is a parabola in x^1-t plane. From Eq. (22) we can obtain two solutions $x_1^1(t)$ and $x_2^1(t)$, which give two branch solutions (world lines of vortices). If

$$\frac{d^2t}{(dx^1)^2}\bigg|_{(t^*,\vec{z}_i)}>0,$$

we have the branch solutions for $t>t^*$ [see Fig. 1(a)]; otherwise, we have the branch solutions for $t<t^*$ [see Fig. 1(b)]. These two cases are related to the origin and annihilation of the vortices.

One of the results of Eq. (20), that the velocity of vortices are infinite when they are annihilating, agrees with the fact

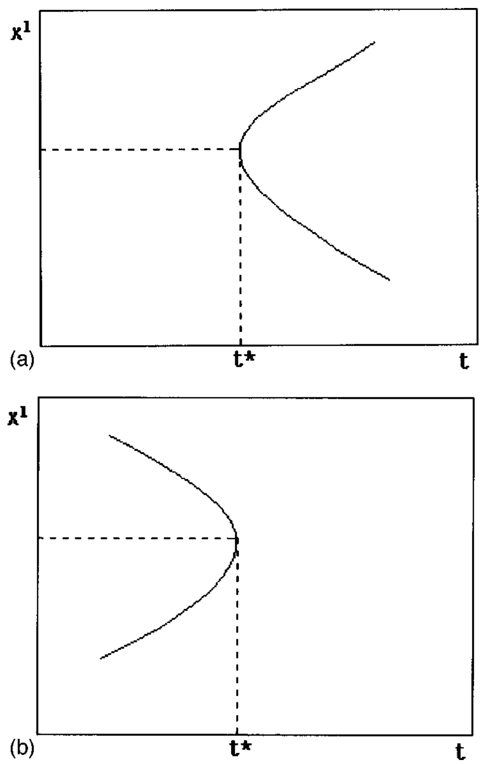

FIG. 1. Projecting the world lines of vortices onto (x^1-t) plane. (a) The branch solutions for Eq. (22) when $d^2t/(dx^1)^2|_{(t^*,\vec{z}_i)}>0$, i.e., a pair of vortices with opposite charges generate at the limit point, i.e., the origin of vortices. (b) The branch solutions for Eq. (22) when $d^2t/(dx^1)^2|_{(t^*,\vec{z}_i)}<0$, i.e., a pair of vortices with opposite charges annihilate at the limit point.

obtained by Bray [16] who has a scaling argument associated with the point defects final annihilation which leads to a large velocity tail. From Eq. (20), we also obtain a new result that the velocity of vortices is infinite when they are generating, which is gained only from the topology of the scalar fields.

Since the topological current is identically conserved, the topological charge of these two generated or annihilated vortices must be opposite at the limit point, i.e.,

$$\beta_{i_1}\eta_{i_1}=-\beta_{i_2}\eta_{i_2}, \tag{23}$$

which shows that $\beta_{i_1}=\beta_{i_2}$ and $\eta_{i_1}=-\eta_{i_2}$. One can see that the fact the Brouwer degree η is indefinite at the limit points implies that it can change discontinuously at limit points along the world lines of vortices (from ± 1 to ∓ 1). It is easy to see from Fig. 1: when $x^1>z_i^1$, $\eta_{i_1}=\pm 1$; when $x^1<z_i^1$, $\eta_{i_2}=\mp 1$.

For a limit point it is required that $D^1(\phi/x)|_{(t^*,\vec{z}_i)}\neq 0$. As to a bifurcation point [17], it must satisfy a more complex condition. This case will be discussed in the following section.

