Metric structures characterized by norm properties in non-Hermitian systems

Xin Wang [1], Hao-Di Liu [2,*] and Libin Fu [1,†]

[1] *Graduate School of China Academy of Engineering Physics, Beijing 100193, China*

[2] *Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, China*

(Received 15 July 2026; accepted 15 October 2026; published 2 November 2026)

For non-Hermitian quantum systems, metric operators provide a way to define the corresponding inner-product structures. In the entirely real-spectrum regime, a positive-definite metric η can be introduced such that the Hamiltonian is η-self-adjoint with respect to the corresponding inner product. In the complex-spectrum regime, different metric constructions can be characterized by their associated norm properties. For finite-dimensional non-Hermitian systems with a complete energy eigenbasis, we compare three representative metrics, namely, the cross-inner-product metric Υ_1, the dynamical metric $\Upsilon_2(t)$, and the pseudo-Hermitian metric Υ_3, and analyze their relationships and differences. For time-independent Hamiltonians, the Υ_1 norm obeys a balance equation governed by the imaginary parts of the eigenvalues, with its long-time behavior dominated by the largest imaginary part among contributions with nonzero initial weights. In contrast, $\Upsilon_2(t)$ preserves the norm through its time dependence, while Υ_3 remains norm-preserving but becomes indefinite in the complex-spectrum regime. For an entirely real spectrum, all three metrics coincide under the same normalization convention when $\Upsilon_2(t_0) = \Upsilon_1$. At exceptional points, the energy eigenbasis is incomplete and Υ_1 is singular and positive semidefinite, whereas a positive-definite $\Upsilon_2(t)$ can be obtained from another complete initial basis, and a Hermitian invertible Υ_3 can still be obtained from its defining equation. For time-dependent Hamiltonians, the $\Upsilon_1(t)$ norm equation acquires a geometric contribution from the evolving biorthogonal basis, while imposing dynamical compatibility on $\Upsilon_3(t)$ makes it coincide with $\Upsilon_2(t)$ for the same initial metric. The corresponding exceptional-point properties are also discussed. A concrete model illustrates these distinctions.

DOI: 10.1103/6rg6-2n55

I. INTRODUCTION

Non-Hermitian systems have attracted increasing attention in recent years [1–4]. They can exhibit a variety of phenomena, including complex spectra, exceptional points [5,6], the non-Hermitian skin effect [7–9], and non-Hermitian topology [10–13]. Related studies have been extended to optical [14–17], microwave [18,19], and electronic systems [20–22].

At the same time, the study of non-Hermitian systems also raises fundamental questions about metric operators and the corresponding inner-product structures. There exists a class of operators that are non-self-adjoint under the standard inner product but nevertheless possess entirely real spectra. To represent such operators as physical observables, an appropriate inner product must be introduced with respect to which they become self-adjoint. Metric operators provide a way to define such an inner-product structure [23,24].

For finite-dimensional diagonalizable non-Hermitian operators with entirely real spectra, such an inner-product structure can be defined by introducing a positive-definite metric [25,26]. There are two common ways to construct such a metric. One approach is to determine the metric from a compatible set of observables that includes the Hamiltonian. The physical inner product is chosen such that all operators in this set become self-adjoint with respect to the same metric [24]. Another approach is to start from a given non-Hermitian Hamiltonian H and seek a positive-definite metric operator η satisfying [25–27]

$$H^\dagger \eta = \eta H. \tag{1}$$

The metric then defines the physical inner product

$$\langle \psi, \phi \rangle_\eta = \langle \psi | \eta | \phi \rangle. \tag{2}$$

With respect to this inner product, H satisfies

$$\langle \psi, H\phi \rangle_\eta = \langle H\psi, \phi \rangle_\eta, \tag{3}$$

and is therefore η-self-adjoint. These two viewpoints lead to the same type of positive-definite metric structure but differ in how the metric is specified.

For non-Hermitian Hamiltonians with complex spectra, different metric-operator constructions have also been introduced. In this case, the Hamiltonian cannot be made self-adjoint with respect to a positive-definite inner product. Instead, these metrics define different inner-product structures and can be distinguished by their associated norm properties.

The first type is based on the cross inner product and introduces a positive-definite metric [28]

$$\Upsilon_1 = \sum_n |\tilde{E}_n\rangle\langle\tilde{E}_n|, \tag{4}$$

where $|\tilde{E}_n\rangle$ denotes the eigenstate of $H^\dagger$. Although Υ_1 is positive definite, the corresponding metric norm is generally

*Contact author: liuhd100@nenu.edu.cn

†Contact author: lbfu@gscaep.ac.cn

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not conserved. Instead, the imaginary parts of the eigenvalues are retained in the resulting norm-balance equation, thereby explicitly characterizing the gain-loss dynamics of the non-Hermitian system. In particular, the Loschmidt echo defined within this framework naturally contains decay contributions from different energy channels [28], consistent with biorthogonal formulations of the Loschmidt echo in non-Hermitian systems [29].

