Available online at www.sciencedirect.com
SCIENCE DIRECT®
Annals of Physics 318 (2005) 419–431
ANNALS of PHYSICS
www.elsevier.com/locate/aop

Various topological excitations in the $SO(4)$ gauge field in higher dimensions

Xin Liu [a,*], Yao-zhong Zhang [a], Yi-shi Duan [b], Li-bin Fu [c]

[a] *Department of Mathematics, University of Queensland, Qld 4072, Australia*
[b] *Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, PR China*
[c] *Max-Planck-Institute for the Physics of Complex Systems, Nöthnitzer Strasse 38, 01187 Dresden, Germany*

Received 24 November 2004; accepted 12 February 2005
Available online 11 March 2005

Abstract

Following the original analysis of Zhang and Hu for the 4-dimensional generalization of Quantum Hall effect, there has been much work from different viewpoints on the higher dimensional condensed matter systems. In this paper, we discuss three kinds of topological excitations in the $SO(4)$ gauge field of condensed matter systems in 4-dimension—the instantons and anti-instantons, the 't Hooft–Polyakov monopoles, and the 2-membranes. Using the ϕ-mapping topological theory, it is revealed that there are 4-, 3-, and 2-dimensional topological currents inhering in the $SO(4)$ gauge field, and the above three kinds of excitations can be directly and explicitly derived from these three kinds of currents, respectively. Moreover, it is shown that the topological charges of these excitations are characterized by the Hopf indices and Brouwer degrees of ϕ-mapping.

PACS: 02.40.−k; 11.15.−q; 73.43.Cd

Keywords: Topological excitations; *SO*(4) gauge field; Higher dimensional condensed matter physical systems

* Corresponding author. Fax: +61 7 3365 1477.
E-mail address: xinliu@maths.uq.edu.au (X. Liu).

0003-4916/$ - see front matter
doi:10.1016/j.aop.2005.02.001

1. Introduction

In 2001, Zhang and Hu [1] constructed a remarkable 4-dimensional generalization of the Quantum Hall (QH) effect, which reveals the interesting property of 4-dimensional QH system that the ground state is separated from all excited states by a finite energy gap, and the density correlation functions decay gaussianly [2–4]. This generalized higher dimensional system was analyzed from many viewpoints and extended in different directions [5,6], including the realization of this system within string theory [7], the connection between this system and the twistor theory [8], the generalization of the QH effect onto the $\mathbf{CP}_n$ manifolds [9,10], the matrix descriptions of even dimensional fuzzy spherical branes in matrix theory [11], and the relationship between the 4-dimensional QH liquid and the non-commutative geometry on S^4 [12]. More recently, based on this higher dimensional theory, the dissipationless quantum spin current in hole-doped semiconductors at room temperature with strong spin–orbit coupling was predicted theoretically (which was an important progress achieved in the research of spintronics) [13,14], and the theoretical framework and experimental realization of the higher dimensional Bose–Einstein condensates were also discussed in [15,16].

In this paper, we will focus on the topological excitations in the generalized higher dimensional systems. In [3,4,7], it has been pointed out that in the higher dimensional condensed matter physical systems there exist various topological excitations, including the $SU(2)$ instantons (i.e., the Yang monopoles [17] which are obtained by the second Hopf mapping: $S^7 \to S^4$), and the membranes of different dimensions. With respect to the fact that the topological excitations are in themselves the singularities on the manifold, we hope to be able to derive these excitations directly and explicitly from the geometric distributions of the physical basic fields on the base manifold. Adopting this viewpoint, in this paper we will use the ϕ-mapping topological theory [18–23] to discuss three kinds of topological excitations in the $SO(4)$ gauge field of 4-dimensional condensed matter physical systems—the instantons and anti-instantons [24], the 't Hooft–Polyakov monopoles [25,26], and the 2-membranes [3,4]. In Section 2, based on the group theoretical relation $SO(4) = SU(2) \otimes SU(2)$, the $SO(4)$ gauge field is fractionalized into two parts, $SU(2)_+$ and $SU(2)_-$. In Section 3, it is revealed that there is a 4-dimensional topological current inhering in the $SU(2)_\pm$ gauge field, while the instanton and anti-instanton structures are directly derived from this topological current, whose topological charges underlie the second Chern number and the signature of the 4-dimensional manifold. In Section 4, we first generalize the 't Hooft–Polyakov theory so as to obtain the $U(1)$ subfield tensor in the $SO(4)$ gauge field, and then show that from this $U(1)$ tensor one can derive both the 3- and 2-dimensional topological currents, and the 't Hooft–Polyakov monopoles and 2-membranes exist, respectively, in these two kinds of topological currents. Moreover, it is shown that the topological charges of the three kinds of excitations are all characterized by the Hopf indices and Brouwer degrees of ϕ-mapping.

2. The fractionalization of the $SO(4)$ gauge field

The symmetry group of the generalized 4-dimensional QH system is $SO(5)$. This symmetry is broken to $SO(4)$ under the given confining potential [1,2]. In this paper, we will emphasize on this $SO(4)$ gauge field of higher dimensional condensed matter systems. In this section, based on the group theoretical relation $SO(4) = SU(2) \otimes SU(2)$, we discuss the fractionalization of $SO(4)$ field by employing the $SO(4)$ generators to construct two $SU(2)$ generator sets, and then using them to rewrite the $SO(4)$ field into two $SU(2)$ sub-field parts.

Let the base $\mathscr{M}$ be a compact oriented 4-dimensional manifold with metric $g_{\mu\nu}(\mu,\nu = 0,1,2,3)$, and $\mathscr{P}(\mathscr{M}, SO(4), \pi)$ the principal $SO(4)$ bundle on $\mathscr{M}$. The Dirac 4-spinor wave function $\Psi(x)$ is the section of the associated bundle $\mathscr{P}\times_{SO(4)}\mathbf{C}^4$: $\Psi = (\,\Phi_1\ \Phi_2\,)^{\mathrm{T}}$, where $\Phi_1(x)$ and $\Phi_2(x)$ are two 2-spinors. The covariant derivative of $\Psi(x)$ is defined as $D\Psi = d\Psi - \omega\Psi$, with ω the $SO(4)$ gauge potential, i.e., the connection of $\mathscr{P} : \omega = \frac{1}{2}\omega^{\alpha\beta} I_{\alpha\beta}$. The $SO(4)$ gauge field tensor F is given by

$$F = d\omega - \omega \wedge \omega = \tfrac{1}{2}F^{\alpha\beta} I_{\alpha\beta}. \tag{1}$$