IV. THE BIFURCATION OF VORTICES VELOCITY FIELD

In this section we have the restrictions of Eq. (8) at the bifurcation points $(t^*,\vec{z}_i)$,

$$D\left(\frac{\phi}{x}\right)\bigg|_{z_i}=0, \quad D^1\left(\frac{\phi}{x}\right)\bigg|_{z_i}=0, \tag{24}$$

which leads to an important fact that the function relationship between t and x^1 is not unique in the neighborhood of the bifurcation point $(t^*,\vec{z}_i)$. It is easy to see that

$$V^1=\frac{dx^1}{dt}=\frac{D^1(\phi/x)}{D(\phi/x)}\bigg|_{z_i}, \tag{25}$$

which under Eq. (24) directly shows that the direction of the integral curve of Eq. (25) is indefinite at $(t^*,\vec{z}_i)$, i.e., the velocity field of vortices is indefinite at the point $(t^*,\vec{z}_i)$. This is why the very point $(t^*,\vec{z}_i)$ is called a bifurcation point.

Assume that the bifurcation point $(t^*,\vec{z}_i)$ has been found from Eqs. (8) and (24). We know that, at the bifurcation point $(t^*,\vec{z}_i)$, the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is 1. In addition, according to the ϕ-mapping theory, the Taylor expansion of the solution of Eq. (8) in the neighborhood of the bifurcation point $(t^*,\vec{z}_i)$ can be expressed as [8]

$$A(x^1-x_i^1)^2+2B(x^1-x_i^1)(t-t^*)+C(t-t^*)^2=0 \tag{26}$$

which leads to

$$A\left(\frac{dx^1}{dt}\right)^2+2B\frac{dx^1}{dt}+C=0 \tag{27}$$

and

$$C\left(\frac{dt}{dx^1}\right)^2+2B\frac{dt}{dx^1}+A=0, \tag{28}$$

where A, B, and C are three constants. The solutions of Eq. (27) or Eq. (28) give different directions of the branch curves (world lines of vortices) at the bifurcation point. There are four possible cases, which will show the physical meanings of the bifurcation points.

Case 1 $(A\neq 0)$. For $\Delta=4(B^2-AC)>0$ from Eq. (27) we get two different directions of the velocity field of vortices

$$\frac{dx^1}{dt}\bigg|_{1,2}=\frac{-B\pm\sqrt{B^2-AC}}{A}, \tag{29}$$

which is shown in Fig. 2, where two world lines of two vortices intersect with different directions at the bifurcation point. This shows that two vortices encounter and then depart at the bifurcation point.

Case 2 $(A\neq 0)$. For $\Delta=4(B^2-AC)=0$ from Eq. (27) we obtain only one direction of the velocity field of vortices

$$\frac{dx^1}{dt}\bigg|_{1,2}=-\frac{B}{A}, \tag{30}$$

which includes three important cases. (a) Two world lines tangentially contact, i.e., two vortices tangentially encounter

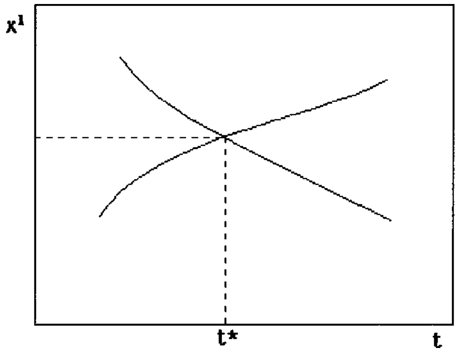

FIG. 2. Projecting the world lines of vortices onto (x^1-t) plane. Two world lines intersect with different directions at the bifurcation point, i.e., two vortices encounter at the bifurcation point.

at the bifurcation point [see Fig. 3(a)]. (b) Two world lines merge into one world line, i.e., two vortices merge into one vortex at the bifurcation point [see Fig. 3(b)]. (c) One world line resolves into two world lines, i.e., one vortex splits into two vortices at the bifurcation point [see Fig. 3(c)].

