General Correlation Functions of the Clauser-Horne-Shimony-Holt Inequality for Arbitrarily High-Dimensional Systems

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We generalize the correlation functions of the Clauser-Horne-Shimony-Holt (CHSH) inequality to arbitrarily high-dimensional systems. Based on this generalization, we construct the general CHSH inequality for bipartite quantum systems of arbitrarily high dimensionality, which takes the same simple form as CHSH inequality for two dimensions. This inequality is optimal in the same sense as the CHSH inequality for two-dimensional systems, namely, the maximal amount by which the inequality is violated consists of the maximal resistance to noise. We also discuss the physical meaning and general definition of the correlation functions. Furthermore, by giving another specific set of the correlation functions with the same physical meaning, we realize the inequality presented by Collins *et al.* [Phys. Rev. Lett. **88**, 040404 (2002)].

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Since the foundations of quantum mechanics were laid, one of the most remarkable aspects of quantum mechanics is its predicted correlations. The quantum correlations between outcomes of measurements performed on quantum entangled states of systems composed of several parts have no classical analog. Historically, this became known as the Einstein-Podolsky-Rosen paradox [1] and was formulated in terms of measurable quantities by Bell [2] as the now famous Bell inequalities. Subsequently, a vast amount of literature has covered lots of aspects, ranging from philosophy to experimental physics. One of the most common forms of Bell inequalities is known as the Clauser-Horne-Shimony-Holt (CHSH) inequality [3] which is described in terms of correlation functions by considering the correlations between measurements performed on two entangled spin-1/2 particles.

Recently, two kinds of inequalities [4,5] have been found that generalize the CHSH inequality to systems of higher dimension. The authors of Ref. [4] developed a new Bell inequality, denoted here as CGLMP inequality, for arbitrarily high-dimensional systems in terms of joint probabilities. Based on this inequality, the authors gave the analytic description of previous numerical results [6]. For the two-dimensional systems (systems composed of two spin-1/2 particles), this inequality reduces to the familiar CHSH inequality. As an alternative to this inequality, the authors of Ref. [5] obtained an inequality for three-dimensional systems in terms of correlation functions. These two inequalities are equivalent for three-dimensional systems [7,8].

In this Letter, inspired by the previous efforts [4,5], we generalize the correlation functions of the CHSH inequality for bipartite two-dimensional systems to arbitrarily high-dimensional systems. Then we construct a new Bell inequality for the arbitrarily high-dimensional system by using these correlation functions. The new inequality is not only of the same form, but also optimal in the same sense as the CHSH inequality, i.e., the maximal amount by which the inequality is violated consists with the maximal resistance to noise. Furthermore, we give a physical interpretation of the correlation function, and discuss the possible equivalent definitions of the correlation functions with the same physical meaning. By employing a specific set of correlation functions, we obtain the CGLMP inequality.

The scenario of the inequality involves two parties: Alice, can carry out two possible measurements, A_1 or A_2, on one of the particles, whereas the other party, Bob, can carry out two possible measurements, B_1 or B_2, on the other one. For the composed systems of d-dimensional parties (or bipartite systems of spin S particles with the relation $d = 2S + 1$), each measurement may have d possible outcomes: $A_1, A_2, B_1, B_2 = 0, \dots, d-1$. The joint probabilities are denoted by $P(A_i, B_j)$, which are required to satisfy the normalization condition: $\sum_{m,n=0}^{d-1} P(A_i = m, B_j = n) = 1$.

The CHSH inequality [3] for two entangled spin-1/2 particles reads

$$\langle A_1B_1\rangle + \langle A_1B_2\rangle - \langle A_2B_1\rangle + \langle A_2B_2\rangle \leq 2, \qquad (1)$$

where the functions $\langle A_iB_j\rangle$ are the expectation values of products $A_i \otimes B_j$ measured on pairs, known as the correlation functions. The inequality will never be violated by a local hidden variable theory, but will be maximally violated with the factor $\sqrt{2}$ by quantum predications of a maximally entangled state. On the other hand, the correlation functions can be expressed in terms of joint probabilities by

$$\langle A_iB_j\rangle = \sum_{m=0}^{1}\sum_{n=0}^{1}(-1)^{n+m}P(A_i = m, B_j = n). \qquad (2)$$

