Prog. Theor. Phys. Vol. 101, No. 2, February 1999, Letters

Topological Structure in the $O(n)$ Symmetric Time-Dependent-Ginzburg-Landau Model

Yishi DUAN,[*] Tao XU[**] and Libin FU

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China

(Received September 11, 1998)

A new topological current in the $O(n)$ symmetrical time-dependent-Ginzburg-Landau model is discussed. Using ϕ-mapping theory, this topological current is studied. It is shown that the inner structure of topological defects can be classified by Brouwer degrees and Hopf indices of the ϕ-mapping.

The topology properties of physical systems play important roles in studying many physical problems. ϕ-mapping topological current theory is a powerful method to investigate the topological invariants and structure of physical systems. It has been used to study the topological current of monopoles,[1] topological string theory,[2] topological characteristics of dislocations and disclination continuums,[3,4] topological structure of defects of space-time in the early universe as well as its topological bifurcations,[5,6] and the topological structure of the Gauss-Bonnet-Chern Theorem.[7,8]

In the present paper, we investigate a new topological current in the $O(n)$ symmetrical time-dependent-Ginzburg-Landau (TDGL) model, which has been widely used in condensed matter physics.[9-11] We study the inner structure of the topological current using the method of ϕ-mapping, quantize the topological charge, and classify the topological defects by Brouwer degrees and Hopf indices of the ϕ-mapping. We shall see the inevitability of the topological current introduced in this paper, which gives all the important topological properties of the TDGL model, including the vortex density $\rho = \delta(\phi) D(\frac{\phi}{x})$ and other conjectures proposed by Liu and Mazenko.[12]

It is well-known that the time-dependent-Ginzburg-Landau model for a nonconserved n-component order parameter $\vec{\phi}(x,t) = (\phi^1(x,t), \cdots, \phi^n(x,t))$ is governed by the Langevin equation

$$\frac{\partial \vec{\phi}}{\partial t} = \vec{K} = -\Gamma \frac{\delta F}{\delta \phi} + \vec{\xi}, \tag{1}$$

where Γ is a kinetic coefficient, F is a Ginzburg-Landau effective free energy assumed to be of the form

$$F = \int d^n x \left[\frac{c}{2} (\nabla \phi)^2 + V\left(\left| \vec{\phi} \right| \right) \right], \tag{2}$$

with $c > 0$, and the potential is assumed to be of the degenerated double-well form.

---

*) E-mail: ysduan@lzu.edu.cn

**) Corresponding author. E-mail: itp5@lzu.edu.cn

Also, $\vec{\xi}$ is a thermal noise, which is related to Γ by the Fluctuation-Dissipation Theorem.

We consider a system in which there are topologically stable point defects. A unit vector is defined as

$$n^a = \frac{\phi^a}{\|\phi\|}, \quad \|\phi\|^2 = \phi^a \phi^a, \quad a = 1, \cdots, n$$

for discussion of this physical system. It can be proved that there exists a topological current

$$j^\mu = \frac{1}{A\left(S^{n-1}\right)(n-1)!} \epsilon^{\mu\mu_1,\cdots,\mu_n} \epsilon_{a_1,\cdots,a_n} \partial_{\mu_1} n^{a_1}, \cdots, \partial_{\mu_n} n^{a_n}, \tag{3}$$

where $A\left(S^{n-1}\right)$ is the area of the $(n-1)$-dimensional unit sphere S^{n-1}:

$$A(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

It is clear that this topological current is identically conserved, i.e.,

$$\partial_\mu j^\mu = 0. \tag{4}$$

This continuity equation reflects the fact that the topological charge, i.e. the vortex charge, is conserved. From Eq. (3), we can define

$$j^0 = \rho$$

as the topological charge, i.e. the vortex charge density. It must be pointed out that the topological current introduced above exists naturally and inevitably in the TDGL model.

Using the identities

$$\partial_\mu n^a = \frac{1}{\phi} \partial_\mu \phi^a + \phi^a \partial_\mu \frac{1}{\phi},$$

$$\frac{\partial}{\partial \phi^a} \frac{1}{\phi} = -\frac{\phi^a}{\phi^3},$$

we can write Eq. (3) as

$$j^\mu = C_n \epsilon^{\mu\mu_1,\cdots,\mu_n} \epsilon_{a_1,\cdots,a_n} \partial_{\mu_1} \phi^a, \partial_{\mu_2} \phi^{a_2}, \cdots, \partial_{\mu_n} \phi^{a_n} \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^{a_1}} (G_n(||\phi||)),$$

where C_n is a constant,

$$C_n = \begin{cases} -\frac{1}{A(S^{n-1})(n-2)(n-1)!} & \text{for} \quad n > 2, \\ \frac{1}{2\pi} & \text{for} \quad n = 2, \end{cases}$$

and $G_n(||\phi||)$ is a generalized function,

$$G_n(||\phi||) = \begin{cases} \frac{1}{||\phi||^{n-2}} & \text{for} \quad n > 2, \\ \ln ||\phi|| & \text{for} \quad n = 2. \end{cases}$$

