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TOPOLOGICAL CURRENT OF POINT DEFECTS AND ITS BIFURCATION

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From the topological properties of a three-dimensional vector order parameter, the topological current of point defects is obtained. One shows that the charge of point defects is determined by Hopf indices and Brouwer degrees. The evolution of point defects is also studied. One concludes that there exist crucial cases of branch processes in the evolution of point defects when the Jacobian $D(\frac{\phi}{x}) = 0$.

1. Introduction

In recent years, an outstanding development in the theory of condensed matter is to study the defect by topology. It has provided new insights and spectacular predictions. In particular there has been progress in the study of defects associated with a nonconserved n-component vector order parameter field $\boldsymbol{\phi}(\mathbf{r}, t)$.[1,2] In studying such objects in physics, the question on how one can define quantity like the density of defects arises. In the important conjecture proposed by Halperin, Liu and Mazenko[3,4] the density of such system has been written as

$$\rho = \delta(\boldsymbol{\phi}) D(\phi/x)\,, \tag{1}$$

which is the fundamental equation for such problem.

In this letter, we will present a new topological current involved in four-dimensional system by the use of the ϕ-mapping topological current theory, which is important in studying the topological invariant and structure of physics systems and has been used to study topological current of magnetic monopole,[5] topological string theory,[6] topological characteristics of dislocations and disclinations continuum,[7] topological structure of the defects of space–time in the early universe as well as its topological bifurcation,[8,9] topological structure of Gauss–Bonnet–Chern theorem[10,11] and topological structure of the London equation in superconductor.[12] It is shown that the topological charges of defects can be classified by Brower degree and Hopf index of the ϕ-mapping. It must be pointed out that

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the existence of this topological current is inevitable and necessary, which carries all the important topological properties of this physical system, and includes the defect density by other conjectures given by Liu and Mazenko.[4] Further research shows that there exist the crucial cases of branch process in the evolution of the point defects when the Jacobian $D(\frac{\phi}{x}) = 0$. We calculate the different branches of the worldlines of defects by using the implicit function theorem. It is found that the worldlines of point defects will split or merge at the critical points.

2. The Topological Current of Point Defects

Consider a four-dimensional system with a nonconserved three-component vector order parameter field $\boldsymbol{\phi}(\mathbf{r}, t)$. In order to compose a topological current, we introduce a unit vector field

$$n^a = \frac{\phi^a}{\|\phi\|}, \qquad a = 1, 2, 3 \tag{2}$$

where

$$\|\phi\|^2 = \phi^a \phi^a .$$

$\boldsymbol{\phi}(x)$ is just the vector order parameter. The topological current of this system is given by

$$j^\mu = \frac{1}{8\pi} \varepsilon^{\mu\nu_1 v_2 v_3} \varepsilon_{a_1 a_2 a_3} \partial_{\nu_1} n^{a_1} \partial_{\nu_2} n^{a_2} \partial_{\nu_3} n^{a_3} . \tag{3}$$

It is clear that the topological current is identically conserved, i.e.

$$\partial_\mu j^\mu = 0 . \tag{4}$$

The topological charge density correspondingly will be defined as

$$\rho = j^0 . \tag{5}$$

So, Eq. (4) is just the continuity equation of this system. Considering the following formula

$$\partial_\mu n^a = \frac{1}{\phi} \partial_\mu \phi^a + \phi^a \partial_\mu \frac{1}{\phi},$$

we can write Eq. (3) as

$$j^\mu = \frac{1}{8\pi} \varepsilon^{\mu\mu_1\mu_2\mu_3} \varepsilon_{a_1 a_2 a_3} \partial_{\mu_1} \phi^a \partial_{\mu_2} \phi^{a_2} \partial_{\mu_3} \phi^{a_3} \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^{a_1}} \left(\frac{1}{\|\phi\|} \right) .$$

