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Collective excitations of a Bose-Einstein condensate in an anharmonic trap

Guan-Qiang Li,[1,2] Li-Bin Fu,[1] Ju-Kui Xue,[2] Xu-Zong Chen,[3] and Jie Liu[1,*]

[1] *Institute of Applied Physics and Computational Mathematics, P. O. Box 8009 (28), 100088 Beijing, People's Republic of China*

[2] *Physics and Electronics Engineering College, Northwest Normal University, 730070 Lanzhou, People's Republic of China*

[3] *Key Laboratory for Quantum Information and Measurements, Ministry of Education, School of Electronics Engineering and Computer Science, Peking University, Beijing 100871, People's Republic of China*

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We investigate the collective excitations of a one-dimensional Bose-Einstein condensate with repulsive interaction between atoms in a quadratic plus quartic trap. Using variational approaches, the coupled equations of motions for the center-of-mass coordinate of the condensate and its width are derived. Then, two low-energy excitation modes are obtained analytically. The frequency shift induced by the anharmonic distortion, and the collapse and revival of the collective excitations originating from the nonlinear coupling between the two modes, are discussed.

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I. INTRODUCTION

One of the most important characters of an interacting quantum many-body system is its response to external perturbations, where collective excitation modes represent a very effective tool for exploring the role of interactions and testing theoretical schemes. For this reason, measurements of the collective modes in trapped gases of alkali-metal atoms [1–3] were carried out soon after the discovery of Bose-Einstein condensates (BECs). For dilute degenerate gases, the essential physics of the BEC ground state is included in the Gross-Pitaevskii equation (GPE). The nonlinearity originating from the interatomic interaction is included in the equation through a mean-field term proportional to the condensate density. The study about the collective excitations of the condensation has been investigated extensively by using various theoretical methods [4–8]. The remarkable agreement between measured frequencies and theoretical predictions is one of the most important achievements in the investigation of these new systems. Meanwhile, because of the emergence of nonlinearity, a lot of interesting phenomena in the collective excitations of BECs, such as frequency shift [9], mode coupling [9–12], damping [13,14], collapse and revival of oscillations [15,16], and the onset of stochastic motions for strong driving amplitude [17–19], have been paid much attention.

Recently, the collective dynamics of a one-dimensional (1D) trapped ultracold Bose gases has attracted considerable attention, since experiments on trapped Bose gases at low temperature have pointed out the occurrence of characteristic 1D feature [20–22]. In previous work the studies of the collective excitations of BECs in the magnetic traps are mainly limited to the harmonic case. However, in the practical situation of experiments, the trap usually is not purely harmonic. With this concern, we study the collective excitations of a one-dimensional BEC in a harmonic trap with a quartic distortion. Our aim is to understand how the distortion affects the collective excitations of BECs.

Our study is facilitated by variational approaches. Using a Gaussian trial function, the GPE is transformed into a set of second-order ordinary differential equations about some parameters that characterize the condensate wave function. Then we derived the expressions for the two low-energy oscillation modes analytically, and the nonlinear coupling between the two modes is revealed. In particular, we find that a very small anharmonic distortion may cause a significant frequency shift of the excitation modes when the atomic interaction is strong. Finally, we demonstrate that the anharmonic distortion may give rise to the collapse and revival of the collective excitations.

The paper is organized as follows. In Sec. II we derive the governing equations for the center-of-mass coordinate of the condensate and its width. In Sec. III we discuss the collective modes and the frequency shift caused by the anharmonic distortion. In Sec IV, we demonstrate the collapse and revival of the collective excitations in an anharmonic potential. The final section is our conclusion.