The second type requires norm conservation. In this case, a dynamical metric $\Upsilon_2(t)$ is introduced [30–33], satisfying

$$\partial_t \Upsilon_2 = \frac{i}{\hbar}(\Upsilon_2 H - H^\dagger \Upsilon_2). \tag{5}$$

Within this framework, the time dependence of $\Upsilon_2(t)$ compensates for the norm variation induced by the non-Hermitian dynamics, thus ensuring norm conservation [34,35]. In addition, within the generalized inner-product structure defined by $\Upsilon_2(t)$, several fundamental no-go theorems of quantum information remain valid for finite-dimensional non-Hermitian quantum systems [32].

The third type applies the pseudo-Hermitian metric construction to spectra containing complex-conjugate eigenvalue pairs by introducing a time-independent Hermitian metric Υ_3 [36], satisfying

$$H^\dagger \Upsilon_3 = \Upsilon_3 H. \tag{6}$$

The metric maps each right eigenstate to the left eigenstate associated with the complex-conjugate eigenvalue. In contrast to Υ_1 and $\Upsilon_2(t)$, Υ_3 is generally not positive definite [37], and its generalized completeness relation takes a different form.

These metrics arise from different constructions, but a direct comparison of their time dependence, positive definiteness, norm equations, and generalized completeness relations remains lacking. Therefore, this work aims to clarify their relationships and differences. We first briefly introduce the biorthogonal basis and the pseudo-Hermitian metric for entirely real spectra, and then consider three representative metrics for complex spectra. For time-independent and time-dependent non-Hermitian Hamiltonians, we separately analyze the connections and distinctions among these metrics in detail. Meanwhile, we further examine the applicability of the three metric constructions at exceptional points. A concrete model further illustrates these results, particularly how the imaginary parts of the eigenvalues are manifested differently in the three metric constructions.

II. METRIC STRUCTURES AND NORM EQUATIONS IN NON-HERMITIAN SYSTEMS

A. Biorthogonal representation and pseudo-Hermitian metric

For a diagonalizable non-Hermitian Hamiltonian H, the right and left eigenstates satisfy [38–41]

$$\begin{aligned} H|E_n\rangle &= E_n|E_n\rangle, \\ H^\dagger|\tilde{E}_n\rangle &= E_n^*|\tilde{E}_n\rangle, \end{aligned} \tag{7}$$

together with the biorthonormality and completeness relations

$$\langle\tilde{E}_n|E_m\rangle = \delta_{nm}, \quad \sum_n |E_n\rangle\langle\tilde{E}_n| = \mathbb{I}. \tag{8}$$

An arbitrary state $|\psi(t)\rangle$ can therefore be expanded in the biorthogonal basis as

$$|\psi(t)\rangle = \sum_n c_n(t)|E_n\rangle, \quad c_n(t) = \langle\tilde{E}_n|\psi(t)\rangle. \tag{9}$$

When the spectrum of H is entirely real, H is pseudo-Hermitian and admits a positive-definite metric operator η satisfying

$$H^\dagger \eta = \eta H. \tag{10}$$

With respect to the associated η-inner product, H is self-adjoint and the corresponding norm is conserved,

$$N_\eta(t) = \langle\psi(t)|\eta|\psi(t)\rangle, \quad \frac{dN_\eta(t)}{dt} = 0. \tag{11}$$

When the spectrum becomes complex, no positive-definite metric can render H self-adjoint, and the corresponding η-norm $N_\eta(t)$ is generally not conserved. Therefore, we discuss the metric structures for complex spectra in detail in the following subsection.

B. Metric structures for time-independent non-Hermitian Hamiltonians

For a finite-dimensional non-Hermitian system with a complex spectrum, whose Hamiltonian is time independent and diagonalizable, various metric operators have been introduced to define norms with different properties. However, the relations and distinctions between these metrics have not yet been fully clarified. To this end, we consider the following three representative metric operators [28,31–33,36]:

$$\begin{aligned} \Upsilon_1 &= \sum_n |\tilde{E}_n\rangle\langle\tilde{E}_n|, \\ \Upsilon_2(t) &= U_l(t,t_0)\Upsilon_2(t_0)U_l^\dagger(t,t_0), \\ \Upsilon_3 &= \sum_{n_0} |\tilde{E}_{n_0}\rangle\langle\tilde{E}_{n_0}| + \sum_{n_+} \left(|\tilde{E}_{n_+}\rangle\langle\tilde{E}_{n_-}| + |\tilde{E}_{n_-}\rangle\langle\tilde{E}_{n_+}|\right), \end{aligned} \tag{12}$$

with

$$U_l(t,t_0) = \mathcal{T}\exp\left(\frac{-i}{\hbar}\int_{t_0}^{t} H^\dagger(\tau)d\tau\right), \tag{13}$$

where $\mathcal{T}$ denotes the time-ordering operator. Although the Hamiltonian considered here is time independent, $U_l(t,t_0)$ is written in this general form, which also applies directly to time-dependent Hamiltonians. The choice of $\Upsilon_2(t_0)$ depends on the complete basis selected at the initial time. If the eigenstates of the Hamiltonian are chosen as the initial basis, then

$$\Upsilon_2(t_0) = \sum_n |\tilde{E}_n\rangle\langle\tilde{E}_n| = \Upsilon_1. \tag{14}$$