If we define

$$\varepsilon^{a_1 a_2 a_3} D^\mu \left(\frac{\phi}{x} \right) = \varepsilon^{\mu\mu_1\mu_2\mu_3} \partial_{\mu_1} \phi^{a_1} \partial_{\mu_2} \phi^{a_2} \partial_{\mu_3} \phi^{a_3} ,$$

in which $J^0(\frac{\phi}{x})$ is just the usual three-dimensional Jacobian determinant

$$D^0\left(\frac{\phi}{x}\right) = D\left(\frac{\phi}{x}\right) = \begin{pmatrix} \frac{\partial\phi^1}{\partial x^1} & \frac{\partial\phi^1}{\partial x^2} & \frac{\partial\phi^1}{\partial x^3} \\ \frac{\partial\phi^2}{\partial x^1} & \frac{\partial\phi^2}{\partial x^2} & \frac{\partial\phi^2}{\partial x^3} \\ \frac{\partial\phi^3}{\partial x^1} & \frac{\partial\phi^3}{\partial x^2} & \frac{\partial\phi^3}{\partial x^3} \end{pmatrix},$$

and make use of the three-dimensional Laplacian Green's function relation

$$\Delta_\phi\left(\frac{1}{\|\phi\|}\right) = -4\pi\delta^3(\boldsymbol{\phi}),$$

where Δ_ϕ is the three-dimensional Laplacian operator in ϕ space, we do obtain the δ function like current

$$j^\mu = \delta^3(\boldsymbol{\phi})D^\mu\left(\frac{\phi}{x}\right). \tag{6}$$

When $\mu = 0$, we know that the density of j^μ can be written as

$$\rho = \delta^3(\boldsymbol{\phi})D\left(\frac{\phi}{x}\right). \tag{7}$$

From this expression, we find that J^μ does not vanish only when $\boldsymbol{\phi} = 0$, i.e.

$$\phi^a(t, x^1, x^2, x^3) = 0, \qquad a = 1, 2, 3. \tag{8}$$

Suppose that the vector field $\boldsymbol{\phi}$ possesses l zeros (i.e., there exist l point defects) denoted as z_i $(i = 1, \ldots, l)$. According to the implicit function theorem,[13] when the zero points $\mathbf{z}_i$ are the regular points of $\boldsymbol{\phi}$ which requires the Jacobian

$$D\left(\frac{\phi}{x}\right)\Big|_{z_i} = D^0\left(\frac{\phi}{x}\right)\Big|_{z_i} \neq 0, \tag{9}$$

the solutions of Eq. (8) can be generally obtained:

$$\mathbf{x} = \mathbf{z}_i(t), \qquad i = 1, 2, \ldots, l \tag{10}$$

which represent the worldlines of l point defects moving in space. From Eq. (8), it is easy to prove that

$$D^\mu\left(\frac{\phi}{x}\right)\Big|_{z_i} = D\left(\frac{\phi}{x}\right)\Big|_{z_i}\frac{dx^\mu}{dt}. \tag{11}$$

So, the topological current (3) can be rigorously expressed as

$$J^\mu = \delta^3(\boldsymbol{\phi})D\left(\frac{\phi}{x}\right)\frac{dx^\mu}{dt}. \tag{12}$$

As we proved in Ref. 9, we have

$$J^\mu = \sum_{i=1}^{l}\beta_i\eta_i\delta^3(\mathbf{x} - \mathbf{z}_i)\frac{dx^\mu}{dt}\Big|_{z_i}, \tag{13}$$

where the positive integer β_i is the Hopf index and $\eta_i = \operatorname{sign} D(\phi/x)_{z_i} = \pm 1$ is Brouwer degree.[5,7] According to Eq. (13), the density of topological charge can be rewritten as

$$\rho = J^0 = \sum_{i=1}^{l} \beta_i \eta_i \delta^3(\mathbf{x} - \mathbf{z}_i)\,. \tag{14}$$