Case 3 $(A=0, C\neq 0)$. For $\Delta=4(B^2-AC)=0$ from Eq. (28) we have

$$\left.\frac{dt}{dx^1}\right|_{1,2}=\frac{-B\pm\sqrt{B^2-AC}}{C}=0, \quad -\frac{2B}{C}. \tag{31}$$

There are two important cases: (a) One world line resolves into three world lines, i.e., one vortex splits into three vortices at the bifurcation point [see Fig. 4(a)]. (b) Three world lines merge into one world line, i.e., three vortices merge into one vortex at the bifurcation point [see Fig. 4(b)].

Case 4 $(A=C=0)$. Equation (27) and Eq. (28) give respectively

$$\frac{dx^1}{dt}=0, \quad \frac{dt}{dx^1}=0. \tag{32}$$

This case is obvious, see Fig. 5, and is similar to case 3.

The above solutions reveal the evolution of the vortices. Besides the encountering of the vortices, i.e., two vortices encounter and then depart at the bifurcation point along different branch curves [see Fig. 2 and Fig. 3(a)], it also includes splitting and merging of vortices. When a multicharged vortex moves through the bifurcation point, it may split into several vortices along different branch curves [see Fig. 3(c), Fig. 4(a), and Fig. 5(b)]. On the contrary, several vortices can merge into one vortex at the bifurcation point [see Fig. 3(b) and Fig. 4(b)].

The identical conversation of the topological charge shows the sum of the topological charge of these final vortices must be equal to that of the original vortices at the bifurcation point, i.e.,

$$\sum_i \beta_{l_i}\eta_{l_i}=\sum_f \beta_{l_f}\eta_{l_f} \tag{33}$$

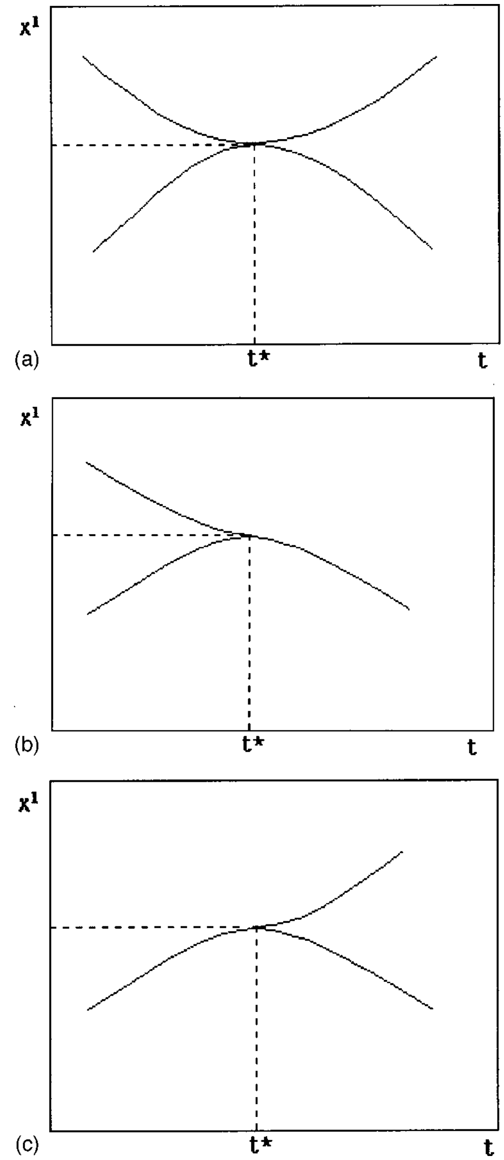

FIG. 3. (a) Two world lines tangentially contact, i.e., two vortices tangentially encounter at the bifurcation point. (b) Two world lines merge into one world line, i.e., two vortices merge into one vortex at the bifurcation point. (c) One world line resolves into two world lines, i.e., one vortex splits into two vortices at the bifurcation point.

for fixed l. Furthermore, from the above studies, we see that the generation, annihilation, and bifurcation of vortices are not gradually changed, but suddenly changed at the critical points.