The subscripts 0 and $\pm$ label the real eigenvalues and the complex-conjugate eigenvalue pairs, respectively, together with their corresponding eigenvectors. We next discuss the three metrics separately and then compare their connections and distinctions. The first approach describes the norm evolution through a balance equation. The corresponding metric is defined as

$$\Upsilon_1 = \sum_n |\tilde{E}_n\rangle\langle\tilde{E}_n|. \tag{15}$$

Under this metric, the corresponding norm satisfies

$$\frac{dN_{\Upsilon_1}}{dt} = \frac{d}{dt}\langle\psi|\Upsilon_1|\psi\rangle = \frac{2}{\hbar}\sum_n |c_n(t)|^2 \,\mathrm{Im}\, E_n. \tag{16}$$

Equation (16) directly shows the contribution of each eigenstate to the rate of change of the norm. Defining the weight of the nth eigenstate as

$$N_n(t) = |c_n(t)|^2 = N_n(t_0)\exp\left(\frac{2}{\hbar}\,\mathrm{Im}\, E_n(t-t_0)\right), \tag{17}$$

each eigenstate separately satisfies

$$\frac{dN_n}{dt} = \frac{2}{\hbar}\,\mathrm{Im}\, E_n\, N_n. \tag{18}$$

Therefore, an eigenvalue with $\mathrm{Im}\, E_n > 0$ contributes a source term, whereas an eigenvalue with $\mathrm{Im}\, E_n < 0$ contributes a sink term. When $\mathrm{Im}\, E_n = 0$, the corresponding eigenstate weight is conserved. The total norm balance equation is obtained by summing Eq. (18) over all eigenstates

$$\frac{dN_{\Upsilon_1}}{dt} = \sum_n \frac{dN_n}{dt}. \tag{19}$$

If the initial state has a nonzero component in the eigenstate whose eigenvalue has the largest imaginary part, this eigenstate dominates the long-time evolution of the generalized norm. More precisely, denoting this state by n_*, one has

$$N_{\Upsilon_1}(t) \sim N_{n_*}(t_0)\exp\left(\frac{2}{\hbar}\,\mathrm{Im}\, E_{n_*}(t-t_0)\right), \quad t\to\infty, \tag{20}$$

provided that

$$\mathrm{Im}\, E_{n_*} > \mathrm{Im}\, E_m \tag{21}$$

for all $m \neq n_*$ such that $N_m(t_0) \neq 0$. Hence, the long-time behavior of the generalized norm is governed by the eigenvalue with the largest imaginary part among the initially occupied eigenstates. This asymptotic behavior is analogous to the long-time dominance of the eigenstate with the largest imaginary part of its eigenvalue in the parity-time ($\mathcal{PT}$) broken phase reported in Ref. [42].

When the entire spectrum is real

$$\mathrm{Im}\, E_n = 0 \quad \text{for all } n, \tag{22}$$

every source or sink term in Eq. (16) vanishes separately. In this case, Υ_1 satisfies

$$H^\dagger\Upsilon_1 = \Upsilon_1 H, \tag{23}$$

and the norm-balance equation degenerates to the pseudo-Hermitian conservation law,

$$\frac{dN_{\Upsilon_1}}{dt} = 0. \tag{24}$$

This real-spectrum case should be distinguished from a state-dependent cancellation between gain and loss. In the latter case, the positive and negative terms in Eq. (16) may cancel for a particular state or at a particular instant, while the individual eigenstate weights remain nonconservative. By contrast, when the entire spectrum is real, the contribution from each eigenstate vanishes independently. The metric Υ_1 explicitly retains the gain and loss associated with the imaginary parts of the energy spectrum. Consequently, its corresponding norm is generally not conserved when complex eigenvalues are present.

The second approach maintains that the norm should be conserved under all circumstances. Consequently, a time-dependent positive-definite metric $\Upsilon_2(t)$ is constructed that satisfies

$$\partial_t\Upsilon_2 = \frac{i}{\hbar}[\Upsilon_2 H - H^\dagger\Upsilon_2]. \tag{25}$$

Under the new inner product $\langle\psi|\Upsilon_2(t)|\psi\rangle$, we have

$$\frac{d}{dt}N_{\Upsilon_2} = \frac{d}{dt}\langle\psi|\Upsilon_2|\psi\rangle = 0, \tag{26}$$

which implies the norm conservation regardless of whether the spectrum is real or complex.