It is clearly that the inner structure of this system is characterized by Hopf indices β_i and Brouwer degrees η_i, which are topological invariants. From discussions above, we see that the density $\rho(x)$ is similar to a system of l classical point-like particles with topological charge $\beta_i\eta_i$ moving in the four-dimensional space–time. The topological charge $\beta_i\eta_i$ is also called the topological charge of the ith point defect and $\rho(x)$ can be regarded as the density of defects. And solution (10) can be regarded as the trajectory of the ith defect. From formula (14), we obtain the total charge of the system:

$$Q = \int \rho(x) d^3x = \sum_{i=1}^{l} \beta_i \eta_i\,. \tag{15}$$

The result (7) has also been carried out by Halperin,[3] and exploited by Liu and Mazenko[4]: the first ingredient is the rather obvious result

$$\sum_{\alpha} \delta(\mathbf{r} - \mathbf{r}_\alpha(t)) = \delta(\boldsymbol{\phi}(\mathbf{r}, t)) \left| D\left(\frac{\phi}{x}\right) \right|,$$

where the second factor on the right-hand side is just the Jacobian of the transformation from the variable $\boldsymbol{\phi}$ to $\mathbf{r}$. This is combined with the less obvious result

$$\eta_\alpha = \operatorname{sgn} D(\phi/x)|_{\mathbf{r}_\alpha}$$

to give

$$\rho(\mathbf{r}, t) = \sum_{\alpha} \eta_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha(t)) = \delta(\boldsymbol{\phi}) D\left(\frac{\phi}{x}\right). \tag{16}$$

Here we see that the result (16) obtained by Halperin, Liu and Mazenko is not complete. They did not consider the cases $\beta_l \neq 1$ and $D(\phi/x) = 0$, i.e. η_l is indefinite. It is interesting to discuss what will happen and what does it correspond to in physics when $D(\phi/x) = 0$.

3. The Bifurcation of Worldlines of Point Defects

As being discussed before, the zeros of the smooth vector $\boldsymbol{\phi}$ (locations of defects) play important roles in studying the evolution of point defect. In this section, we will study the properties of the zero points, in other words, the properties of the solutions of the following equations

$$\begin{cases} \phi^1(x^0, x^1, x^2, x^3) = 0\,, \\ \phi^2(x^0, x^1, x^2, x^3) = 0\,, \\ \phi^3(x^0, x^1, x^2, x^3) = 0\,. \end{cases} \tag{17}$$

As we know, if the Jacobian determinant

$$D\left(\frac{\phi}{x}\right)\bigg|_{z_i} = D^0\left(\frac{\phi}{x}\right)\bigg|_{z_i} \neq 0\,,$$

we will have the isolated solutions (10) of Eq. (17). The isolated solutions are called regular points. It is easy to see that the results in Sec. 2 is based on this condition. However, when this condition fails, the above results will change in some way, and will lead to the branch process of topological density and give rise to the bifurcation. We denote one of zero points as $(t^*, \mathbf{z}_i)$.

3.1. *The branch process at a limit point*

It is well known that when the Jacobian $D(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)} = 0$, the usual implicit function theorem is of no use. But if the Jacobian

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\mathbf{z}_i)} = \frac{\partial(\phi^1,\phi^2,\phi^3)}{\partial(t,x^2,x^3)}\bigg|_{(t^*,\mathbf{z}_i)} \neq 0\,,$$

we can use the Jacobian $D^1(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)}$ instead of $D(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)}$, for the purpose of using the implicit function theorem.[13] Then we have a unique solution of Eqs. (17) in the neighborhood of the point $(t^*, \mathbf{z}_i)$

$$\begin{aligned} t &= t(x^1)\,, \\ x^i &= x^i(x^1)\,, \qquad i = 2,3\,, \end{aligned} \tag{18}$$

with $t^* = t(x^1)$. And we call the critical points $(t^*, \mathbf{z}_i)$ the limit points. In the present case, it is easy to know that

$$\frac{dx^1}{dt}\bigg|_{(t^*,\mathbf{z}_i)} = \frac{D^1(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)}}{D(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)}} = \infty\,, \tag{19}$$

i.e.

$$\frac{dt}{dx^1}\bigg|_{(t^*,\mathbf{z}_i)} = 0\,.$$

Then we have the Taylor expansion of Eq. (18) at the point $(t^*, \mathbf{z}_i)$

$$\begin{aligned} t &= t^* + \frac{dt}{dx^1}\bigg|_{(t^*,\mathbf{z}_i)}(x^1 - z_i^1) + \frac{1}{2}\frac{d^2t}{(dx^1)^2}\bigg|_{(t^*,\mathbf{z}_i)}(x^1 - z_i^1)^2 \\ &= t^* + \frac{1}{2}\frac{d^2t}{(dx^1)^2}\bigg|_{(t^*,\mathbf{z}_i)}(x^1 - z_i^1)^2\,. \end{aligned}$$