More recently, the authors of Ref. [5] gave a CHSH-type inequality for three-dimensional systems (for spin-1 particles), which reads

$$I = \mathrm{Re}[\bar{Q}_{11} + \bar{Q}_{12} - \bar{Q}_{21} + \bar{Q}_{22}] + \frac{1}{\sqrt{3}}\mathrm{Im}[\bar{Q}_{11} - \bar{Q}_{12} - \bar{Q}_{21} + \bar{Q}_{22}] \le 2, \tag{3}$$

where the correlation functions $\bar{Q}_{ij}$ are defined as follows:

$$\bar{Q}_{ij} = \sum_{m,n=0}^{2} \alpha^{n+m} P(A_i = m, B_j = n), \tag{4}$$

in which $\alpha = e^{i2\pi/3}$. Let us reform it as follows. Since the joint probabilities are real, we can simplify (3) to

$$I = Q_{11} + Q_{12} - Q_{21} + Q_{22} \le 2 \tag{5}$$

by defining $Q_{ij} = \mathrm{Re}[\bar{Q}_{ij}] + 1/\sqrt{3}\mathrm{Im}[\bar{Q}_{ij}]$ for $i \ge j$, and $Q_{12} = \mathrm{Re}[\bar{Q}_{12}] - 1/\sqrt{3}\mathrm{Im}[\bar{Q}_{12}]$. Obviously, (5) has the same form of the CHSH inequality (1). Furthermore, from (4), we can prove that the new correlation functions Q_{ij} can be written in the following form:

$$Q_{ij} \equiv \frac{1}{S}\sum_{m=0}^{d-1}\sum_{n=0}^{d-1} f^{ij}(m,n) P(A_i = m, B_j = n), \tag{6}$$

in which $S = 1$, the spin of the particle for the three-dimensional system, $f^{ij}(m,n) = S - M[\varepsilon(i-j)(m+n), d]$, and $\varepsilon(x)$ is the sgn function: $\varepsilon(x) = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0 \end{cases}$. $M(x,d)$ is defined as follows: $M(x,d) = (x \bmod d)$ and $0 \le M(x,d) \le d-1$. Comparing (6) with (2), we find that the correlation functions for two entangled spin-1/2 particles can also be expressed with (6) by substituting $S = 1/2$ and $d = 2$, correspondingly.

Obviously, the formula (6) generalizes the correlation function to arbitrarily dimensional systems.

At the same time, we assume the CHSH inequality expression for arbitrarily dimensional systems takes the same form as the CHSH inequality for two-dimensional systems, namely

$$I_d = Q_{11} + Q_{12} - Q_{21} + Q_{22}. \tag{7}$$

I_d is upper bounded by 4. This follows immediately from the fact that the extreme values of Q_{ij} are ± 1. However, these four functions are strongly correlated, so I_d can never reach this value. In a local hidden variable theory, only three of the four pairs of operators: (A_1, B_1), (A_1, B_2), (A_2, B_1), and (A_2, B_2) can be freely chosen, the last one is constrained. We can prove the maximum value of I_d for local hidden variable theories is 2, i.e., $I_d \le 2$.

The proof consists of enumerating all the possible relations between pairs of operators allowed by the local hidden variable theory. Defining $r_{11} \equiv A_1 + B_1$, $r_{12} \equiv A_1 + B_2$, $r_{21} \equiv A_2 + B_1$, and $r_{22} \equiv A_2 + B_2$. Obviously, they obey the constraint

$$r_{11} + r_{22} = r_{12} + r_{21}. \tag{8}$$

The correlation functions (6) for a given choice of r_{11}, r_{12}, r_{21}, and r_{22} are $Q_{11} = g_1(r_{11})$, $Q_{12} = g_2(r_{12})$, $Q_{21} = g_1(r_{21})$, and $Q_{22} = g_1(r_{22})$, where $g_{1,2}(x)$ are given by

$$g_1(x) = \frac{S - M(x,d)}{S}, \qquad g_2(x) = \frac{M(x,d) - S - 1}{S}. \tag{9}$$

Then we immediately have

$$I_d = \frac{M(r_{12},d) + M(r_{21},d) - M(r_{11},d) - M(r_{22},d) - 1}{S}. \tag{10}$$

Now, we consider different cases according to the values of r_{11}, r_{12}, r_{21}, and r_{22}.