In this sense, $\Upsilon_2(t)$ can be viewed as a fiber metric compatible with the dynamics whose time dependence compensates the norm variation generated by the non-Hermitian Hamiltonian. The formal solution to Eq. (25) can be written as

$$\Upsilon_2(t) = U_l(t,t_0)\Upsilon_2(t_0)U_l^\dagger(t,t_0). \tag{27}$$

If we introduce a transformation matrix $T(t)$ satisfying

$$\partial_t T(t) = \frac{i}{\hbar}[T(t)H - HT(t)], \tag{28}$$

then the metric (see Appendix 1 for details)

$$G(t) = T^\dagger(t)\Upsilon_2(t)T(t) = U_l(t,t_0)G(t_0)U_l^\dagger(t,t_0) \tag{29}$$

also satisfies the same dynamical equation as $\Upsilon_2(t)$. However, once a complete set of basis vectors $\{|n\rangle\}$ is chosen, the metric $\Upsilon_2(t_0)$ takes a uniquely determined form. If the chosen complete basis is the eigenbasis of H, the initial metric can be chosen as

$$\Upsilon_2(t_0) = \Upsilon_1 = \sum_n |\tilde{E}_n\rangle\langle\tilde{E}_n|. \tag{30}$$

The corresponding dynamical metric can then be written as

$$\Upsilon_2(t) = \sum_n |L_n(t)\rangle\langle L_n(t)| = \sum_n e^{-\frac{2}{\hbar}\mathrm{Im}(E_n)(t-t_0)}|\tilde{E}_n\rangle\langle\tilde{E}_n|, \tag{31}$$

where $|L_n(t)\rangle = U_l|\tilde{E}_n\rangle$. In particular, when the spectrum is entirely real

$$\Upsilon_2(t) = \eta = \sum_n |\tilde{E}_n\rangle\langle\tilde{E}_n|. \tag{32}$$

Furthermore, $\Upsilon_2(t)$ is also naturally derived based on the properties of the evolution operator in biorthogonal quantum mechanics and the condition for the conservation of the biorthogonal inner product (see Appendix 2 for details).

The third approach also requires the conservation of the metric inner product, while assuming that the metric operator is Hermitian, invertible, and time independent. Accordingly, it satisfies

$$H^\dagger\Upsilon_3 = \Upsilon_3 H. \tag{33}$$

Unlike the positive-definite metric $\Upsilon_2(t)$, Υ_3 is not necessarily positive definite. The metrics $\Upsilon_2(t)$ and Υ_3 are based

TABLE I. Properties of different metric operators for time-independent Hamiltonians.

Metric	Time dependent	Positive definite	Norm equation	Completeness relation
Υ_1	No	Yes	$\frac{d}{dt}\langle\psi\|\Upsilon_1\|\psi\rangle = \frac{2}{\hbar}\sum_n \|c_n\|^2 \operatorname{Im} E_n$	$\sum_n \|E_n\rangle\langle E_n\| \Upsilon_1 = \mathbb{I}$
$\Upsilon_2(t)$	Yes	Yes, if $\Upsilon(t_0) > 0$	$\frac{d}{dt}\langle\psi\|\Upsilon_2(t)\|\psi\rangle = 0$	$\sum_n \|n(t)\rangle\langle n(t)\| \Upsilon_2(t) = \mathbb{I}$
Υ_3	No	Not necessarily	$\frac{d}{dt}\langle\psi\|\Upsilon_3\|\psi\rangle = 0$	$\left[\sum_{n_0} \|E_{n_0}\rangle\langle E_{n_0}\| + \sum_{n_+}(\|E_{n_+}\rangle\langle E_{n_-}\| + \|E_{n_-}\rangle\langle E_{n_+}\|)\right]\Upsilon_3 = \mathbb{I}$

on different constructions and lead to different generalized completeness relations.

Specifically, the authors of Ref. [32] introduced a generalized inner product $\langle\langle\psi_1(t)|\psi_2(t)\rangle\rangle$, where the new state vector $[|\psi(t)\rangle\rangle]$ is formally identical to the ordinary state vector $[|\psi(t)\rangle]$, whereas the dual vectors are not simply the Hermitian conjugates of the ordinary vectors, but are also subject to a linear map $\Upsilon_2(t)$:

$$\langle\langle\psi(t)| = \langle\psi(t)|\,\Upsilon_2(t). \tag{34}$$

Within this framework, the generalized completeness relations associated with the three metrics can be written as

$$\sum_n |E_n\rangle\langle E_n|\,\Upsilon_1 = \mathbb{I}, \tag{35}$$

$$\sum_n |n(t)\rangle\langle n(t)|\,\Upsilon_2(t) = \mathbb{I}, \tag{36}$$

$$\left[\sum_{n_0} |E_{n_0}\rangle\langle E_{n_0}| + \sum_{n_+}(|E_{n_+}\rangle\langle E_{n_-}| + |E_{n_-}\rangle\langle E_{n_+}|)\right]\Upsilon_3 = \mathbb{I}, \tag{37}$$

where $|n(t)\rangle = U_r(t,t_0)|n(t_0)\rangle$, and $\{|n(t_0)\rangle\}$ is any complete basis of the Hilbert space.