II. VARIATIONAL APPROACH AND GOVERNING EQUATIONS

We consider dilute degenerate bosons confined in a cigar-shaped trap and assume that the system is far from the Tonks-Girardeau regime [23]. Then the BEC can be well described by the dimensionless 1D GPE,

$$i\frac{\partial\psi(x,t)}{\partial t}=\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+V(x)+g|\psi(x,t)|^2\right)\psi(x,t), \qquad (1)$$

where the coordinate x is measured in units of $\sqrt{\hbar/m\omega_x}$ and time is in units of $1/\omega_x$. ω_x is the x-component frequency of the harmonic trap. $\psi(x,t)$ is the macroscopic wave function of the condensate normalized so that $\int|\psi(x)|^2dx=1$; $g=4N\pi\alpha_{1d}a_s/\sqrt{\hbar/m\omega_x}$ characterizes the interatomic interaction and is defined in terms of the s-wave scattering length a_s (below we shall be concerned with repulsive BECs for which $a_s>0$); $\alpha_{1d}=\int|\varphi(y,z)|^4dy\,dz/(\int|\varphi(y,z)|^2dy\,dz)^{5/2}$ is a coefficient which compensates for the loss of two dimensions [24]. In the above expressions, N is the total number of atoms and

*Electronic address: Liu_Jie@iapcm.ac.cn

$\varphi(y,z)$ is the ground-state wave function of the lateral dimensions.

The trapping potential we consider takes the form

$$V(x) = \frac{1}{2}(x^2 + \lambda x^4). \tag{2}$$

The quartic term in the potential denotes the anharmonicity of the trap. In [25], the authors created such a quartic confinement with a blue-detuned Gaussian laser directed along the axial direction. In their case, the nonrotating condensate was cigar shaped and the strength of the quartic admixture was $\lambda \simeq 10^{-3}$. Here, we regard λ as a controllable parameter and assume that the anharmonicity is weak, i.e., $|\lambda| \ll 1$.

The problem of solving Eq. (1) can be restated as a variational problem corresponding to the minimization of the action related to the Lagrangian density [6],

$$\ell = \frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \frac{1}{2}|\nabla\psi|^2 + V(x)|\psi|^2 + \frac{g}{2}|\psi|^4, \tag{3}$$

where the asterisk denotes a complex conjugate. In order to obtain the dynamics of the condensate in the trapping potential we will find the extremum of Eq. (3) with a set of trial functions. In our case, a natural choice of the trial function is a Gaussian, i.e., we take

$$\psi(x,t) = \eta(t)e^{-[x-\chi(t)]^2/2w(t)^2 + ix\alpha(t) + ix^2\beta(t)}. \tag{4}$$

At a given time t, this function defines a Gaussian distribution centered at the position χ with width w. The other variational parameters η, α, and β are all real variables. Inserting (4) into (3) we can calculate a grand Lagrangian by integrating the Lagrangian density over the whole coordinate space, $L = \int_{-\infty}^{+\infty}\ell\, dx$. Then, from the Lagrange equations, we obtain the evolution equations for all variational parameters.

The dynamical equations for the center of mass and width of the condensate are derived as follows:

$$\ddot{\chi} + \chi + 2\lambda\chi^3 + 3\lambda\chi w^2 = 0, \tag{5}$$

$$\ddot{w} + w + 3\lambda w^3 + 6\lambda\chi^2 w = \frac{1}{w^3} + \frac{p}{w^2}, \tag{6}$$

where the effective interaction $p \equiv g/\sqrt{2\pi}$, which comes from the nonlinear interaction between the particles.

The other variational parameters can be obtained from the center coordinate and the width through the equations

$$\sqrt{\pi}|\eta(t)|^2 w(t) = 1, \tag{7}$$

$$\beta = \frac{\dot{w}}{2w}, \quad \alpha = \dot{\chi} - \chi\frac{\dot{w}}{w}. \tag{8}$$

The first one corresponds to the normalization condition of the wave function, $\int|\psi(x,t)|^2 dx = 1$. Therefore, once we know the behavior of the center and width of the condensate, we can calculate the evolution of the rest of the parameters, and then completely characterize the dynamics of the condensate.