Therefore

$$t - t^* = \frac{1}{2} \frac{d^2 t}{(dx^1)^2}\bigg|_{(t^*, \mathbf{z}_i)} (x^1 - z_i^1)^2 \tag{20}$$

which is a parabola in the $x^1 - t$ plane. From Eq. (20), we can obtain two worldlines of two-point defects $x_1^1(t)$ and $x_2^1(t)$, which give the branch solutions of the system (17). If $\frac{d^2t}{(dx^1)^2}|_{(t^*,\mathbf{z}_i)} > 0$, we have the branch solutions for $t > t^*$, otherwise, we have the branch solutions for $t > t^*$. It is clear that these two cases are related to the origin and annihilation of point defects. Since the topological charge of point defect is identically conserved, the topological charge of these two must be opposite at the zero point, i.e.

$$\beta_{i1}\eta_{i1} = -\beta_{i2}\eta_{i2}\,. \tag{21}$$

One of the results of Eq. (19) — the velocity of point defects is infinite when they are annihilating — agrees with that obtained by Bray[14] who has a scaling argument associated with point defects final annihilation which leads to large velocity tail. From Eq. (19), we also get a new result that the velocity of point defects is infinite when they are generating, which is gained only from the topology of the three-dimensional vector order parameter.

For a limit point, it also requires the $D^1(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)} \neq 0$. As to a bifurcation point,[15] it must satisfy a more complement condition. This case will be discussed in the following subsections in detail.

3.2. *The branch process at a bifurcation point*

In this subsection, we have the restrictions of the system (17) at the bifurcation point $(t^*, \mathbf{z}_i)$:

$$\begin{cases} D\left(\dfrac{\phi}{x}\right)\bigg|_{(t^*,\mathbf{z}_i)} = 0\,, \\ D^1\left(\dfrac{\phi}{x}\right)\bigg|_{(t^*,\mathbf{z}_i)} = 0\,. \end{cases} \tag{22}$$

These will lead to an important fact that the function relationship between t and x^1 is not unique in the neighborhood of the bifurcation point $(\mathbf{z}_i, t^*)$. It is easy to see from equation

$$\frac{dx^1}{dt}\bigg|_{(t^*,\mathbf{z}_i)} = \frac{D^1(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)}}{D(\frac{\phi}{x})|_{(t^*,\mathbf{z}_i)}} \tag{23}$$

which under the restraint (22) directly shows that the direction of the worldlines of point defects is indefinite at the point $(\mathbf{z}_i, t^*)$. This is why the point $(\mathbf{z}_i, t^*)$ is called a bifurcation point of the system (17).

Next, we will find a simple way to search for the different directions of all branch curves at the bifurcation point. Assume that the bifurcation point $(\mathbf{z}_i, t^*)$ has been found from Eqs. (17) and (22). We know that, at the bifurcation point $(\mathbf{z_i}, t^*)$, the rank of the Jacobian matrix $[\frac{\partial\phi}{\partial x}]$ is smaller than 3. First, we suppose the rank of the Jacobian matrix $[\frac{\partial\phi}{\partial x}]$ is 2 (the case of a more smaller rank will be discussed later). Suppose that the 2×2 submatrix $J_1(\frac{\phi}{x})$ is

$$J_1\left(\frac{\phi}{x}\right) = \begin{pmatrix} \dfrac{\partial\phi^1}{\partial x^2} & \dfrac{\partial\phi^1}{\partial x^3} \\[2ex] \dfrac{\partial\phi^2}{\partial x^2} & \dfrac{\partial\phi^2}{\partial x^3} \end{pmatrix}, \tag{24}$$

and its determinant $D^1(\frac{\phi}{x})$ does not vanish. The implicit function theorem says that there exist one and only one function relation

$$x^i = f^i(x^1, t)\,, \qquad i = 2, 3\,. \tag{25}$$

We denoted the partial derivatives as

$$f_1^i = \frac{\partial f^i}{\partial x^1}\,, \qquad f_t^i = \frac{\partial f^i}{\partial t}\,, \qquad f_{11}^i = \frac{\partial^2 f^i}{\partial x^1 \partial x^1}\,,$$

$$f_{1t}^i = \frac{\partial^2 f^i}{\partial x^1 \partial x^t}\,, \qquad f_{tt}^i = \frac{\partial f^i}{\partial x^t \partial x^t}\,.$$

