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MEASURE ENTANGLEMENT OF BIPARTITE SYSTEM BY A NEW NONLOCAL EFFECT

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The state of a bipartite system may be changed by a cyclic operation applied on one of its subsystem. The change is a nonlocal effect, and can be detected only by measuring the two parts jointly. By employing the Hilbert-Schmidt metric, we can quantify such nonlocal effects via measuring the distance between initial and final state. By making use of the new nonlocal effect we can measure the entanglement of bipartite system. For qubit pair, we show that this measurement is equivalent to degree of entanglement and consists with the Bell inequality.

1. Introduction

Nowadays, quantum entanglement has become a more powerful resource in a number of applications[1,2,3,4]. One of the most important problem is the characterization and classification of mixed entangled states. The most prominent criterion for deciding whether a given state is entangled or not is known as positive partial transpose (PPT) test[5,6]. On the other hand, entanglement witness (EW) is also a very useful tool for experimental detection of entanglement.[7,8]

In this paper, we generalize the nonlocal effect manifested by the maximally entangled state in the quantum dense coding process to any state of a bipartite system. By employing the Hilbert-Schmidt distance, we quantify this nonlocal effect. Furthermore, by making use of this nonlocal effect we suggest a measure of entanglement for bipartite system. As examples, we give detailed calculation for qubit and qutrit pair. For qubit pair, the new measurement is equivalent to degree of entanglement and consist with Bell inequality.

2. Nonlocal Effect Induced by Local Cyclic Operation

Assuming Alice and Bob share a system compounded by two particles A (in Alice's hand) and B (in Bob's hand), and the sate is denoted by the density operator ρ. Then, the subsystems are described by the reduced density operators, $\rho_0^A = tr_B(\rho_0)$ and $\rho_0^B = tr_A(\rho_0)$ respectively. Bob applies a local unitary operation U^B on the particle in his hand which satisfies

$$[\rho_0^B, U^B] = 0. \tag{1}$$

Obviously, the subsystem is not changed by such an operation. For convenience, we denote the operation satisfies condition (1) as a local cyclic operation.

In general, the whole system will not always return to its initial state after Bob carried out the local cyclic evolution, i.e., $\rho_0 \neq \left(I \otimes U^B\right) \rho_0 \left(U^{B\dagger} \otimes I\right)$. The change between the final and initial states can not be detected locally.

To denote the difference between the initial and final states, we employ the Hilbert-Schmidt metric,[4] $D(\rho_1||\rho_2) = Tr|\rho_1 - \rho_2|^2$, to measure the distance between quantum states ρ_1 and ρ_2, where $|X| = \sqrt{X^+X}$. The Hilbert-Schmidt metric $D(\rho_1||\rho_2) \geq 0$ with the equality saturated iff $\rho_1 = \rho_2$.[9,10]

We quantify the shift between the initial state ρ_0 and the final state $\rho_f = \left(I \otimes U^B\right) \rho_0 \left(U^{B\dagger} \otimes I\right)$ by

$$d(\rho_0, U^B) = \sqrt{D(\rho_0||\rho_f)/2}. \tag{2}$$

By considering $Tr(\rho_0^2) = Tr(\rho_f^2)$, we can obtain

$$d(\rho_0, U^B) = \sqrt{Tr(\rho_0^2) - Tr(\rho_0\rho_f)}. \tag{3}$$

Obviously, $d(\rho_0, U^B(\tau)) \leq 1$ and $d(\rho_0, U^B) = 1$ only when the initial state is a pure state and it is orthonormal with the final state. In fact, for $\rho_0 = |\psi\rangle\langle\psi|$ is a pure state, we can have $d(\rho_0, U^B) = \sqrt{1 - F(\rho_0, \rho_f)}$,where $F(\rho_0, \rho_f) = \langle\psi|\rho_\tau|\psi\rangle$ is just the Bures fidelity.[4]

In Ref. 11, we proved that $d(\rho_0, U^B)$ is zero for product states but nonzero for *classically correlated* states.[12] However, for qubit pair, the value of $d(\rho_0, U^B)$ for disentangled states is limited by $1/\sqrt{2}$, while the entangled states can exceed this limit and reach 1 for maximally entangled states. Hence, this effect can be used to measure entanglement for some states.

3. Measure Entanglement of Bipartite System

It is well-known, for bipartite system, any pure state has the Schmidt decomposition form, i.e.,

$$|\psi\rangle = \sum_i \lambda_i |\varphi_i\rangle_A \otimes |\psi_i\rangle_B, \tag{4}$$

where $|\varphi_i\rangle_A$ and $|\psi_i\rangle_B$ are orthonormal basises. Then, the reduced density operator ρ_B can be expressed as

$$\rho_B = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|, \tag{5}$$

in the above formula the subscript B has been omitted for convenience. It is not difficult to prove that a local cyclic operation on the subsystem, which satisfy $U^B U^{\dagger B} = I$ and $U^B \rho_B U^{\dagger B} = \rho_B$, can be written as

$$U^B = \sum_i e^{i\phi_i} |\psi_i\rangle\langle\psi_i|, \tag{6}$$

which is the general expression for local cyclic operation. Hence,

$$|\psi^f\rangle = I \otimes U^B|\psi\rangle = \sum_i \lambda_i e^{i\phi_i}|\varphi_i\rangle_A \otimes |\psi_i\rangle_B. \tag{7}$$

Obviously, from (3), we obtain

$$d(|\psi\rangle, U^B) == 2\sqrt{\sum_{i<j}\left|\lambda_i\lambda_j \sin\frac{\phi_{ij}}{2}\right|^2}, \tag{8}$$

where $\phi_{ij} = \phi_i - \phi_j$.