Comparing the above equations with that of a pure harmonic potential, we find the emergence of two new terms in Eqs. (5) and (6), i.e., the third and fourth terms in the left side. Obviously, the third term is directly from the distortion of potential, whereas, the fourth one represents the response of a coherent wave to the distortion of the potential, manifesting a coupling between the motion of the center and the width.

III. COLLECTIVE MODES AND FREQUENCY SHIFT DUE TO THE ANHARMONIC DISTORTION

When we consider the contribution from the quartic distortion, i.e., $\lambda \neq 0$ in the potential $V(x)$, nonlinear coupling between the oscillations of the center and width emerges. The equilibrium points of Eqs. (5) and (6) correspond to stable or unstable stationary states of the condensate. They satisfy the following equations:

$$\chi_0 + 2\lambda\chi_0^3 + 3\lambda\chi_0 w_0^2 = 0, \tag{9}$$

$$(1 + 6\lambda\chi_0^2)w_0 + 3\lambda w_0^3 = \frac{1}{w_0^3} + \frac{p}{w_0^2}. \tag{10}$$

There is only one stable equilibrium point for $\lambda > 0$, that is,

$$\chi_0 = 0, \tag{11}$$

$$w_0 + 3\lambda w_0^3 = \frac{1}{w_0^3} + \frac{p}{w_0^2}. \tag{12}$$

For $\lambda < 0$ there are several equilibrium points; one of them is stable and the others are unstable. The stable equilibrium point also satisfies Eqs. (11) and (12).

Expanding Eqs. (5) and (6) around the equilibrium points defined by Eqs. (11) and (12) and making a routine diagonalizing process, we can obtain the following frequencies for low-energy excitation modes:

$$\omega_{1,2} = \left[1 + \left(1 \mp \frac{1}{2}\right)6\lambda w_0^2 + (1 \mp 1)\left(\frac{3}{2w_0^4} + \frac{p}{w_0^3}\right)\right]^{1/2}, \tag{13}$$

which are related to the coupled variation of the center and width of the condensate [4–8]. When $\lambda = 0$, ω_1 corresponds to the dipole oscillation $(m=1)$ characterizing the motion of the center of mass, and ω_2 is the frequency of the variation of the condensate width; it is just the low-lying collective mode $(m=0)$.

When the potential is not perfectly harmonic, i.e., $\lambda \neq 0$, the contribution from the quartic term will give rise to a shift on the frequencies. When $\lambda > 0$ the frequency will be blueshifted and when $\lambda < 0$ the frequency will be redshifted. This is true for both single particles and BECs. However, it is interesting that for BECs the frequency shift is enhanced dramatically by the atomic interaction. The frequencies of two low-energy excitations as functions of p for the above parameters are plotted in Fig. 1.

From Eq. (13), we see that the contribution to the frequency shift comes from the second term in the right-hand side, i.e., $\sim\lambda w_0^2$, which is due to the response of the coherent wave to the distortion of the potential. On the other hand, the width w_0 of the wave function will be broadened by the atom