Here, $\alpha,\beta = 1,2,3,4$ are the $SO(4)$ group indices, and $I_{\alpha\beta}$ is the $SO(4)$ generator which satisfies the Lie bracket $[I_{\gamma\delta}, I_{\lambda\rho}] = C^{\alpha\beta}_{\gamma\delta\lambda\rho} I_{\alpha\beta}$, with $C^{\alpha\beta}_{\gamma\delta\lambda\rho}$ the structure constant: $C^{\alpha\beta}_{\gamma\delta\lambda\rho} = \delta_{\gamma\rho}\delta^{\alpha}_{\delta}\delta^{\beta}_{\lambda} + \delta_{\delta\lambda}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\rho} - \delta_{\gamma\lambda}\delta^{\alpha}_{\delta}\delta^{\beta}_{\rho} - \delta_{\delta\rho}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\lambda}$. $I_{\alpha\beta}$ can be given in terms of the 4×4 γ-matrices: $I_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha, \gamma_\beta]$, where γ_α's satisfy the definition of Clifford algebra: $\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} I$. A realization of $\gamma^{1,2,3,4}$ and the corresponding γ^5 in the Kramers form is given by

$$\gamma^a = \begin{pmatrix} 0 & -\mathrm{i}\sigma^a \\ \mathrm{i}\sigma^a & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2}$$

where σ^a $(a = 1,2,3)$ are the Pauli matrices. It can be seen that this γ^5-matrix is chiral with eigenvalues $+1$ and -1, corresponding to which the eigenstates are denoted as $(\,\Phi_+\ 0\,)^{\mathrm{T}}$ and $(\,0\ \Phi_-\,)^{\mathrm{T}}$, respectively, where Φ_+ and Φ_- are two 2-spinors. (The topology of $\Phi_\pm$ will be discussed in Section 3.)

From the $SO(4)$ generator $I_{\alpha\beta}$, one can construct other two sets of basis, I^a_+ and I^a_-:

$$I^a_+ = -\tfrac{1}{2}\left(\tfrac{1}{2}\epsilon_{abc} I_{bc} + I_{a4}\right), \quad I^a_- = -\tfrac{1}{2}\left(\tfrac{1}{2}\epsilon_{abc} I_{bc} - I_{a4}\right) \quad (a,b,c = 1,2,3). \tag{3}$$

It can be proved that I^a_+ and I^a_- are actually

$$I^a_+ = \begin{pmatrix} \frac{\sigma^a}{2\mathrm{i}} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$I^a_- = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma^a}{2\mathrm{i}} \end{pmatrix},$$

which satisfy the commutation relations

$$[I^a_\pm, I^b_\pm] = \epsilon_{abc} I^c_\pm, \quad [I^a_+, I^b_-] = 0, \tag{4}$$

where no sum is assumed for "±". (4) means I_+^a's and I_-^a's are, respectively, the generator sets of two separate $SU(2)$ sub-groups in the $SO(4)$ group, therefore the $so(4)$ Lie algebra is fractionalized into two $su(2)$ sub-algebras, $so(4) = su(2) \oplus su(2)$. In the following, one denotes the two $SU(2)$ sub-groups, which, respectively, correspond to I_+^a and I_-^a, as $SU(2)_+$ and $SU(2)_-$, and their principal bundles on $\mathscr{M}$ as $\mathbf{P}_\pm \equiv \mathbf{P}(\mathscr{M}, SU(2)_\pm, \pi)$.

Next, we would express the $SO(4)$ gauge field in two $SU(2)$ parts. It can be proved that for an arbitrary $so(4)$ vector $u = \frac{1}{2}u^{\alpha\beta}I_{\alpha\beta} = \frac{1}{2}u^{ab}I_{ab} + u^{a4}I_{a4}$, u can be written as an $su(2)_+$ vector plus an $su(2)_-$ one: $u = u_+ + u_- = u_+^a I_+^a + u_-^a I_-^a$, where $u_\pm^a = u^a \pm u^{a4} = \frac{1}{2}\epsilon^{abc}u_{bc} \pm u^{a4}$. Therefore, the $SO(4)$ gauge potential ω, which is an $so(4)$ vector, can be written in two parts: $\omega = \omega_+ + \omega_-$ with $\omega_\pm = \omega_\pm^a I_\pm^a$, while the $SO(4)$ field tensor F is also written in two $SU(2)$ parts:

$$F = F_+ + F_-, \tag{5}$$

where$F_\pm = d\omega_\pm - [\omega_\pm, \omega_\pm] = F_\pm^a I_\pm^a$. According to the topological theory of 4-dimensional manifolds [24], the $SU(2)_\pm$ field tensor F_+ and F_- should satisfy the self- and antiself-duality equations, respectively:

$$^*F_+ = F_+, \quad ^*F_- = -F_-, \tag{6}$$

where $^*F_\pm$ is the Hodge dual tensor of $F_\pm$.

Therefore, the $SO(4)$ field tensor F is divided into two $SU(2)$ parts, F_+ and F_-. In the following sections, various kinds of topological excitations in the $SO(4)$ gauge field will be derived from these $SU(2)$ sub-field tensors F_+ and F_-.

3. The instantons and anti-instantons

In this section, it is shown that there is a 4-dimensional topological current inhering in the second Chern classes in the $SO(4)$ gauge field, and the instanton and anti-instanton structures are derived from this 4-dimensional topological current.

Since the base manifold $\mathscr{M}$ is 4-dimensional, in the study of its topology we will begin with the topological signature. It is known that on the compact oriented 4-manifold $\mathscr{M}$, under the action of Hodge star, the space of harmonic 2-forms $H^2(\mathscr{M};\mathbf{R})$ divides into a direct sum decomposition: $H^2(\mathscr{M};\mathbf{R}) = H_+^2(\mathscr{M};\mathbf{R}) \oplus H_-^2(\mathscr{M};\mathbf{R})$, where $H_+^2(\mathscr{M};\mathbf{R})$ and $H_-^2(\mathscr{M};\mathbf{R})$ consist of self- and antiself-dual harmonic 2-forms, respectively [24]. Let $b_+ = \dim H_+^2(\mathscr{M};\mathbf{R})$ and $b_- = \dim H_-^2(\mathscr{M};\mathbf{R})$, one has the second Betti number of $\mathscr{M}$ given by $b_2 = b_+ + b_-$, and the so-called topological signature of 4-manifold $\mathscr{M}$ defined as [24,27,28]

$$\tau(\mathscr{M}) = b_+ - b_-. \tag{7}$$

In topology, $\tau(\mathscr{M})$ is a topological invariant which plays an important role in the classification of 4-dimensional manifolds.