From Eqs. (17) and (25) we have for $a = 1, 2, 3$

$$\phi^a = \phi^a(x^1, f^2(x^1, t), f^3(x^1, t), t) = 0 \tag{26}$$

which give

$$\frac{\partial\phi^a}{\partial x^1} = \phi_1^a + \sum_{j=2}^{3} \frac{\partial\phi^a}{\partial f^j}\frac{\partial f^j}{\partial x^1} = 0\,, \tag{27}$$

$$\frac{\partial\phi^a}{\partial t} = \phi_t^a + \sum_{j=2}^{3} \frac{\partial\phi^a}{\partial f^j}\frac{\partial f^j}{\partial t} = 0\,, \tag{28}$$

from which we can get the first-order derivatives of f^i: f_1^i and f_t^i. Differentiating Eq. (27) with respect to x^1 and t respectively we get

$$\sum_{j=2}^{3} \phi_j^a f_{11}^j = -\sum_{j=2}^{3}\left[2\phi_{j1}^a f_1^j + \sum_{k=2}^{3}(\phi_{jk}^a f_1^k) f_1^j\right] - \phi_{11}^a\,, \qquad a = 1, 2, 3 \tag{29}$$

$$\sum_{j=2}^{3} \phi_j^a f_{1t}^j = -\sum_{j=2}^{3}\left[\phi_{jt}^a f_1^j + \phi_{j1}^a f_t^j + \sum_{k=2}^{3}(\phi_{jk}^a f_t^k) f_1^j\right] - \phi_{1t}^a\,, \qquad a = 1, 2, 3\,. \tag{30}$$

And the differentiation of Eq. (28) with respect to t gives

$$\sum_{j=2}^{3} \phi_j^a f_{tt}^j = -\sum_{j=2}^{3} \left[2\phi_{jt}^a f_t^j + \sum_{k=2}^{3} (\phi_{jk}^a f_t^k) f_t^j \right] - \phi_{tt}^a \,, \qquad a = 1,2,3 \tag{31}$$

where

$$\phi_{jk}^a = \frac{\partial^2 \phi^a}{\partial x^j \partial x^k} \,, \qquad \phi_{jt}^a = \frac{\partial^2 \phi^a}{\partial x^j \partial t} \,. \tag{32}$$

The differentiation of Eq. (28) with respect to x^1 gives the same expression as Eq. (30). By making use of the Gaussian elimination method to Eqs. (30)–(32) we can find the second-order derivatives f_{11}^i, f_{1t}^i and f_{tt}^i. The above discussion does not affect the last component $\phi^3(\mathbf{x}, t)$. In order to find the different values of dx^1/dt at the bifurcation point $(\mathbf{z}_i, t^*)$, let us investigate the Taylor expansion of $\phi^3(\mathbf{x}, t)$ in the neighborhood of $(\mathbf{z}_i, t^*)$. Substituting Eq. (25) into $\phi^3(\mathbf{x}, t)$ we have the function of two variables x^1 and t

$$F(x^1, t) = \phi^3(x^1, f^2(x^1, t), f^3(x^1, t), t) \tag{33}$$

which according to Eq. (17) must vanish at the bifurcation point

$$F(z_i^1, t^*) = 0 \,. \tag{34}$$

From Eq. (33) we have the first-order partial derivatives of $F(x^1, t)$

$$\frac{\partial F}{\partial x^1} = \phi_1^3 + \sum_{j=2}^{3} \phi_j^3 f_1^j \,, \qquad \frac{\partial F}{\partial t} = \phi_t^3 + \sum_{j=2}^{3} \phi_j^3 f_t^j \,. \tag{35}$$