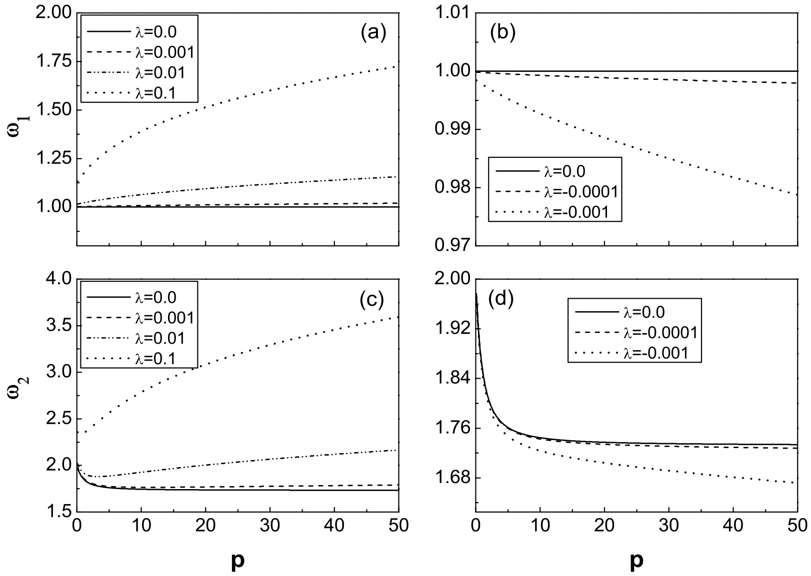

FIG. 1. Frequencies of two low-energy excitations as functions of effective interaction p for different λ.

interaction. In Fig. 2 we show this effect by plotting the dependence of w_0 on λ for different interaction parameters.

From the above discussion, we know that, although the anharmonic distortion is very small, the frequency shift may be large due to magnification effects from the atom interaction. To demonstrate it, in Fig. 3 we plot the frequencies of the dipole motion of the BEC wave packet for different atom interactions and anharmonic parameters. It is clearly shown that the atom interaction will give rise to 20% or more shift of frequency when there is only 2% of anharmonic distortion (see Fig. 3, data at $p=20$). We expect that these phenomena can be observed in future experiments.

IV. COLLAPSE AND REVIVAL OF THE COLLECTIVE EXCITATIONS

Another promising direction is to investigate the collapse and revival of the collective excitations [15,16], which are directly induced by the nonlinear coupling effect between two oscillation modes. In previous work, the nonlinear coupling originates from the intrinsic interaction between particles in the system. Here, we show a different mechanism for the nonlinear coupling that is due to the nontrivial anharmonic corrections to the trap.

In our case, qualitative results can be obtained by analyzing Eqs. (5) and (6). The oscillation of the center of the condensate couples with the motion of the width through the anharmonic parameter λ. It is noted that the coupled motion can be triggered by putting a small shift on the center of the condensate from its equilibrium point. The consequent motions of the width of the condensate excited by the shift are illustrated in Fig. 4 for different anharmonicity parameters. In all the above calculations, the initial shift is set as $\Delta\chi=0.1$. Figure 4 clearly shows the collapse and revival of oscillation patterns for the condensate width with respect to time, induced by the anharmonicity.

Moreover, the changes of collapse and revival are not monotonic with increasing λ, and the revival period can be effectively controlled by adjusting the anharmonicity param-

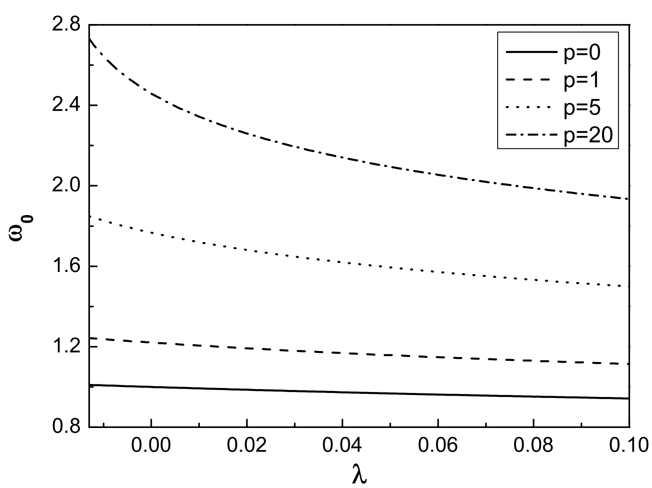

FIG. 2. Equilibrium width w_0 as functions of λ for different effective interaction p.