Although the three metrics arise from different constructions, direct connections exist among them. When the initial basis associated with $\Upsilon_2(t)$ is chosen as the energy eigenbasis of H, the initial metric can be chosen as $\Upsilon_2(t_0) = \Upsilon_1$. In particular, for an entirely real spectrum, the three metrics are equivalent under the same normalization convention

$$\Upsilon_2(t) = \Upsilon_1 = \Upsilon_3. \tag{38}$$

In general, however, they represent different metric structures and lead to different norm equations and completeness relations. Their main properties and distinctions are summarized in Table I, while the relations among the three metrics are presented in Table II.

It should be noted that a metric operator has an overall normalization redundancy

$$\Upsilon \to c\Upsilon, \quad c > 0. \tag{39}$$

This redundancy can be fixed by imposing a normalization convention on the corresponding biorthogonal eigenstates [43].

The analysis and conclusions presented above apply to finite-dimensional non-Hermitian systems whose Hamiltonians possess a complete energy eigenbasis. At an exceptional point, the energy eigenstates become incomplete, so the preceding conclusions cannot be applied directly.

For Υ_1, the incompleteness of the energy eigenstates implies

$$\operatorname{rank}\Upsilon_1 \leqslant \dim \operatorname{span}\{|\widetilde{E}_n\rangle\} < \dim \mathcal{H}, \tag{40}$$

where $\mathcal{H}$ denotes the Hilbert space of the system. Thus, Υ_1 is necessarily singular.

At the exceptional point, the right and left energy eigenstates $\{|E_n\rangle, |\tilde{E}_n\rangle\}$ no longer form complete bases. Several right eigenstates coalesce into a single eigenstate $|E_{\mathrm{EP}}\rangle$, which satisfies [44,45]

$$\langle\tilde{E}_n|E_{\mathrm{EP}}\rangle = 0 \quad \text{for all } n, \tag{41}$$

and

$$\langle E_{\mathrm{EP}}|\Upsilon_1|E_{\mathrm{EP}}\rangle = \sum_n |\langle\widetilde{E}_n|E_{\mathrm{EP}}\rangle|^2 = 0. \tag{42}$$

Nevertheless, for any state $|\psi\rangle$,

$$\langle\psi|\Upsilon_1|\psi\rangle = \sum_n |\langle\widetilde{E}_n|\psi\rangle|^2 \geqslant 0. \tag{43}$$

Hence, Υ_1 is positive semidefinite.

For $\Upsilon_2(t)$, choosing $\Upsilon_2(t_0) = \Upsilon_1$ gives

$$\Upsilon_2(t) = U_l(t,t_0)\Upsilon_1 U_l^\dagger(t,t_0). \tag{44}$$

Since $U_l(t,t_0)$ is invertible, this transformation preserves the rank and positive semidefiniteness of Υ_1. Thus, $\Upsilon_2(t)$ is also singular and positive semidefinite. However, $\Upsilon_2(t)$ is defined by dynamical compatibility, allowing another complete initial basis to be chosen. In particular, an orthonormal initial basis gives

$$\Upsilon_2(t_0) = I, \quad \Upsilon_2(t) = U_l(t,t_0)U_l^\dagger(t,t_0) > 0. \tag{45}$$

Hence, a positive-definite $\Upsilon_2(t)$ can still be obtained at an exceptional point.

For Υ_3, Eq. (6) gives its definition, whereas Eq. (12) gives its energy-eigenstate expression. At an exceptional point, the latter yields a singular matrix. However, the loss of diagonalizability does not necessarily destroy pseudo-Hermiticity, and a Hermitian invertible Υ_3 can still exist through Eq. (6).

TABLE II. Relations among the three metric operators for time-independent Hamiltonians.

Metrics	Condition
$\Upsilon_1 = \Upsilon_2(t_0)$	The initial basis associated with $\Upsilon_2(t)$ is chosen as the energy eigenbasis of H.
$\Upsilon_1 = \Upsilon_2(t) = \Upsilon_3$	The initial basis associated with $\Upsilon_2(t)$ is chosen as the energy eigenbasis of H and the energy spectrum is entirely real.

In summary, at an exceptional point, the energy-eigenstate expressions for Υ_1 and Υ_3 in Eq. (12) yield singular matrices. Since Υ_1 is defined through the energy eigenstates, it is necessarily singular and positive semidefinite. In contrast, the original definitions of $\Upsilon_2(t)$ and Υ_3 do not require a complete energy eigenbasis, allowing a positive-definite $\Upsilon_2(t)$ and a Hermitian invertible Υ_3 to be constructed [46].