Case 1. Both r_{11} and r_{22} are less than d. From (8), there are two cases for the rest: (i) none of r_{12} and r_{21} is larger than d, (ii) one of them is larger than d. Then from (10), if none of r_{12} and r_{21} is larger than d, we get $I_d = [r_{12} + r_{21} - (r_{11} + r_{22} - d) - 1]/S = 2$ (keeping in mind $d = 2S + 1$); if one of r_{12} and r_{21} is larger than d, then $I_d = -1/S$.

Case 2. One of r_{11} and r_{22} is larger than d. There are three cases for the rest: (i) none of r_{12} and r_{21} is larger than d, (ii) one of them is larger than d, and (iii) both r_{12} and r_{21} are larger than d. On can find that there are three possible results for I_d: $I_d = 2$, $I_d = -1/S$, and $I_d = -2(S+1)/S$.

Case 3. Both r_{11} and r_{22} are larger than d. Then (8) implies the following: (i) both r_{12} and r_{21} are also larger than d, (ii) one of them is larger than d. From (10), one finds I_d is either $I_d = -1/S$ or $I_d = 2$.

Thus, for all possible choices of r_{ij}, $I_d \le 2$ for local realism. (Note that for $d = 2$, not all the possibilities enumerated above can occur. One can prove that the only possible values are $I_2 = \pm 2$.) Here, we must point out that the proof is also valid for nondeterministic local hidden variable theories for the convexity of the correlation polytope.

Let us now consider the maximum value that can be attained for the Bell expression I_d for quantum measurements on an entangled quantum state. For the maximally entangled state of two d-dimensional systems $\psi = (1/\sqrt{d}\sum_{j=0}^{d-1} |l\rangle_A |l\rangle_B$, we therefore first recall the optimal measurements performed on such a state described in [4,6]. Let the operators A_i $i = 1, 2$, measured by Alice and B_j, $j = 1, 2$, measured by Bob, have the nondegenerate eigenvectors

$$|m\rangle_{A_i} = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1} \exp\left[i\frac{2\pi}{d} l(m + \alpha_i)\right]|l\rangle_A, \qquad |n\rangle_{B_j} = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1} \exp\left[i\frac{2\pi}{d} l(n + \beta_j)\right]|l\rangle_B, \tag{11}$$

where $\alpha_1 = 0$, $\alpha_2 = 1/2$, $\beta_1 = 1/4$, and $\beta_2 = -1/4$. Thus the joint probabilities are [4]

$$P_{QM}(A_i = m, B_j = n) = \frac{1}{2d^3 \sin^2[\pi(m+n+\alpha_i+\beta_j)/d]}. \tag{12}$$

These joint probabilities have several symmetries. First of all we can have the relation

$$P_{QM}(A_i = m, B_j = n) = P_{QM}(A_i = m \pm c, B_j = n \mp c)$$

for any integer c.

For convenience, in the following we use the symbol $\doteq$ to denote equality modulus d. Let us define the probabilities $P(A_i \pm B_j \doteq k)$ by

$$P(A_i \pm B_j \doteq k) = \sum_{m=0}^{d-1} P(A_i = m, B_j = (k \mp m \bmod d)). \tag{13}$$