V. CONCLUSION

First, we obtain the inner topological structure of the Chern-Simons vortex. The multicharged vortex has been found at the every zero point of the Higgs scalar field ϕ under the condition that the Jacobian determinate $D(\phi/x)\neq 0$. One also shows that the charge of the vortex is determined by Hopf indices and Brouwer degrees. Second, we

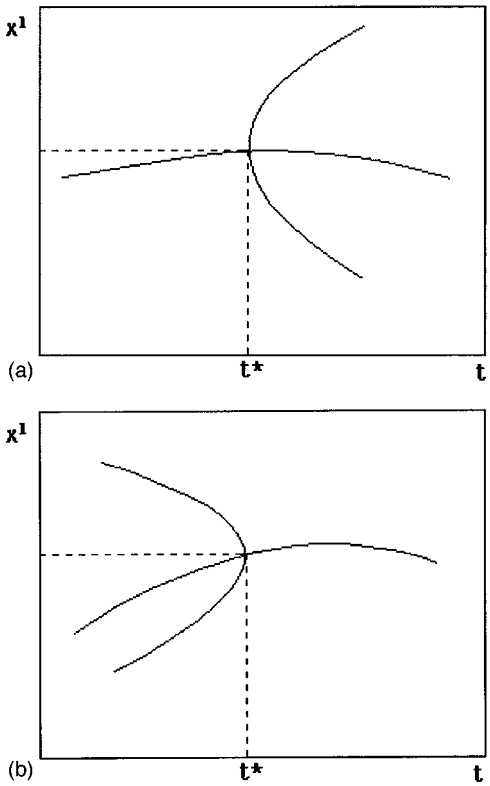

FIG. 4. Two important cases of Eq. (31). (a) One world line resolves into three world lines, i.e., one vortex splits into three vortices at the bifurcation point. (b) Three world lines merge into one world line, i.e., three vortices merge into one vortex at the bifurcation point.

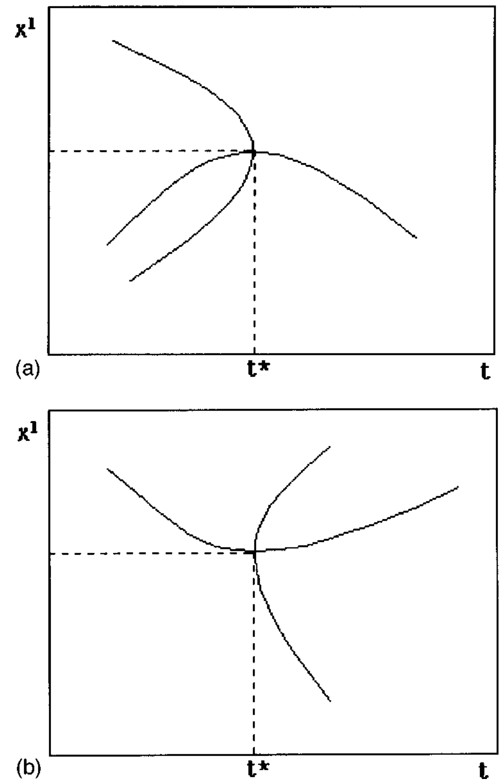

FIG. 5. Two world lines intersect normally at the bifurcation point. This case is similar to Fig. 4. (a) Three vortices merge into one vortex at the bifurcation point. (b) One vortex splits into three vortices at the bifurcation point.

conclude that there exist crucial cases of branch processes in the evolution of the vortices when $D(\phi/x)=0$, i.e., η_i is indefinite. This means that the vortices generate or annihilate at the limit points and encounter, split, or merge at the bifurcation points of the Higgs scalar fields, which shows that the vortices system is unstable at these branch points. Here we must point out that there exist two restrictions of the evolution of vortices. One restriction is the conservation of the topological charge of the vortices during the branch process [see Eqs. (23) and (33)], the other restriction is the number of different directions of the world lines of vortices is at most four at the bifurcation points [see Eqs. (27) and (28)]. Perhaps the former was known before, but the latter is pointed out for the first time.

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