The Hirzebruch theorem shows that $\tau(\mathscr{M})$ can be given by the characteristic classes on $\mathscr{M}$ [24]

$$\tau(\mathscr{M}) = \frac{1}{3}\int_{\mathscr{M}} P_1(\mathscr{M}) = -\frac{1}{24\pi^2}\int_{\mathscr{M}} \mathrm{Tr}(F \wedge F), \tag{8}$$

where $P_1(\mathscr{M})$ is the first Pontrjagin class of $\mathscr{M}$, $P_1(\mathscr{M}) = -\frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F)$. From (5) and (4), one sees that $\tau(\mathscr{M})$ can be written in two $SU(2)$ parts

$$\tau(\mathscr{M}) = -\frac{1}{24\pi^2}\int_{\mathscr{M}} \mathrm{Tr}(F_+ \wedge F_+) - \frac{1}{24\pi^2}\int_{\mathscr{M}} \mathrm{Tr}(F_- \wedge F_-). \tag{9}$$

Noticing the definition of the second Chern class of $\mathbf{P}_\pm$: $C_2(\mathbf{P}_\pm) = \frac{1}{8\pi^2}\int_{\mathscr{M}}\mathrm{Tr}(F_\pm \wedge F_\pm)$, the expression (9) is just

$$\tau(\mathscr{M}) = -\frac{1}{3}\int_{\mathscr{M}} C_2(\mathbf{P}_+) - \frac{1}{3}\int_{\mathscr{M}} C_2(\mathbf{P}_-) = -\frac{1}{3}c_{2+} - \frac{1}{3}c_{2-}, \tag{10}$$

where $c_{2\pm} = \int_{\mathscr{M}} C_2(\mathbf{P}_\pm)$ is the second Chern number. In the following, we would reveal the inner structure of the signature $\tau(\mathscr{M})$, and derive the instanton and anti-instanton structures by studying the topology of the physical basic fields on $\mathscr{M}$.

The basic field $\Phi_\pm(x)$ is the 2-spinor wave function in the $SU(2)_\pm$ sub-field, i.e., the section of the associated bundle $\mathbf{P}_\pm \times_{SU(2)_\pm} \mathbf{C}^2$: $\Phi_\pm = \left(\phi^0_\pm + \mathrm{i}\phi^1_\pm \;\; \phi^2_\pm + \mathrm{i}\phi^3_\pm\right)^{\mathrm{T}}$, where $\phi^A_\pm \in \mathbf{R}$ $(A = 0,1,2,3)$, with $\phi^A_\pm\phi^A_\pm = \|\phi_\pm\|^2 = \Phi^\dagger_\pm\Phi_\pm$. From $\phi^A_\pm$ a unit vector $m^A_\pm$ is introduced: $m^A_\pm = \phi^A_\pm/\|\phi_\pm\|$ with $m^A_\pm m^A_\pm = 1$. It is seen that the zero points of $\phi^A_\pm$ are just the singular points of $m^A_\pm$. From $\Phi_\pm$, we introduce a normalized spinor

$$\hat{\Phi}_\pm\colon\ \hat{\Phi}_\pm = \frac{1}{\sqrt{\Phi^\dagger_\pm\Phi_\pm}}\Phi_\pm = \left(m^0_\pm + \mathrm{i}m^1_\pm \;\; m^2_\pm + \mathrm{i}m^3_\pm\right)^{\mathrm{T}}.$$

It has been proved in our previous work [20,22] that when specially choosing $\hat{\Phi}_\pm$ as a parallel field which satisfies $D_\mu\hat{\Phi}_\pm = 0$, one can express the second Chern class $C_2(\mathbf{P}_\pm)$ as

$$C_2(\mathbf{P}_\pm) = \frac{1}{12\pi^2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ABCD}\partial_\mu m^A_\pm\partial_\nu m^B_\pm\partial_\lambda m^C_\pm\partial_\rho m^D_\pm \mathrm{d}^4x, \tag{11}$$

where $\mu,\nu,\lambda,\rho = 0,1,2,3$ denote the base $\mathscr{M}$. According to the ϕ-mapping topological theory, the 4-dimensional topological current is defined as [23]

$$J_\pm = \frac{1}{12\pi^2\sqrt{g}}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ABCD}\partial_\mu m^A_\pm\partial_\nu m^B_\pm\partial_\lambda m^C_\pm\partial_\rho m^D_\pm, \tag{12}$$

where $g = \det(g_{\mu\nu})$, therefore $C_2(\mathbf{P}_\pm) = J_\pm\sqrt{g}\,\mathrm{d}^4x$.

Making use of the Green function relation in $\phi_\pm$-space: $\partial_A\partial_A(1/\|\phi_\pm\|^2) = -4\pi^2\delta^4(\vec{\phi}_\pm)$, one proves that $J_\pm$ can be written in the δ-function form

$$J_\pm = \frac{1}{\sqrt{g}}\delta^4(\vec{\phi}_\pm)D\left(\frac{\phi_\pm}{x}\right), \tag{13}$$

where $D(\phi_\pm/x)$ is the Jacobi determinant, $\epsilon^{ABCD}D(\phi_\pm/x) = \epsilon^{\mu\nu\lambda\rho}\partial_\mu\phi^A_\pm\partial_\nu\phi^B_\pm\partial_\lambda\phi^C_\pm\partial_\rho\phi^D_\pm$. The expression (13) provides the important conclusion:

$$J_\pm \begin{cases} = 0 & \text{iff } \vec{\phi}_\pm \neq 0, \\ \neq 0 & \text{iff } \vec{\phi}_\pm = 0, \end{cases} \tag{14}$$

so it is necessary to study the zero points of $\vec{\phi}_\pm$ to determine the non-zero solutions of $J_\pm$. The implicit function theory shows [29] that under the regular condition

$D(\phi_\pm/x) \neq 0$, the general solutions of the zero point equations $\phi^A(x^0, x^1, x^2, x^3) = 0$ $(A = 0, 1, 2, 3)$ can be expressed as

$$x^\mu = x^\mu_{j\pm} \quad (\mu = 0, 1, 2, 3;\ j = 1, \ldots, M_\pm), \tag{15}$$

which represents $M_\pm$ isolated 4-dimensional singular points on base $\mathscr{M}$. Since the $SU(2)$ sub-field tensors F_+ and F_-, respectively, satisfy the self- and antiself-duality equations (6), these 4-dimensional singular point solutions are M_+ instantons and M_- anti-instantons in the $SO(4)$ gauge field.