Then, from (12) we have

$$P_{QM}(A_1 + B_1 \doteq c) = dP_{QM}(A_1 = c, B_1 = 0). \tag{14}$$

Let us define $S_z = S - M(\varepsilon(i-j)(m+n), d)$. One can prove that $S_z = -S, -S+1, \ldots, S-1, S$ for all the possible values of m and n. By denoting $q(S_z) = P_{QM}(A_1 + B_1 \doteq S - S_z) = \{1/[2d^2 \sin^2[\pi(S - S_z + 1/4)/d]]\}$, and using the above formula, we can prove: $Q_{11} = Q_{12} = Q_{22} = -Q_{21} = Q_d$, where

$$Q_d = \frac{1}{S} \sum_{S_z=-S}^{S} S_z q(S_z). \tag{15}$$

Then, we can obtain the quantum prediction of Bell expression for the maximally entangled state $I_d(QM)$, namely

$$I_d(QM) = 4Q_d. \tag{16}$$

One can prove that this result is the same as that obtained in [4], and it consists with the numerical work in [6]. So, the measurements defined by (11) are optimal for the maximally entangled states and $I_d(QM)$ is the strongest violation of Bell expression I_d for the maximally entangled states of bipartite d-dimensional systems. From the numerical work of [6] and analytical result of [4], we conclude that the general CHSH inequality (7) is optimal in the same sense as the CHSH inequality is optimal for two-dimensional systems.

At the same time, we obtain the optimal correlation matrix $\hat{Q} = \{Q_{ij}\}$ for the maximally entangled state for d-dimensional systems, which is $\hat{Q} = Q_d \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. This matrix is well known for $d = 2$ [$Q_2 = (\sqrt{2}/2)$].

In fact, one can formulate other versions of Bell inequality for a given experimental setup. In the interesting paper [9], Pitowsky and Svozil have presented the general method for the derivation of all Bell inequalities for each given experimental setup, in which two specific cases have also been discussed. Actually, the inequality suggested in Ref. [5] was obtained by searching the optimal inequality on the correlation polytope corresponding to the unbiased six-port beam splitters measurements.

The CHSH inequality for two-dimensional system is the most popular form of Bell inequality among physicists, due to its simplicity and optimality. Another important reason is that it demonstrates the nature of quantum correlations clearly. As shown in (1), the correlation functions of CHSH inequality for two-dimensional systems have explicit physical meaning, namely, the correlation functions are the expectation values of products $A_i \otimes B_j$ measured on pairs. Obviously, the general correlation functions (for arbitrary dimensionality) cannot be understood in such a sense. Nevertheless, the correlation functions, in general, imply some kind of correlation between the measurements on pairs.

Indeed, the normalization condition of joint probabilities can be rewritten as $\sum_{k=0}^{d-1} P(A_i + B_j \doteq k) = 1$. Naturally, the d real numbers $P(A_i + B_j \doteq k)$ $(k = 0, 1, \ldots, d-1)$ can be regarded as the probabilities of eigenvalues S_z $(S_z = -S, -S+1, \ldots, S)$ of a spin S system, $\tilde{P}(S_z)$, under the relations

$$\tilde{P}(S_z) = P(A + B \doteq S - S_z). \tag{17}$$

We define the correlation function of the two measurements A and B as follows:

$$C(A, B) = \sum_{S_z=-S}^{S} S_z \tilde{P}(S_z) = \langle S_z \rangle_{AB}. \tag{18}$$

The meaning of the above formula is apparent, i.e., the correlation of the two measurements A and B can be interpreted as the average of spin projection for the imaginary system with spin S defined by (17).

In analogy, we can imagine another system which is dual to the former one by

$$\tilde{P}^-(S_z) = P(A + B \doteq -(S - S_z)). \tag{19}$$

Consequently, we have another correlation function. To distinguish these two, we substitute $\tilde{P}^+(S_z)$ for $\tilde{P}(S_z)$ in Eq. (17). Then the two kinds of correlation functions can be labeled as $C^\pm(A, B) = \sum_{S_z=-S}^{S} S_z \tilde{P}^\pm(S_z) = \langle S_z \rangle_{AB}^\pm$.

It is not difficult to prove that the CHSH inequality can be rewritten by this kind of correlation function as

$$\frac{1}{S}[C^+(A_1, B_1) + C^-(A_1, B_2) - C^+(A_2, A_1) + C^+(A_2, B_2)] \leq 2. \tag{20}$$

Comparing with (7), we have $Q_{11} = C^+(A_1, B_1)/S$, $Q_{21} = C^+(A_2, B_1)/S$, $Q_{22} = C^+(A_2, B_2)/S$, and $Q_{12} = C^-(A_1, B_2)/S$.