C. Metric structures for time-dependent Hamiltonians

In the previous section, we discussed several types of metric operators for time-independent non-Hermitian Hamiltonians. We now extend the discussion to time-dependent Hamiltonians. Consider a time-dependent non-Hermitian Hamiltonian $H(t)$, whose biorthogonal basis also evolves in time. Consequently, the metric operator Υ_1 also becomes time-dependent, i.e., $\Upsilon_1 = \Upsilon_1(t)$. In this case, starting from Eq. (16), an additional contribution arises due to the explicit time dependence of the metric operator

$$\begin{aligned}\langle\psi|\frac{\partial}{\partial t}\Upsilon_1(t)|\psi\rangle &= \sum_{nm} c_m^* c_n(\langle E_m|\dot{\tilde{E}}_n\rangle + \langle\dot{\tilde{E}}_m|E_n\rangle)\\ &= 2\sum_{nm}\mathrm{Re}(c_m^* c_n A_{m\tilde{n}}),\end{aligned} \tag{46}$$

where Berry connection $A_{m\tilde{n}} = \langle E_m|\dot{\tilde{E}}_n\rangle$. Accordingly, the norm balance equation takes the form

$$\frac{\partial}{\partial t}\langle\psi|\Upsilon_1(t)|\psi\rangle = \frac{2}{\hbar}\sum_n |c_n|^2\,\mathrm{Im}\,E_n + 2\sum_{nm}\mathrm{Re}(c_m^* c_n A_{m\tilde{n}}). \tag{47}$$

Compared with the time-independent Hamiltonian case, the norm balance equation acquires an additional contribution associated with the non-Hermitian Berry connection (1-form), which arises from the geometric effect induced by the time evolution of the instantaneous biorthogonal eigenbasis [40,47,48].

The additional geometric contribution substantially modifies the interpretation of the norm evolution. The norm equation is no longer determined solely by the imaginary parts of the instantaneous eigenvalues. The evolution of the instantaneous biorthogonal basis can itself produce an increase or decrease of the norm, giving rise to geometric amplification or attenuation. In particular, even when the instantaneous spectrum is entirely real, the norm associated with $\Upsilon_1(t)$ is not necessarily conserved because the geometric contribution may remain finite. In addition, the geometric contribution affects the gain–loss dynamics and changes the relative weights of the instantaneous eigenstates. Consequently, the long-time dominance inferred from the largest imaginary part of the eigenvalues in the time-independent case does not generally extend to time-dependent systems; the accumulated geometric contribution and the coupling between instantaneous eigenstates must also be taken into account.

Moreover, for Υ_3, the static condition $H^\dagger\Upsilon_3 = \Upsilon_3 H$ is insufficient to ensure norm conservation in the time-dependent case. Therefore, Υ_3 must be promoted to a time-dependent operator that satisfies the evolution equation [33]

$$\frac{\partial\Upsilon_3}{\partial t} = \frac{i}{\hbar}[\Upsilon_3 H - H^\dagger\Upsilon_3]. \tag{48}$$

This generalized metric satisfies the same evolution equation as $\Upsilon_2(t)$. Therefore, within the time-dependent framework, $\Upsilon_2(t)$ and $\Upsilon_3(t)$ can be described by the same dynamical metric, $\Upsilon(t) = \Upsilon_2(t) = \Upsilon_3(t)$.

A further connection between $\Upsilon_1(t)$ and $\Upsilon(t)$ can also be established by choosing the initial basis. If the initial basis associated with $\Upsilon(t)$ is chosen as the instantaneous energy eigenbasis of $H(t_0)$, then

$$\Upsilon(t_0) = \Upsilon_1(t_0). \tag{49}$$

Therefore, under a common choice of the initial metric, $\Upsilon_1(t)$ and $\Upsilon(t)$ coincide at $t = t_0$, although they generally evolve differently thereafter. The main properties and relations of these two metric constructions are summarized in Table III.

At an exceptional point, as in the time-independent case, the instantaneous energy eigenstates are incomplete, and $\Upsilon_1(t)$ is singular and positive semidefinite. However, the dynamical metric $\Upsilon(t)$ remains positive definite when constructed using a complete initial basis. To illustrate the above discussion in a concrete system, we now consider the following model.