It is easy to prove that for maximally entangled state $|\psi_{\max}\rangle$, i.e., for $\lambda_i = 1/\sqrt{N}(i = 1, 2, \cdots, N)$, we can always find optimal local cyclic operation U^B_{op}, for which $d(|\psi_{\max}\rangle, U^B_{op}) = 1$. We suggest that for any pure state of bipartite system we can measure its entanglement with $d(|\psi\rangle, U^B_{op})$. We denote it as $d_{op}(|\psi\rangle) = d(|\psi\rangle, U^B_{op})$.

Example 1. For the state of two qubits,

$$|\psi\rangle = k_1|00\rangle + k_2|11\rangle, \tag{9}$$

with $|k_1|^2 + |k_2|^2 = 1$. From Eq. (6), we have the local cyclic unitary operation

$$U^B = e^{i\phi_0}|0\rangle\langle 0| + e^{i\phi_1}|1\rangle\langle 1|. \tag{10}$$

From Eq.(3) and (8), one can have

$$d(\rho_0, U^B) = 2|k_1k_2 \sin\frac{\phi_{01}}{2}|. \tag{11}$$

Obviously, $d_{op}(\rho_0) = 2|k_1||k_2|$ with $\phi_{01} = \pi$, which just equals to the degree of entanglement for pure state of two qubits suggested in Refs. 13, 14, 15. The definition of entanglement degree consists with the violation of Bell inequality. The optimal form of Bell inequality for the entangled qubits is known as the Clauser-Horne-Shimony-Holt (CHSH) inequality.[16] It has been shown by Gisin[17] that any entangled pure state of qubit pair can violate the CHSH inequality and the maximum violation is $B_{\max}(\psi) = 2\sqrt{1 + 4|k_1k_2|^2}$. Obviously, $B_{\max}(\psi) = 2\sqrt{1 + d^2_{op}(\psi)}$. Therefore, the nonlocal effect can be used to quantify the entanglement of pure state of qubit pair.

Example 2. For the state of two entangled qutrits

$$|\psi\rangle = k_1|00\rangle + k_2|11\rangle + k_3|22\rangle, \tag{12}$$

with $|k_1|^2 + |k_2|^2 + |k_3|^2 = 1$. Similarly, the unitary operation applied on the subsystem can be expressed as

$$U^B = e^{i\phi_0}|0\rangle\langle 0| + e^{i\phi_1}|1\rangle\langle 1| + e^{i\phi_2}|2\rangle\langle 2|. \tag{13}$$

Then, we get

$$d(\rho_0, U^B) = 2\sqrt{|k_1|^2|k_2|^2 \sin^2\phi_{01}/2 + |k_1|^2|k_3|^2 \sin^2\phi_{02} + |k_2|^2|k_3|^2 \sin^2\phi_{12}/2}. \tag{14}$$

For the maximally entanglement state, with $|k_1| = |k_2| = |k_3| = \frac{\sqrt{3}}{3}$, we can obtain , $d(|\psi\rangle_{\max}) = 1$ with $\phi_0 = 0$, $\phi_1 = \frac{2\pi}{3}, \phi_2 = \frac{4\pi}{3}$. Therefore, $U_{op} = |0\rangle\langle 0| + e^{i\frac{2\pi}{3}}|1\rangle\langle 1| + e^{i\frac{4\pi}{3}}|2\rangle\langle 2|$. Then for state of form (12), we have

$$d_{op}(|\psi\rangle) = \sqrt{3(|k_1|^2|k_2|^2 + |k_1|^2|k_3|^2 + |k_2|^2|k_3|^2)}. \tag{15}$$

In analog to qubit pair, we define the degree of entanglement for qutrit pair as $P_E = d_{op}(|\psi\rangle)$. It is easy to prove, $0 \leq P_E \leq 1$, and $P_E = 1$ only for maximally entangled states and $P_E = 0$ for disentangled states.

4. Conclusions

In conclusion, we have introduced a new quantum nonlocal effect induced by local cyclic operation and quantified it with Hilbert-Schmidt distance. As applications of this nonlocal effect, we use it to measure entanglement of bipartite system. We find that, for two qubits in a pure state, the new measurement just equals to the degree of entanglement and consist with Bell inequality. Similarly, we suggest the degree of entanglement for pure state of two qutrits by making use of the nonlocal effect.

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