In fact, the labels of d possible outcomes for each side (A and B) are arbitrary. So, in general, if we have a mapping: $g:(A, B) \rightharpoonup (0, 1, \ldots, d-1)$, namely, $g(A, B) = c$ $(c = 0, 1, \ldots, d-1)$, which is a one to one mapping for fixed value of A (or B). Let $P(g(A, B) = k)$ be the sum of all the joint probabilities $P(A, B)$ which satisfy $g(A, B) = k$. We can define the correlation functions of the pair measurements A and B as $C^{\pm}(A, B) = \sum_{S_z=-S}^{S} S_z \widetilde{P}^{\pm}(S_z) = \langle S_z \rangle_{AB}^{\pm}$ with $\widetilde{P}^{\pm}(S_z) = P(g(A, B) \doteq \pm(S - S_z))$. Then from (20), a CHSH-type inequality can be established by employing these correlation functions.

Especially, let us consider $g(A, B) = (A - B) \mathrm{mod}\, d$. Defining $\widetilde{P}^{\pm}(S_z) = P(A - B \doteq \pm(S - S_z))$, one can prove that the sum $\sum_{S_z=-S}^{S} S_z \widetilde{P}^{\pm}(S_z)$ can be split into two parts as $\sum_{S_z=S_0}^{S} S_z[\widetilde{P}^{\pm}(S_z) - \widetilde{P}^{\pm}(-S_z)]$, where $S_0 = 1/2$ for even dimension (fermions) or $S_0 = 1$ for odd dimension (bosons). Then from (13) and (20), and letting $k = S - S_z$, we can get

$$Q_{ij} \equiv \sum_{k=0}^{[d/2]-1} \left(1 - \frac{k}{S}\right)[P(A_i - B_j \doteq k\varepsilon(i - j)) - P(A_i - B_j \doteq (-k - 1)\varepsilon(i - j))], \tag{21}$$

in which $[d/2]$ denotes the integer part of $d/2$ and we have used the formula $2S - k \doteq -k - 1$. Then from the expression of (7), we can obtain the CGLMP inequality [4]. So, the CGLMP inequality can be converted into the standard form of CHSH inequality for arbitrarily high dimensionality by introducing the general correlation functions.

From the above discussion, we know that the correlation function has specific physical meaning, namely, each pair of measurements can be cast into the spin projection of an imaginary spin S system, and the correlation function is the expectation value of the spin projection of the imaginary system. Especially, for the bipartite system composed by two spin $1/2$ particles, the correlation functions can also be expressed by the expectation values of products $A_i \otimes B_j$ measured on pairs. We do not know so far, for the arbitrarily high-dimensional systems, if the correlations functions can be expressed by expectation values of products of some kinds of general measurements of pairs. However, we think the correlation functions of arbitrary dimensionality are worth further study.

In summary, we have constructed the general CHSH inequality for arbitrarily high-dimensional systems by generalizing the correlation functions of the CHSH inequality for bipartite two-dimensional systems to arbitrarily high-dimensional systems. The general CHSH inequality is of the same form and optimal in the same sense as the CHSH inequality for two-dimensional systems. We also discussed the physical meaning of the correlation functions and gave a general description of the correlation functions. The facts that the Bell inequality as well as the correlation functions have the unified forms and physical meaning for arbitrary dimensionality, suggest that the quantum correlations should have some common properties for arbitrary dimensionality, which will be useful for discussing entanglement of systems of higher dimensionality as they have had for two-dimensional systems [10]. Furthermore, the general description of correlation functions makes it possible for us to construct some alternative CHSH-type inequalities which may be convenient to realize in experimental setups for higher dimensional systems. We hope that the general CHSH inequality and the general correlation functions presented here will draw much more attention of physicists on studying Bell inequality and entanglement of systems of large dimensionality.

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