III. EXAMPLE

We consider a two-level non-Hermitian system described by the following 2×2 Hamiltonian:

$$H = \begin{pmatrix} re^{i\theta} & s_1 \\ s_2 & re^{-i\theta}\end{pmatrix}, \tag{50}$$

where $r, \theta \in \mathbb{R}$ and $s_1, s_2 > 0$. The eigenvalues of the system H are

$$E_\pm = r\cos\theta \pm \sqrt{s_1 s_2 - r^2\sin^2\theta}, \tag{51}$$

and the corresponding biorthonormal eigenbases are

$$\begin{aligned}|E_+\rangle &= \frac{1}{\sqrt{2\lambda\cos\alpha}}\begin{pmatrix}\lambda e^{i\alpha/2}\\ e^{-i\alpha/2}\end{pmatrix},\\ |E_-\rangle &= \frac{1}{\sqrt{2\lambda\cos\alpha}}\begin{pmatrix}\lambda e^{-i\alpha/2}\\ -e^{i\alpha/2}\end{pmatrix},\\ \langle\tilde{E}_+| &= \frac{1}{\sqrt{2\lambda\cos\alpha}}\begin{pmatrix}e^{i\alpha/2}\\ \lambda e^{-i\alpha/2}\end{pmatrix}^T,\\ \langle\tilde{E}_-| &= \frac{1}{\sqrt{2\lambda\cos\alpha}}\begin{pmatrix}e^{-i\alpha/2}\\ -\lambda e^{i\alpha/2}\end{pmatrix}^T,\end{aligned} \tag{52}$$

where $\sin\alpha = \frac{r\sin\theta}{\sqrt{s_1 s_2}}$, $\cos\alpha = \sqrt{1-\frac{r^2}{s_1 s_2}\sin^2\theta}$, and $\lambda = \sqrt{\frac{s_1}{s_2}}$.

Clearly, based on the eigenvalues, the system can be divided into three distinct regions: $s_1 s_2 > r^2\sin^2\theta$, $s_1 s_2 < r^2\sin^2\theta$, and $s_1 s_2 = r^2\sin^2\theta$, which correspond to the $\mathcal{PT}$ unbroken region, the $\mathcal{PT}$ broken region, and the exceptional point, respectively. We then compare the metrics, focusing specifically on how the imaginary parts of the eigenvalues are manifested in the different metric constructions within the $\mathcal{PT}$-broken region.

TABLE III. Comparison between the metric operators $\Upsilon_1(t)$ and $\Upsilon(t)$ for time-dependent Hamiltonians.

Metric property	$\Upsilon_1(t)$	$\Upsilon(t)=\Upsilon_2(t)=\Upsilon_3(t)$
Positive definiteness	Yes	Yes, if $\Upsilon(t_0)>0$
Norm equation	$\frac{\partial}{\partial t}\langle\psi\|\Upsilon_1(t)\|\psi\rangle=\frac{2}{\hbar}\sum_n \|c_n\|^2 \operatorname{Im} E_n+2\sum_{nm}\operatorname{Re}(c_m^* c_n A_{m\tilde{n}})$	$\frac{\partial}{\partial t}\langle\psi\|\Upsilon(t)\|\psi\rangle=0$
Completeness	$\sum_n \|E_n(t)\rangle\langle E_n(t)\|\Upsilon_1(t)=\mathbb{I}$	$\sum_n \|n(t)\rangle\langle n(t)\|\Upsilon(t)=\mathbb{I}$
Relation	$\Upsilon(t_0)=\Upsilon_1(t_0)=\sum_n \|\tilde{E}_n(t_0)\rangle\langle\tilde{E}_n(t_0)\|$ [The initial basis is the instantaneous eigenbasis of $H(t_0)$]	

A. $\mathcal{PT}$ unbroken region

In the $\mathcal{PT}$-symmetric region $s_1 s_2 > r^2\sin^2\theta$, the eigenvalues $E_\pm = r\cos\theta \pm \sqrt{s_1 s_2 - r^2\sin^2\theta}$ are purely real. The Hamiltonian H satisfies pseudo-Hermiticity, i.e., there exists a Hermitian positive-definite metric operator [49]

$$\eta = \frac{1}{\cos\alpha}\begin{pmatrix} \lambda^{-1} & -i\sin\alpha \\ i\sin\alpha & \lambda \end{pmatrix} \tag{53}$$

such that $H^\dagger = \eta H \eta^{-1}$. Consequently, the three metrics are completely equivalent in this region, represented by the same physical metric.

First, the terms associated with the complex energy spectrum in Υ_3 vanish and the metric becomes

$$\Upsilon_1 = \Upsilon_3 = \sum_n |\tilde{E}_n\rangle\langle\tilde{E}_n| = \frac{1}{\cos\alpha}\begin{pmatrix} \lambda^{-1} & -i\sin\alpha \\ i\sin\alpha & \lambda \end{pmatrix}. \tag{54}$$

Second, we choose the complete basis to be the energy eigenstates, which uniquely determine $\Upsilon_2(t_0)$. Assuming that the system is initially in the eigenstates, the quantum states at time t are given by

$$\begin{aligned} |\mathbf{1}(t)\rangle &= e^{-\frac{i}{\hbar}E_+ t}|E_+\rangle = \frac{e^{-\frac{i}{\hbar}E_+ t}}{\sqrt{2\lambda\cos\alpha}}\begin{pmatrix} \lambda e^{i\alpha/2} \\ e^{-i\alpha/2}\end{pmatrix}, \\ |\mathbf{2}(t)\rangle &= e^{-\frac{i}{\hbar}E_- t}|E_-\rangle = \frac{e^{-\frac{i}{\hbar}E_- t}}{\sqrt{2\lambda\cos\alpha}}\begin{pmatrix} \lambda e^{-i\alpha/2} \\ -e^{i\alpha/2}\end{pmatrix}. \end{aligned} \tag{55}$$