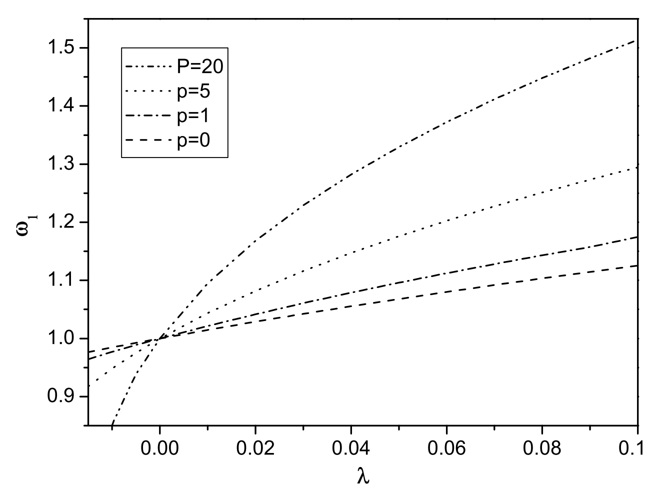

FIG. 3. Frequencies of mass center motions in anharmonic trap as functions of λ for BECs with different atom interactions p.

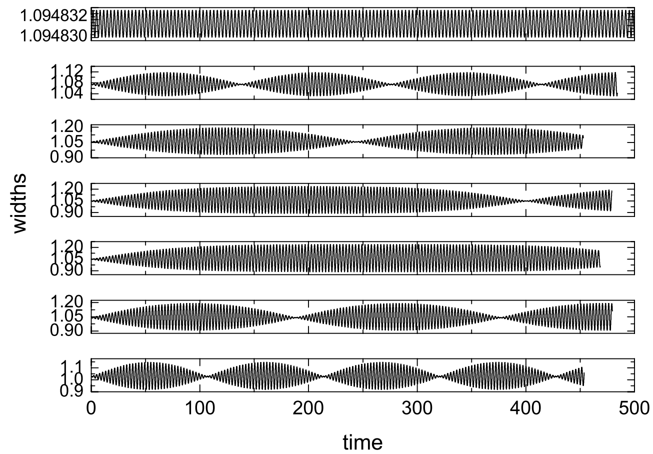

FIG. 4. Variations of the condensate width w for the parameter $p=0.4$ and different values of the anharmonic parameter λ. From top to bottom $\lambda=0, 0.02, 0.04, 0.05, 0.053, 0.06, 0.08$.

eter. In fact, by linearizing Eqs. (5) and (6) around the equilibrium point, we have $\ddot{\chi}'+\omega_1^2\chi'=0$ and $\ddot{w}'+\omega_2^2 w'=-6\lambda w_0 \chi'^2$. Obviously, the motion of the width behaves like a periodically driven oscillator. Since the frequency of the "external force" is $2\omega_1$, and the intrinsic frequency of the width oscillation is ω_2, the linear combination of the two frequencies gives the frequency of the collapse and revival, that is, $|2\omega_1-\omega_2|$. Detailed analysis also suggests that the frequency of the collapse and revival increases monotonically with the nonlinear parameter p, but is not monotonic with increasing λ. It will vanish for some parameters, e.g., at $\lambda \approx 0.053$ for $p=0.4$. This is confirmed by our numerical simulations, as in Fig. 4; around $\lambda \approx 0.053$ for $p=0.4$, the period of collapse and revival becomes much longer.

V. CONCLUSION

We have investigated collective excitations of a Bose-Einstein condensate in an anharmonic trap using variational approaches and obtained analytical expressions for the frequencies of the low-energy excitations. It is shown that the two low-energy excitation modes, corresponding to variations of the center and width of the condensate, couple with each other. The blueshift and redshift on the excitation frequency caused by the anharmonic distortion are revealed and found to be more dramatic in the case of strong atomic interaction. Furthermore, the collapse and revival of collective excitations in the anharmonic potential is discussed. We hope our theoretical results will stimulate experiments in this direction.

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