Furthermore, in δ-function theory [30] it can be proved that

$$J_\pm = \frac{1}{\sqrt{g}} \sum_{j=1}^{M_\pm} \beta_{j\pm} \eta_{j\pm} \delta^4(x^\mu - x^\mu_{j\pm}), \tag{16}$$

where $\eta_{j\pm}$ is the Brouwer mapping degree, $\eta_{j\pm} = \mathrm{sign}[D(\phi_\pm/x)]_{x^\mu_{j\pm}} = 1, -1$; and $\beta_{j\pm}$ a positive integer which is called the topological Hopf index (which means that when x^μ covers the neighborhood of $x^\mu_{j\pm}$ once, $\phi^A_\pm$ covers the corresponding region in the $\phi_\pm$-space $\beta_{j\pm}$ times). In (16), $\beta_{j\pm}\eta_{j\pm}$ forms the topological charges of the instantons and anti-instantons. Then, using (10) and (16) one obtains the second Chern numbers

$$c_{2\pm} = \int_{\mathscr{M}} J_\pm \sqrt{g}\, \mathrm{d}^4 x = \sum_{j=1}^{M_\pm} \beta_{j\pm} \eta_{j\pm} \tag{17}$$

and the topological signature

$$\tau(\mathscr{M}) = -\frac{1}{3} \sum_{j=1}^{M_+} \beta_{j+} \eta_{j+} - \frac{1}{3} \sum_{j=1}^{M_-} \beta_{j-} \eta_{j-}. \tag{18}$$

The expressions (16)–(18) reveal the inner structure of the second Chern class, Chern number, and the topological signature, which mean that the topological effect of the second Chern class is indeed caused by the inherent instantons and anti-instantons with topological charges $\beta_{j\pm}\eta_{j\pm}$, while the Chern number, and then the signature $\tau(\mathscr{M})$, is just characterized by the topological numbers of these instantons and anti-instantons, $\beta_{j\pm}$ and $\eta_{j\pm}$.

It should be addressed that according to the Atiyah–Singer index theorem [24], the numbers of the instantons and anti-instantons, M_+ and M_-, should be determined by the self- and antiself-duality equations (6). The relationship between $M_\pm$ and the expression (6) will be discussed in detail in our further work.

4. The ’t Hooft–Polyakov monopoles and 2-membranes

In this section, the ’t Hooft–Polyakov monopoles and 2-membranes induced by the $U(1)$ sub-field tensor in the $SO(4)$ gauge field are discussed.

’t Hooft and Polyakov proposed [25,19] that the $U(1)$ sub-field tensor in $SU(2)$ gauge field, $f^{SU(2)}$, is written as

$$f^{SU(2)} = (F^{SU(2)}, n^{SU(2)}) + (n^{SU(2)}, [Dn^{SU(2)}, Dn^{SU(2)}]), \tag{19}$$

where $F^{SU(2)}$ is the $SU(2)$ field tensor, and $n^{SU(2)}$ is the unit vector of $su(2)$ Lie algebra. Here, we will generalize this theory to discuss the $U(1)$ sub-field tensor in $SO(4)$ gauge field. First, it is known that in the $so(4)$ Lie algebra space, the inner product between two arbitrary $so(4)$ vectors, say, $u = \frac{1}{2}u^{\alpha\beta}I_{\alpha\beta}$ and $v = \frac{1}{2}v^{\alpha\beta}I_{\alpha\beta}$, is given by: $(u, v) = \frac{1}{4}g_{\alpha\beta\gamma\delta}u^{\alpha\beta}v^{\gamma\delta} = \frac{1}{2}u^{\alpha\beta}v^{\alpha\beta}$, where $g_{\alpha\beta\gamma\delta}$ is the metric tensor of $SO(4)$ group manifold defined from the structure constant $C^{\alpha\beta}_{\gamma\delta\lambda\rho}$: $g_{\alpha\beta\gamma\delta} = \frac{1}{4}C^{\lambda\rho}_{\alpha\beta\sigma\tau}C^{\sigma\tau}_{\lambda\rho\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$. Using (4), the inner product is expressed as two parts: $(u, v) = \frac{1}{2}(u^a_+v^a_+ + u^a_-v^a_-)$. Then, generalizing (19), one can define the $U(1)$ sub-field tensor in the $SO(4)$ gauge field as

$$f_\pm = (F, n_\pm) + (n_\pm, [Dn_\pm, Dn_\pm]), \tag{20}$$

where $n_\pm$ is the unit vector in $SU(2)_\pm$ sub-group space, $n^a_\pm = \hat{\Phi}^\dagger_\pm I_a \hat{\Phi}_\pm$ $(a = 1, 2, 3)$ with $n^a_\pm n^a_\pm = 1$; and $Dn_\pm = dn_\pm - [\omega_\pm, n_\pm]$ is the covariant derivative of $n_\pm$. Noticing the definitions $F^a_\pm = d\omega^a_\pm + \epsilon^{abc}\omega^b_\pm\omega^c_\pm$ and $Dn^a_\pm = dn^a_\pm + \epsilon^{abc}\omega^b_\pm n^c_\pm$, (20) becomes

$$f_{\pm\mu\nu} = (\partial_\mu A_{\pm\nu} - \partial_\nu A_{\pm\mu}) - K_{\pm\mu\nu} \tag{21}$$

with

$$K_{\pm\mu\nu} = \epsilon_{abc}n^a_\pm\partial_\mu n^b_\pm\partial_\nu n^c_\pm, \tag{22}$$

where $A_{\pm\mu} = \omega^a_{\pm\mu}n^a_\pm$ is a $U(1)$ gauge potential, and $K_{\pm\mu\nu}$ a topological term describing the non-uniform distribution of $n^a_\pm$ at large distance (see [19] and references therein). Since in the 4-dimensional space there exist different kinds of non-uniform distributions at large distance for unit vector $n^a_\pm$, there are different topological excitations which can be derived from the topological term $K_{\pm\mu\nu}$. In the following, we will discuss the 't Hooft–Polyakov monopole and 2-membrane structures from the $K_{\pm\mu\nu}$ tensor.

4.1. The 't Hooft–Polyakov monopoles

In this subsection, it is shown that there is a 3-dimensional topological current which can be derived from $K_{\pm\mu\nu}$, and the monopole structures are inhering in the 3-dimensional topological current.