We define the $(n+1)$-dimensional Jacobian as

$$\epsilon^{a_1,\cdots,a_n} D^\mu\left(\frac{\phi}{x}\right) = \epsilon^{\mu\mu_1,\cdots,\mu_n}\partial_{\mu_1}\phi^{a_1}\partial_{\mu_2}\phi^{a_2},\cdots,\partial_{\mu_n}\phi^{a_n},$$

and $D^0(\frac{\phi}{x})$ is just the usual n-dimensional Jacobian determinant,

$$D^0\left(\frac{\phi}{x}\right) = D\left(\frac{\phi}{x}\right) = \frac{\partial(\phi^1,\cdots,\phi^n)}{\partial(x^1,\cdots,x^n)}. \tag{5}$$

We obtain the δ-function-like current

$$j^\mu = \delta\left(\phi\right) D^\mu\left(\frac{\phi}{x}\right) \tag{6}$$

via the formula of n-dimensional Laplacian Green's function in ϕ-space,

$$\triangle_\phi(G_n(||\phi||)) = -\frac{4\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)}\delta(\vec{\phi}),$$

where $\triangle_\phi$ is the n-dimensional Laplacian operator. Then, the density of the topological current j^μ can be written as

$$\rho = \delta(\phi)D\left(\frac{\phi}{x}\right). \tag{7}$$

This is the important conjecture of Liu and Mazenko. [12] From the above expressions and Eq. (6) the continuity Eq. (4) can be written as

$$\frac{\partial\rho}{\partial t} + \partial_a\left[\delta(\phi)D^a\left(\frac{\phi}{x}\right)\right] = 0, \qquad a = 1,\cdots,n. \tag{8}$$

From Eqs. (7) and (8), we see clearly that $j^\mu \neq 0$, and $\rho \neq 0$ only when $\vec{\phi} = 0$, which implies the singularity of the topological current j^μ. The inner structure of the topological current is labelled by the zeroes of $\vec{\phi}$. Thus, we search for the solutions of the equations

$$\phi^a\left(x\right) = 0, \qquad a = 1,\cdots,n, \tag{9}$$

by means of the Implicit Function Theorem, and further give the dynamic form of the topological current j^μ. Suppose the function $\phi^a\left(x\right)$ possesses l zeroes. One can clearly see that these zeroes are regular points of the ϕ-mapping and that the Jacobian satisfies

$$D\left(\frac{\phi}{x}\right) \neq 0. \tag{10}$$

There is one and only one system of continuous functions of $x^0 = t$,

$$\vec{x} = \vec{z}_i(t), \qquad i = 1,\cdots,l,$$

which is the trajectory of the i-th singular manifold L_i in X. Let M be a submanifold in X, and let M_i be a neighborhood of $\vec{z}_i\left(t\right) = \left(t, \vec{z}_i\left(t\right)\right)$ on M with boundary

∂M_i. Then, the generalized winding number can be defined by the Gauss map $n : \partial M_i \to S^{n-1}$

$$W_i = \frac{1}{A\left(S^{n-1}\right)(n-1)!}\int_{\partial M_i} n^*(\epsilon_{a_1,\cdots,a_n} n^{a_1} dn^{a_2}\wedge,\cdots,\wedge dn^{a_n}). \tag{11}$$

Here, n^* denotes the pull-back of map n. In topological terms, this means that, when the point x covers ∂M_i once, the unit vector $n^a(x)$ will cover the unit sphere S^{n-1}, or $\phi^a(x)$ covers the corresponding region W_i times, which is a topological invariant and is also called the degree of the Gauss map. It is well-known that W_i corresponds to the first homotopy group $\pi_1(S^{n-1}) = Z$.