Using Eqs. (27) and (28), the first equation of (22) is expressed as

$$D\left(\frac{\phi}{x}\right)\Big|_{(\mathbf{z}_i, t^*)} = \begin{vmatrix} -\sum_{j=2}^{3} \phi_j^1 f_1^j & \phi_2^1 & \phi_3^1 \\ -\sum_{j=2}^{3} \phi_j^2 f_1^j & \phi_2^2 & \phi_3^2 \\ \phi_1^3 & \phi_2^3 & \phi_3^3 \end{vmatrix}_{(\mathbf{z}_i, t^*)} = 0 \tag{36}$$

which by Cramer's rule can be written as

$$D\left(\frac{\phi}{x}\right)\Big|_{(\mathbf{z}_i, t^*)} = \frac{\partial F}{\partial x^1} \det J_1\left(\frac{\phi}{x}\right)\Big|_{(\mathbf{z}_i, t^*)} = 0 \,.$$

Since $\det J_1(\frac{\phi}{x})|_{(\mathbf{z}_i, t^*)} \neq 0$, the above equation gives

$$\frac{\partial F}{\partial x^1}\Big|_{(\mathbf{z}_i, t^*)} = 0 \,. \tag{37}$$

With the same reasons, we have

$$\left.\frac{\partial F}{\partial t}\right|_{(\mathbf{z}_i,t^*)} = 0\,. \tag{38}$$

The second-order partial derivatives of the function F can be found out easily to be

$$\frac{\partial^2 F}{(\partial x^1)^2} = \phi^3_{11} + \sum_{j=2}^{3}\left[2\phi^3_{1j}f^j_1 + \phi^3_j f^j_{11} + \sum_{k=2}^{3}(\phi^3_{kj}f^k_1)f^j_1\right], \tag{39}$$

$$\frac{\partial^2 F}{\partial x^1 \partial t} = \phi^3_{1t} + \sum_{j=2}^{3}\left[\phi^3_{1j}f^j_t + \phi^3_{tj}f^j_1 + \phi^3_j f^j_{1t} + \sum_{k=2}^{3}(\phi^3_{jk}f^k_t)f^j_1\right], \tag{40}$$

$$\frac{\partial^2 F}{\partial t^2} = \phi^3_{tt} + \sum_{j=2}^{3}\left[2\phi^3_{jt}f^j_t + \phi^3_j f^j_{tt} + \sum_{k=2}^{3}(\phi^3_{jk}f^k_t)f^j_t\right], \tag{41}$$

which at $(\mathbf{z}_i, t^*)$ are denoted by

$$A = \left.\frac{\partial^2 F}{(\partial x^1)^2}\right|_{(\mathbf{z}_i,t^*)}, \qquad B = \left.\frac{\partial^2 F}{\partial x^1 \partial t}\right|_{(\mathbf{z}_i,t^*)}, \qquad C = \left.\frac{\partial^2 F}{\partial t^2}\right|_{(\mathbf{z}_i,t^*)}. \tag{42}$$

Then take note of Eqs. (34), (37), (38) and (42) the Taylor expansion of $F(x^1, t)$ in the neighborhood of the bifurcation point $(\mathbf{z}_i, t^*)$ can be expressed as

$$F(x^1, t) = \frac{1}{2}A(x^1 - z^1_i)^2 + B(x^1 - z^1_i)(t - t^*) + \frac{1}{2}C(t - t^*)^2 \tag{43}$$

which by Eq. (33) is the expression of $\phi^3(\mathbf{x}, t)$ in the neighborhood of $(\mathbf{z}_i, t^*)$. The expression (43) is reasonable, which shows that at the bifurcation point $(\mathbf{z}_i, t^*)$, one of Eqs. (17), $\phi^3(\mathbf{x}, t) = 0$, is satisfied, i.e.

$$A(x^1 - z^1_i)^2 + 2B(x^1 - z^1_i)(t - t^*) + C(t - t^*)^2 = 0\,. \tag{44}$$

Dividing Eq. (44) by $(t - t^*)^2$ and taking the limit $t \to t^*$ as well as $x^1 \to z^1_i$ respectively we get

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + C = 0\,. \tag{45}$$

In the same way we have

$$C\left(\frac{dt}{dx^1}\right)^2 + 2B\frac{dt}{dx^1} + A = 0\,, \tag{46}$$

where A, B and C are three constants. The solutions of Eq. (45) or Eq. (46) give different directions of the branch curves (worldlines of point defects) at the bifurcation point. There are four possible cases, which will show the physical meanings of the bifurcation points.