In the classical electromagnetic theory, the Maxwell equations are

$$\partial^\nu f^{U(1)}_{\mu\nu} = 4\pi j^E_\mu, \quad \partial_\nu {}^*f^{\mu\nu}_{U(1)} = 0, \tag{23}$$

where $f^{U(1)}_{\mu\nu}$ is the classical $U(1)$ electromagnetic field tensor with j^E_μ the electric current, and ${}^*f^{\mu\nu}_{U(1)}$ is the dual tensor of $f^{U(1)}_{\mu\nu}$: ${}^*f^{\mu\nu}_{U(1)} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}f^{U(1)}_{\lambda\rho}$. The second equation in (23) is known as the Bianchi identity. To include the monopoles in the theory, 't Hooft and Polyakov generalized the electromagnetic field to $SU(2)$ and proposed a $U(1)$ sub-field tensor in $SU(2)$ gauge field, i.e., the tensor $f^{SU(2)}_{\mu\nu}$ in (19) [25]. In terms of $f^{SU(2)}_{\mu\nu}$, the Maxwell equations become

$$\partial^\nu f^{SU(2)}_{\mu\nu} = 4\pi j^E_\mu, \quad \partial_\nu {}^*f^{\mu\nu}_{SU(2)} = -4\pi j^\mu_M, \tag{24}$$

where j^μ_M is the monopole current.

To discuss the monopoles in $SO(4)$ field, we have defined the $U(1)$ sub-field tensor $f_\pm$ in (20) and (21), hence the corresponding Maxwell equations are

$$\partial^{\nu} f_{\pm\mu\nu} = 4\pi j^{E}_{\pm\mu}, \quad \partial_{\nu}{}^{*}f^{\mu\nu}_{\pm} = -4\pi j^{\mu}_{\pm M}, \tag{25}$$

where the electric current $j^{E}_{\pm\mu}$ is purely due to the $U(1)$ potential $A_{\pm\mu}$, and $j^{\mu}_{\pm M}$ due to the topological term $K_{\pm\mu\nu}$: $j^{\mu}_{\pm M} = \frac{1}{4\pi}\partial_{\nu}{}^{*}K^{\mu\nu}_{\pm} = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda\rho}\epsilon_{abc}\partial_{\nu}n^{a}_{\pm}\partial_{\lambda}n^{b}_{\pm}\partial_{\rho}n^{c}_{\pm}$. According to the ϕ-mapping theory, the 3-dimensional topological current is defined as [23]

$$J^{\mu}_{\pm} = \frac{1}{8\pi\sqrt{g}}\epsilon^{\mu\nu\lambda\rho}\epsilon_{abc}\partial_{\nu}n^{a}_{\pm}\partial_{\lambda}n^{b}_{\pm}\partial_{\rho}n^{c}_{\pm} \quad (\mu = 0, 1, 2, 3) \tag{26}$$

so $\frac{1}{\sqrt{g}}j^{\mu}_{\pm M}$ is just the 3-dimensional topological current $J^{\mu}_{\pm}$. It can be seen that $J^{\mu}_{\pm}$ is identically conserved: $\frac{1}{\sqrt{g}}\partial_{\mu}(\sqrt{g}J^{\mu}_{\pm}) = 0$.

Similar to Section 3, one can prove that

$$J^{\mu}_{\pm} = \frac{1}{\sqrt{g}}\delta^{3}(\vec{\varphi}_{\pm})D^{\mu}(\varphi_{\pm}/x), \tag{27}$$

where $D^{\mu}(\varphi_{\pm}/x)$ is the Jacobian vector: $\epsilon^{abc}D^{\mu}(\varphi_{\pm}/x) = \epsilon^{\mu\nu\lambda\rho}\partial_{\nu}\varphi^{a}\partial_{\lambda}\varphi^{b}\partial_{\rho}\varphi^{c}$, and $\varphi^{a}_{\pm}$ is a 3-component vector defined in the $su(2)_{\pm}$ sub-algebra space: $n^{a}_{\pm} = \frac{\varphi^{a}}{\|\varphi_{\pm}\|}$, with $\|\varphi_{\pm}\|^{2} = \varphi^{a}_{\pm}\varphi^{a}_{\pm}$ $(a = 1, 2, 3)$. Obviously, the zero points of $\vec{\varphi}_{\pm}$ field are just the 3-dimensional singular points of $\vec{n}_{\pm}$ field. The expression (27) provides

$$J^{\mu}_{\pm}\begin{cases} = 0 & \text{iff } \vec{\varphi} \neq 0, \\ \neq 0 & \text{iff } \vec{\varphi} = 0, \end{cases}$$

so it is necessary to study the zero points of $\vec{\varphi}_{\pm}$ to determine the non-zero solutions of $J^{\mu}_{\pm}$. Since the number of the zero point equations

$$\varphi^{a}_{\pm}(x) = 0 \quad (a = 1, 2, 3) \tag{28}$$

is three but the base manifold $\mathscr{M}$ is 4-dimensional, one can choose x^{0} as the parameter of the solutions of (28). The implicit function theory shows [29] that under the regular condition $D^{0}(\varphi_{\pm}/x) \neq 0$, the general solutions of (28) can be expressed as

$$x^{1,2,3} = x^{1,2,3}_{\pm l}(x^{0}) \quad (l = 1, 2, \ldots, N_{\pm}), \tag{29}$$

which represents $N_{\pm}$ isolated singular lines on $\mathscr{M}$, with x^{0} the line parameter. These singular lines are just the world lines of $N_{\pm}$ 't Hooft–Polyakov monopoles on 4-manifold $\mathscr{M}$ [25,26].

Next, we should expand $J^{\mu}_{\pm}$ onto these $N_{\pm}$ singular lines. In δ-function theory [30], one can prove that

$$\delta^{3}(\vec{\varphi}_{\pm}) = \sum_{l=1}^{M}\frac{\beta_{\pm l}\eta_{\pm l}}{D^{0}(\varphi/x)_{\vec{x}_{\pm l}}}\delta^{3}(x^{\mu} - x^{\mu}_{\pm l}), \tag{30}$$

then (27) is

$$J^{0}_{\pm} = \frac{1}{\sqrt{g}}\sum_{l=1}^{N_{\pm}}\beta_{\pm l}\eta_{\pm l}\delta^{3}(x^{\mu} - x^{\mu}_{\pm l}),$$

$$J^{i}_{\pm} = \frac{1}{\sqrt{g}}\sum_{l=1}^{N_{\pm}}\beta_{\pm l}\eta_{\pm l}\delta^{3}(x^{\mu} - x^{\mu}_{\pm l})\hat{d}^{i}_{\pm l} \quad (i = 1, 2, 3), \tag{31}$$

where $\hat{d}^i_{\pm l}$ denotes the direction vectors of the world lines, $\hat{d}^i_{\pm l} = \mathrm{d}x^i_\pm/\mathrm{d}x^0_{\pm l} = D^i(\varphi_\pm/x)/D^0(\varphi_\pm/x)$. The expression (31) shows that these monopoles are characterized by the topological charge $\beta_{\pm l}\eta_{\pm l}$, and their world lines are in the direction of $\hat{d}^i_{\pm l}$.