Using Stokes' Theorem in exterior differential form and Eq. (11), we obtain

$$W_i = \int_{M_i} \rho d^n x. \tag{12}$$

It is well-known from ordinary theory of the δ function that

$$\delta(\phi) = \frac{\sum_{i=1}^{l} \delta(x - z_i)}{\left|D(\frac{\phi}{x})\right|_{\vec{x}=\vec{z}_i}}.$$

Consider the case that while x covers the region neighboring the zero point $\vec{z}_i$ once, the function $\vec{\phi}$ covers the corresponding region β_i times. Then we obtain [8)]

$$V^\mu = \frac{dz_i^\mu}{dt} = \frac{D^\mu(\frac{\phi}{x})}{D(\frac{\phi}{x})}\Big|_{x=z_i} \tag{13}$$

and

$$j^\mu = \sum_{i=1}^{l} \beta_i \eta_i \delta(x - z_i)\frac{dz_i^\mu}{dt}, \qquad \mu = 1,\cdots,n, \tag{14}$$

where β_i is a positive integer (the Hopf index of the i-th zeroes) and η_i is the Brouwer degree: [13), 14)]

$$\eta_i = \frac{D(\phi/x)}{|D(\phi/x)|} = \mathrm{sgn}[D(\phi/x)]|_{x=z_i} = \pm 1.$$

From the useful expression (14), we see clearly that the j^μ are exactly the components of the current of l classical point particles with topological charge $g_i = \beta_i \eta_i$ moving in $(n+1)$-dimensional space-time. The topological defects may be generated or annihilated from the zeroes of the order parameter $\vec{\phi}(x,t)$. The zeroes of $\vec{\phi}(x,t)$ map out these vortex lines, i.e. the lines of topological defects are zeroes of the n-component order parameter vector fields.

We define the general velocity

$$V^\mu = \frac{dx^\mu}{dt}, \qquad V^0 = 1$$

and substitute this expression into (6) to rewrite the topological current in a simple and compact form:

$$j^\mu = \rho V^\mu. \tag{15}$$

The expression (13) is very useful because it avoids specifying the positions of vortices explicitly. The positions are implicitly determined by the zeroes of the order parameter field.

The total topological charge of the system is given by

$$G = \int_M j^0 d^n x = \sum_{i=1}^{l} \beta_i \eta_i = \sum_{i=1}^{l} W_i \ . \tag{16}$$

It is clear that the inner structure of this system is characterized by Hopf indices β_i and Brouwer degrees η_i, which are topological invariants. From (8), the total topological charge can be also expressed as

$$G = \int_M \rho d^n x = \deg \phi \int_{\phi(M)} \delta(\phi) d^n \phi = \deg \phi. \tag{17}$$

We see that the total topological charge (i.e. total vortex charge G) is equal to the degree of the ϕ-mapping $\deg \phi$, and from (16) we have the obvious result that $\deg \phi = \sum_{i=1}^{l} \beta_i \eta_i$, i.e. the degree of the ϕ-mapping is equal to the sum of the indices of the vector $\vec{\phi}$ at its zeroes. This fact is closely related to the Poincaré-Hopf Theorem for the Euler characteristic of a compact manifold which is determined by the model. Therefore we can classify topological defects (i.e. vorticies) by Brouwer degrees and Hopf indices.

Acknowledgements

This work was supported by the Foundation of National Nature Science Foundation of China.

1) Y. S. Duan and M. L. Ge, Sci. Sinica **11** (1979), 1072.
C. Gu, Phys. Rep. **C80** (1981), 251.
2) Y. S. Duan and J. C. Liu, in *Proceedings of Johns Hopkins Workshop 11* (World Scientific, Singapore, 1988).
3) Y. S. Duan and S. L. Zhang, Int. J. Eng. Sci. **28** (1990), 689.
4) Y. S. Duan and S. L. Zhang, Int. J. Eng. Sci. **30** (1992), 153.
5) Y. S. Duan, S. L. Zhang and S. S. Feng, J. Math. Phys. **35** (1994), 1.
6) Y. S. Duan, G. H. Yang and Y. Jiang, Gen. Relat. Gravit. **29** (1997), 715.
7) Y. S. Duan and X. H. Meng, J. Math. Phys. **34** (1993), 1149.
Y. S. Duan and X. G. Lee, Helv. Phys. Acta. **68** (1995), 513.
8) Y. S. Duan, Sheng Li and Guohong Yang, Nucl. Phys. **B514** (1998), 705.
9) T. J. Newmann, A. J. Bray and M. A. Moore, Phys. Rev. **B42** (1990), 4514.
10) M. Mondello and N. Goldenfield, Phy. Rev. **A42** (1990), 5865; **A45** (1992), 657.
11) F. Liu and G. F. Mazenko, Phys. Rev. **B45** (1992), 6989; **B46** (1992), 5963.
12) F. Liu and G. F. Mazenko, Phys. Rev. Lett. **78** (1997), 401.
13) J. M. Milnor, *Topology from the differential viewpoint* (Univ. of Virginia press, Charlottesville, VA, 1965).
14) B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry-methods and Applications* (Springer -Verlag, New York, 1984).