Case 1 ($A \neq 0$): for $\Delta = 4(B^2 - AC) > 0$ from Eq. (45) we get two different directions of the velocity field of point defects

$$\left.\frac{dx^1}{dt}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}\,, \tag{47}$$

where two worldlines of two-point defects intersect with different directions at the bifurcation point. This shows that two-point defects encounter and then depart at the bifurcation point.

Case 2 ($A \neq 0$): for $\Delta = 4(B^2 - AC) = 0$ from Eq. (45) we get only one direction of the velocity field of point defects

$$\left.\frac{dx^1}{dt}\right|_{1,2} = -\frac{B}{A}\,, \tag{48}$$

which includes three important cases. (a) Two worldlines tangentially contact, i.e. two-point defects tangentially encounter at the bifurcation point. (b) Two worldlines merge into one worldline, i.e. two-point defects merge into one-point defect at the bifurcation point. (c) One worldline resolves into two worldlines, i.e. one-point defect splits into two-point defects at the bifurcation point.

Case 3 ($A = 0$, $C \neq 0$): for $\Delta = 4(B^2 - AC) = 0$ from Eq. (46) we have

$$\left.\frac{dt}{dx^1}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = 0\,,\ -\frac{2B}{C}\,. \tag{49}$$

There are two important cases: (a) one worldline resolves into three worldlines, i.e. one-point defect splits into three-point defects at the bifurcation point. (b) Three worldlines merge into one worldline, i.e. three-point defects merge into one-point defect at the bifurcation point.

Case 4 ($A = C = 0$): Eqs. (45) and (46) give respectively

$$\frac{dx^1}{dt} = 0\,, \qquad \frac{dt}{dx^1} = 0\,. \tag{50}$$

This case is obvious, similar to case 3.

The above solutions reveal the evolution of the point defects. Besides the encountering of the point defects, i.e. two-point defects encounter and then depart at the bifurcation point along different branch curves, it also includes splitting and merging of point defects. When a multicharged point defect moves through the bifurcation point, it may split into several point defects along different branch curves. On the contrary, several point defects can merge into one-point defect at the bifurcation point. The identical conversation of the topological charge shows the sum of the topological charge of final point defect(s) must be equal to that of the initial point defect(s) at the bifurcation point, i.e.

$$\sum_f \beta_{l_f}\eta_{l_f} = \sum_i \beta_{l_i}\eta_{l_i} \tag{51}$$

for fixed l. Furthermore, from the above studies, we see that the generation, annihilation and bifurcation of point defects do not change gradually, but start at a critical value of arguments, i.e. a sudden change.

3.3. *The branch process at a higher degenerate point*

In the above subsection, we have studied the case that the rank of the Jacobian matrix $[\frac{\partial\phi}{\partial x}]$ of Eqs. (17) is $2 = 3 - 1$. In this subsection, we consider the case that the rank of the Jacobian matrix is $1 = 3 - 2$. Let the $J_2(\frac{\phi}{x}) = \frac{\partial\phi^1}{\partial x^1}$ and suppose that $\det J_2 \neq 0$. With the same reasoning as we obtained Eq. (25), we can have the function relations

$$x^3 = f^3(x^1, x^2, t)\,. \tag{52}$$

Substituting relations (52) into the last two equations of (17), we have the following two equations with three arguments x^1, x^2, t

$$\begin{cases} F_1(x^1, x^2, t) = \phi^2(x^1, x^2, f^3(x^1, x^2, t), t) = 0\,, \\ F_2(x^1, x^2, t) = \phi^3(x^1, x^2, f^3(x^1, x^2, t), t) = 0\,. \end{cases} \tag{53}$$

Calculating the partial derivatives of the function F_1 and F_2 with respect to x^1, x^2 and t, taking note of Eq. (52) and using six similar expressions to Eqs. (37) and (38), i.e.