In the case of the Zhang–Hu 4-dimensional QH system the base manifold $\mathscr{M}$ is an S^4 [1]. Considering an S^2 sub-space Ξ on S^4 [7], and taking the flat limit of S^4: $S^4 \to \mathbf{R}^4$ (which is thought of as a copy of the 2-dimensional QH system [9,6,1]), the winding number $\mathscr{W}_\pm$ on Ξ is given by

$$\mathscr{W}_\pm = \frac{1}{8\pi}\int_\Xi \epsilon_{abc} n^a_\pm \partial_\mu n^b_\pm \partial_\nu n^c_\pm \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{8\pi}\int_\Xi K_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{32}$$

In topology, this means that when x^μ covers Ξ once, the unit vector $n^a_\pm$ will cover the 2-sphere S^2 in $\varphi^a_\pm$ space for $\mathscr{W}_\pm$ times. $\mathscr{W}_\pm$ is a topological invariant and actually a Gauss mapping degree. By making use of the Stokes' theorem, one has

$$\mathscr{W}_\pm = \frac{1}{8\pi}\int_\Delta \epsilon_{abc}\partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \quad (\partial\Delta = \Xi). \tag{33}$$

In terms of the zero component of $J^\mu_\pm$, $\mathscr{W}_\pm$ is given by

$$\mathscr{W}_\pm = \int_\Delta J^0_\pm \, \mathrm{d}^3x = \sum_{l=1}^{N_\pm} \beta_{\pm l}\eta_{\pm l}. \tag{34}$$

This quantized result describes the topological effect of the 't Hooft–Polyakov monopoles in higher dimensional QH system, which shows that the winding number $\mathscr{W}_\pm$ wrapping these monopoles is characterized by their ϕ-mapping topological numbers $\beta_{\pm l}$ and $\eta_{\pm l}$.

4.2. The 2-membranes

In this subsection, it is shown that from $K_{\pm\mu\nu}$ one can also derive a 2-dimensional topological current, and the 2-membrane structures are inhering in this 2-dimensional current.

The $K_{\pm\mu\nu}$ tensor can also be expressed in an Abelian field tensor form [31,19]

$$K_{\pm\mu\nu} = \epsilon_{abc} n^a_\pm \partial_\mu n^b_\pm \partial_\nu n^c_\pm = \partial_\mu W_{\pm\nu} - \partial_\nu W_{\pm\mu}. \tag{35}$$

Here, $W_{\pm\mu}$ is the Wu–Yang potential: $W_{\pm\mu} = \vec{e}_{\pm 1}\cdot\partial_\mu \vec{e}_{\pm 2}$, where $\vec{e}_{\pm 1}$ and $\vec{e}_{\pm 2}$ are two unit vectors, respectively, normal to $n^a_\pm$, i.e., $(\vec{e}_{\pm 1}, \vec{e}_{\pm 2}, \vec{n}_\pm)$ forms an orthogonal frame: $\vec{e}_{\pm 1}\cdot\vec{e}_{\pm 2} = \vec{e}_{\pm 1}\cdot\vec{n}_\pm = \vec{e}_{\pm 2}\cdot\vec{n}_\pm = 0$ and $\vec{e}_{\pm 1}\cdot\vec{e}_{\pm 1} = \vec{e}_{\pm 2}\cdot\vec{e}_{\pm 2} = \vec{n}_\pm\cdot\vec{n}_\pm = 1$.

Consider another 2-component vector in $SU(2)_\pm$ space: $\vec{\xi}_\pm = (\xi^1_\pm, \xi^2_\pm)$, which resides in the plane formed by the unit vectors $\vec{e}_{\pm 1}$ and $\vec{e}_{\pm 2}$, satisfying

$$e^a_{\pm 1} = \frac{\xi^a_\pm}{\|\xi_\pm\|}, \quad e^a_{\pm 2} = \epsilon^{ab}\frac{\xi^b_\pm}{\|\xi_\pm\|} \quad (\|\xi_\pm\|^2 = \xi^a_\pm\xi^a_\pm;\ a, b = 1, 2). \tag{36}$$

It can be proved that this expression of $\vec{e}_{\pm 1}$ and $\vec{e}_{\pm 2}$ satisfies the above restriction of orthogonal frames. Obviously, the zero points of $\vec{\xi}_\pm$ are the 2-dimensional singular points of $\vec{e}_{\pm 1}$ and $\vec{e}_{\pm 2}$. Using $\vec{\xi}_\pm$ field, the Wu–Yang potential can be expressed as $W_{\pm\mu} = \epsilon_{ab}\frac{\xi^a_\pm}{\|\xi_\pm\|}\partial_\mu\frac{\xi^b_\pm}{\|\xi_\pm\|}$, and the field tensor $K_{\pm\mu\nu}$ becomes $K_{\pm\mu\nu} = 2\epsilon^{ab}\partial_\mu\frac{\xi^a_\pm}{\|\xi_\pm\|}\partial_\nu\frac{\xi^b_\pm}{\|\xi_\pm\|}$.

Similar to Section 3, one can prove that

$$\epsilon_{ab}\partial_\mu \frac{\xi^a_\pm}{\|\xi_\pm\|}\partial_\nu \frac{\xi^b_\pm}{\|\xi_\pm\|} = \epsilon_{\mu\nu\lambda\rho}2\pi\delta^2(\vec{\xi}_\pm)D^{\lambda\rho}(\xi_\pm/x) \quad (a,b=1,2), \tag{37}$$

where $D^{\lambda\rho}(\xi_\pm/x) = \frac{1}{2}\epsilon^{\lambda\rho\mu\nu}\epsilon_{ab}\partial_\mu\xi^a_\pm\partial_\nu\xi^b_\pm$ is the Jacobian tensor. Then, $K_{\pm\mu\nu}$ is expressed in a δ-function form: $K_{\pm\mu\nu} = 4\pi\epsilon_{\mu\nu\lambda\rho}\delta^2(\vec{\xi}_\pm)D^{\lambda\rho}(\xi_\pm/x)$. According to the ϕ-mapping theory [23], the 2-dimensional topological current is defined as

$$J^{\mu\nu}_\pm = \frac{1}{2\pi\sqrt{g}}\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ab}\partial_\lambda\frac{\xi^a_\pm}{\|\xi_\pm\|}\partial_\rho\frac{\xi^b_\pm}{\|\xi_\pm\|} = \frac{1}{\sqrt{g}}\delta^2(\vec{\xi}_\pm)D^{\mu\nu}(\xi_\pm/x) \tag{38}$$

so in $K_{\pm\mu\nu}$ there exists a 2-dimensional topological current: $K_{\pm\mu\nu} = 4\pi\sqrt{g}\epsilon_{\mu\nu\lambda\rho}J^{\lambda\rho}_\pm$.