$$\left.\frac{\partial F_j}{\partial x^1}\right|_{(\mathbf{z}_i, t^*)} = 0\,, \qquad \left.\frac{\partial F_j}{\partial x^2}\right|_{(\mathbf{z}_i, t^*)} = 0\,, \qquad \left.\frac{\partial F_j}{\partial t}\right|_{(\mathbf{z}_i, t^*)} = 0\,, \qquad j = 1, 2\,, \tag{54}$$

we have the following forms of Taylor expressions of F_1 and F_2 in the neighborhood of $(\mathbf{z}_i, t^*)$

$$\begin{aligned} F_j(x^1, x^2, t) \approx{} & A_{j1}(x^1 - z_i^1)^2 + A_{j2}(x^1 - z_i^1)(x^2 - z_i^2) + A_{j3}(x^1 - z_i^1)(t - t^*) \\ & + A_{j4}(x^2 - z_i^2)^2 + A_{j5}(x^2 - z_i^2)(t - t^*) \\ & + A_{j6}(t - t^*)^2 = 0\,, \qquad j = 1, 2\,. \end{aligned} \tag{55}$$

In case of $A_{j1} \neq 0$ and $A_{j4} \neq 0$, dividing Eq. (55) by $(t - t^*)^2$ and taking the limit $t \to t^*$, we obtain two quadratic equations of $\frac{dx^1}{dt}$ and $\frac{dx^2}{dt}$

$$\begin{aligned} & A_{j1}\left(\frac{dx^1}{dt}\right)^2 + A_{j2}\frac{dx^1}{dt}\frac{dx^2}{dt} + A_{j3}\frac{dx^1}{dt} + A_{j4}\left(\frac{dx^2}{dt}\right)^2 \\ & \quad + A_{j5}\frac{dx^2}{dt} + A_{j6} = 0\,, \qquad j = 1, 2\,. \end{aligned} \tag{56}$$

Eliminating the variable dx^1/dt, we obtain an equation of dx^2/dt in the form of a determinant

$$\begin{vmatrix} A_{11} & A_{12}v + A_{23} & A_{14}v^2 + A_{15}v + A_{16} & 0 \\ 0 & A_{11} & A_{12}v + A_{13} & A_{14}v^2 + A_{15}v + A_{16} \\ A_{21} & A_{22}v + A_{23} & A_{24}v^2 + A_{25}v + A_{26} & 0 \\ 0 & A_{21} & A_{22}v + A_{23} & A_{24}v^2 + A_{25}v + A_{26} \end{vmatrix} = 0\,, \tag{57}$$

where $v = dx^2/dt$, that is a fourth-order equation of dx^2/dt

$$a_0\left(\frac{dx^2}{dt}\right)^4 + a_1\left(\frac{dx^2}{dt}\right)^3 + a_2\left(\frac{dx^2}{dt}\right)^2 + a_3\left(\frac{dx^2}{dt}\right) + a_4 = 0\,. \tag{58}$$

Therefore we get different directions of the worldlines of point defects at the higher degenerate point bifurcation point. The number of different branch curves is at most four.

At the end of this section, we conclude that in our theory of point defects there exist the crucial case of branch process.[11] Besides the encountering of the point defects, i.e. two-point defects encounter and then depart at the bifurcation point along different worldlines, it also includes splitting and merging of point defects. When a multicharged point defect moves through the bifurcation point, it may split into several point defects along different worldlines. On the contrary, several point defects can merge into one-point defect at the bifurcation point. Since the topological charges of point defects is identically conserved (4), the sum of the topological charge of final point defect(s) must be equal to that of the initial point defect(s) at the bifurcation point.

4. Conclusions

We have studied the evolution of the point defects of a three-dimensional vector order parameter by making use of the ϕ mapping topological current theory. We conclude that there exist crucial cases of branch processes in the evolution of the point defects. This means that the point defects generate or annihilate at the limit points and encounter, split or merge at the bifurcation points of the three-dimensional vector order parameter, which shows that the point defects system is unstable at these branch points. There are two restrictions of the evolution of point defects in this letter. One restriction is the conservation of the topological charge of the point defects during the branch process, the other restriction is the number of different directions of the worldlines of point defects is at most four at the bifurcation points. We would like to point out that all the results in this letter are obtained from the viewpoint of topology without using any particular models or hypothesis.

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