The expression (38) provides

$$J^{\mu\nu}_\pm \begin{cases} = 0 & \text{iff } \vec{\xi} \neq 0, \\ \neq 0 & \text{iff } \vec{\xi} = 0, \end{cases}$$

so it is necessary to study the zero points of $\vec{\xi}_\pm$ to determine the non-zero solutions of $J^{\mu\nu}_\pm$. Since the number of the zero point equations

$$\xi^a(x) = 0 \quad (a = 1,2) \tag{39}$$

is two but the base manifold $\mathscr{M}$ is 4-dimensional, the solutions of (39) will be expressed with two parameters denoting the two surplus dimensions of $\mathscr{M}$. The implicit function theory shows [29] that under the regular condition $D^{\mu\nu}(\xi_\pm/x) \neq 0$, the general solutions of (39) can be expressed as

$$x^\mu = x^\mu_k(\sigma^1_\pm, \sigma^2_\pm) \quad (\mu = 0,1,2,3;\ k = 1,2,\ldots,L_\pm), \tag{40}$$

which represents $L_\pm$ 2-dimensional isolated surfaces $P_{\pm k}$ ($k = 1,2,\ldots,L_\pm$) on $\mathscr{M}$, with $\sigma^1_\pm$ and $\sigma^2_\pm$ the intrinsic coordinates of $P_{\pm k}$. These singular surfaces are the $L_\pm$ 2-membranes in 4-manifold $\mathscr{M}$ [19].

Then, we should expand $J^{\mu\nu}$ onto these $L_\pm$ singular surfaces $P_{\pm k}$'s. First, it can be proved that in the 4-manifold $\mathscr{M}$ there exists another 2-dimensional surface Σ_+ which is transversal to every P_{+k} at the section point p_{+k} (and similarly a Σ_- transversal to every P_{-k}):

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^I_\pm}\frac{\partial x^\nu}{\partial\tau^C_\pm}\bigg|_{p_{\pm k}} = 0 \quad (I = 1,2;\ C = 1,2), \tag{41}$$

where $\tau^1_\pm$ and $\tau^2_\pm$ are the intrinsic coordinates of $\Sigma_\pm$. Thus, on $\Sigma_\pm$ one can prove that [30,23]

$$\delta^2(\vec{\xi}_\pm) = \sum_{k=1}^{N}\beta_{\pm k}\eta_{\pm k}\delta^2(\vec{\tau}_\pm - \vec{\tau}(p_{\pm k})). \tag{42}$$

Second, since every $p_{\pm k}$ is related to a singular surface $P_{\pm k}$, the above two-dimensional δ-function $\delta^2(\vec{\tau} - \vec{\tau}(p_{\pm k}))$ must be expanded to the δ-function on singular surface $P_{\pm k}$ [i.e., $\delta(P_{\pm k})$]. Meanwhile, in δ-function theory it has been given [30,23] that $\delta(P_{\pm k}) = \int_{P_k}\delta^4(x^\mu - x^\mu_k(\sigma_\pm))\sqrt{g_{\pm\sigma}}\,\mathrm{d}^2\sigma_\pm$, where $g_{\pm\sigma}$ is the determinant of the metric

$g_{\pm IJ}$ of $P_k : g_{\pm\sigma} = \det(g_{\pm IJ})(I, J = 1, 2;\ g_{\pm IJ} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma_\pm^I}\frac{\partial x^\nu}{\partial\sigma_\pm^J})$. So, third, $J_\pm^{\mu\nu}$ is expanded onto $L_\pm$ singular surfaces $P_{\pm k}$:

$$J_\pm^{\mu\nu} = \frac{1}{\sqrt{g}} D^{\mu\nu}\left(\frac{\xi_\pm}{x}\right) \sum_{k=1}^{L_\pm} \beta_{\pm k}\eta_{\pm k} \int_{P_k} \delta^4(x^\mu - x_{\pm k}^\mu(\sigma_\pm))\sqrt{g_{\pm\sigma}}\mathrm{d}^2\sigma_\pm. \tag{43}$$

This means that in the $SO(4)$ gauge field there exists the 2-dimensional topological current, while the 2-membranes are inhering in this current with the topological charges $\beta_{\pm k}\eta_{\pm k}$.

In the end of this section, we would briefly discuss the integral of this current $J_\pm^{\mu\nu}$ on $\mathscr{M}$. In analogy with the form of the Nielsen's Lagrangian in classical dual string theory [32] one can define

$$\mathscr{L}_\pm = \sqrt{\tfrac{1}{2} g_{\mu\lambda} g_{\nu\rho} J_\pm^{\mu\nu} J_\pm^{\lambda\rho}}. \tag{44}$$

Then, from (42) it can be proved that $\mathscr{L}_\pm = \frac{1}{\sqrt{g}}\delta^4(\vec{\xi}_\pm)$, and the integral of $J_\pm^{\mu\nu}$ on $\mathscr{M}$ becomes

$$\mathscr{S}_\pm = \int_{\mathscr{M}} \mathscr{L}_\pm \sqrt{g}\,\mathrm{d}^4x = \int_{\mathscr{M}} \delta^4(\vec{\xi}_\pm)\,\mathrm{d}^4x. \tag{45}$$

Noticing (42), we arrive at the important result

$$\mathscr{S}_\pm = \sum_{k=1}^{L_\pm} \beta_{\pm k}\eta_{\pm k} \int_{P_{\pm k}} \sqrt{g_{\pm\sigma}}\,\mathrm{d}^2\sigma_\pm = \sum_{k=1}^{L_\pm} \beta_{\pm k}\eta_{\pm k}\mathscr{S}_{\pm k}, \tag{46}$$

where $\mathscr{S}_{\pm k} = \int_{P_{\pm k}} \sqrt{g_{\pm\sigma}}\,\mathrm{d}^2\sigma_\pm$ is the area of 2-membrane $P_{\pm k}$. It can be seen that the expression (46) takes the same form as the Nambu action [32], and $\mathscr{S}_\pm$ is quantized and characterized by the ϕ-mapping topological numbers $\beta_{\pm k}$ and $\eta_{\pm k}$.

5. Conclusion

In this paper, we use the ϕ-mapping topological theory to directly derive three kinds of topological excitations from the $SO(4)$ gauge field of condensed matter systems in 4-dimension: the instantons and anti-instantons, the 't Hooft–Polyakov monopoles, and the 2-membranes. In Section 2, the $SO(4)$ gauge field is fractionalized into two parts $SU(2)_+$ and $SU(2)_-$. In Section 3, it is revealed that there is a 4-dimensional topological current $J_\pm$ inhering in the $SU(2)_\pm$ gauge field, while the instanton and anti-instanton structures are acquired from $J_\pm$, whose topological charges underlie the second Chern number $c_{2\pm}$ and the signature $\tau(\mathscr{M})$ of 4-manifold $\mathscr{M}$. In Section 4, we generalize the 't Hooft–Polyakov theory and obtain the $U(1)$ sub-field tensor $f_{\pm\mu\nu}$ in the $SO(4)$ gauge field. We show that from the topological term $K_{\pm\mu\nu}$ of $f_{\pm\mu\nu}$, one can derive both the 3-dimensional topological current $J_\pm^\mu$ and the 2-dimensional topological current $J_\pm^{\mu\nu}$, and the 't Hooft–Polyakov monopoles and 2-membranes are, respectively, existing in $J_\pm^\mu$ and $J_\pm^{\mu\nu}$. Moreover, it is shown that the topological charges of these three kinds of excitations are all characterized by the ϕ-mapping topological numbers Hopf indices and Brouwer degrees.

Acknowledgments

X. Liu was financially supported by the IPRS and UQGSS of the University of Queensland and Y.Z. Zhang was supported by Australian Research Council.

References

[1] S.-C. Zhang, J.-P. Hu, Science 294 (2001) 823.
[2] E. Demler, S.-C. Zhang, Ann. Phys. 271 (1999) 83.
[3] B.A. Bernevig, C.-H. Chern, J.-P. Hu, N. Toumbas, S.-C. Zhang, Ann. Phys. 300 (2002) 185.
[4] S.-C. Zhang, Phys. Rev. Lett. 90 (2003) 196801.
[5] S. Bellucci, P.Y. Casteill, A. Nersessian, Phys. Lett. B 574 (2003) 121.
[6] H. Elvang, J. Polchinski, The Quantum Hall Effect on R^4. Available from: <hep-th/0209104>.
[7] M. Fabinger, J. High. Energy Phys. 05 (2002) 037.
[8] G. Sparling, Twistor theory and the four-dimensional Quantum Hall effect of Zhang and Hu. Available from: <cond-mat/0211679>.
[9] D. Karabali, V.P. Nair, Nucl. Phys. B 641 (2002) 533.
[10] V.P. Nair, S. Randjbar-Daemi, Nucl. Phys. B 679 (2004) 427;
D. Karabali, V.P. Nair, Nucl. Phys. B 679 (2004) 447.
[11] P.-M. Ho, S. Ramgoolam, Nucl. Phys. B 627 (2002) 266.
[12] Y.X. Chen, B.Y. Hou, B.Y. Hou, Nucl. Phys. B 638 (2002) 220.
[13] S. Murakami, N. Nagaosa, S.-C. Zhang, Science 301 (2003) 1348.
[14] S.A. Wolf, et al., Science 294 (2001) 1488;
E.I. Rashba, Phys. Rev. B 68 (2003) 241315;
E.I. Rashba, Spin currents in thermodynamic equilibrium? Available from: <cond-mat/0311110>.
[15] C. Wu, J.-P. Hu, S.-C. Zhang, Phys. Rev. Lett. 91 (2003) 186402.
[16] C.-H. Chern, H.-D. Chen, C. Wu, J.-P. Hu, S.-C. Zhang, Phys. Rev. B 69 (2004) 214512;
S. Murakami, N. Nagaosa, S.-C. Zhang, Phys. Rev. B 69 (2004) 235206.
[17] C.N. Yang, J. Math. Phys. 19 (1978) 320;
C.N. Yang, J. Math. Phys. 19 (1978) 2622.
[18] Y.S. Duan, X. Liu, J. Yang, Ann. Phys. 312 (2004) 84.
[19] Y.S. Duan, X. Liu, H. Zhang, Ann. Phys. 308 (2003) 493.
[20] Y.S. Duan, X. Liu, L.B. Fu, Commun. Theor. Phys. 40 (2003) 447.
[21] L.B. Fu, Y.S. Duan, H. Zhang, Phys. Rev. D 61 (2000) 045004.
[22] Y.S. Duan, L.B. Fu, J. Math. Phys. 39 (1998) 4343.
[23] Y.S. Duan, L.B. Fu, G. Jia, J. Math. Phys. 41 (2000) 4379.
[24] C. Nash, Differential topology and quantum field theory, Academic Press, New York, 1991;
J.D. Moore, Lectures on Seiberg–Witten Invariants, Springer-Verlag, New York, 1996;
S.K. Donaldson, P.B. Kronheimer, The geometry of four-manifolds, Clarendon Press, Oxford, New York, 1990;
T. Eguchi, P.B. Gilkey, A.J. Hanson, Phys. Rep. 66 (1980) 213.
[25] G. 't Hooft, Nucl. Phys. B 79 (1974) 276;
G. 't Hooft, Nucl. Phys. B 190 (1981) 455;
A. Polyakov, JETP Lett. 20 (1974) 194.
[26] M. Atiyah, N. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton University Press, Princeton, 1988.
[27] G.E. Volovik, JETP Lett. 75 (2002) 55.
[28] B.P. Dolan, J. High. Energy Phys. 05 (2003) 018.
[29] É. Goursat, A Course in Mathematical Analysis, vol. 1, translated by E.R. Hedrick, Dover, New York, 1904.
[30] J.A. Schouten, Tensor Analysis for Physicists, Clarendon Press, Oxford, 1951.

[31] T.T. Wu, C.N. Yang, Phys. Rev. D 12 (1975) 3845;
T.T. Wu, C.N. Yang, Phys. Rev. D 14 (1975) 437.
[32] H.B. Nielsen, P. Olesen, Nucl. Phys. B 57 (1973) 367;
H.B. Nielsen, P. Olesen, Nucl. Phys. B 61 (1973) 45.