Starting from the generalized completeness condition

$$[|\mathbf{1}(t)\rangle\langle\mathbf{1}(t)| + |\mathbf{2}(t)\rangle\langle\mathbf{2}(t)|]\Upsilon_2(t) = \mathbb{I}, \tag{56}$$

the explicit form of $\Upsilon_2(t)$ can be obtained as

$$\Upsilon_2(t) = [|\mathbf{1}(t)\rangle\langle\mathbf{1}(t)| + |\mathbf{2}(t)\rangle\langle\mathbf{2}(t)|]^{-1}. \tag{57}$$

Substituting the evolved states in Eq. (55) into Eq. (57) yields

$$\Upsilon_2(t) = \frac{1}{\cos\alpha}\begin{pmatrix} \lambda^{-1} & -i\sin\alpha \\ i\sin\alpha & \lambda \end{pmatrix}. \tag{58}$$

In summary, Υ_1, $\Upsilon_2(t)$, and Υ_3 have exactly the same explicit form in the $\mathcal{PT}$ unbroken region, indicating that the metric operators obtained from the three different constructions actually correspond to the same physical inner product.

B. $\mathcal{PT}$ broken region

In the $\mathcal{PT}$ broken region, the energy eigenvalues of the system becomes complex. Nevertheless, the eigenstates can still be expressed as

$$\begin{aligned} \langle\tilde{E}_+| &= \frac{1}{\sqrt{2\lambda\sinh\beta}}\begin{pmatrix} e^{\beta/2} \\ -i\lambda e^{-\beta/2}\end{pmatrix}^T, \\ \langle\tilde{E}_-| &= \frac{1}{\sqrt{2\lambda\sinh\beta}}\begin{pmatrix} -ie^{-\beta/2} \\ -\lambda e^{\beta/2}\end{pmatrix}^T, \end{aligned} \tag{59}$$

with

$$\begin{aligned} \alpha &= \frac{\pi}{2} - i\beta, \quad (\beta>0, \beta\in\mathbb{R}), \\ \cos\alpha &= \sqrt{1-\frac{r^2}{s_1 s_2}\sin^2\theta} = i\sinh\beta. \end{aligned} \tag{60}$$

First, the metric Υ_1 can be readily obtained as

$$\Upsilon_1 = \frac{1}{\sinh\beta}\begin{pmatrix} \lambda^{-1}\cosh\beta & -i \\ i & \lambda\cosh\beta\end{pmatrix}. \tag{61}$$

Second, due to the presence of the complex energy spectrum, the metric Υ_3 contains cross terms between eigenstates of different complex energy levels, and is given by

$$\Upsilon_3 = |\tilde{E}_+\rangle\langle\tilde{E}_-| + |\tilde{E}_-\rangle\langle\tilde{E}_+| = \begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}. \tag{62}$$

Finally, to determine $\Upsilon_2(t_0)$, we choose the eigenstates as the complete initial basis. Their time evolution is given by

$$\begin{aligned} |\mathbf{1}(t)\rangle &= e^{-\frac{i}{\hbar}E_+ t}|E_+\rangle = \frac{e^{-\frac{i}{\hbar}E_+ t}}{\sqrt{2\lambda\sinh\beta}}\begin{pmatrix} \lambda e^{\beta/2} \\ -ie^{-\beta/2}\end{pmatrix}, \\ |\mathbf{2}(t)\rangle &= e^{-\frac{i}{\hbar}E_- t}|E_-\rangle = \frac{e^{-\frac{i}{\hbar}E_- t}}{\sqrt{2\lambda\sinh\beta}}\begin{pmatrix} -i\lambda e^{-\beta/2} \\ -e^{\beta/2}\end{pmatrix}. \end{aligned} \tag{63}$$

Combining with the definition of Υ_2, we obtain

$$\begin{aligned} \Upsilon_2(t) &= \left[|E_+\rangle\langle E_+| e^{\frac{2}{\hbar}\operatorname{Im}E_+ t} + |E_-\rangle\langle E_-| e^{\frac{2}{\hbar}\operatorname{Im}E_- t}\right]^{-1} \\ &= \left[\frac{1}{\sinh\beta}\begin{pmatrix} \lambda\cosh(\phi(t)+\beta) & i\cosh\phi(t) \\ -i\cosh\phi(t) & \lambda^{-1}\cosh(\phi(t)-\beta)\end{pmatrix}\right]^{-1} \\ &= \frac{1}{\sinh\beta}\begin{pmatrix} \lambda^{-1}\cosh(\phi(t)-\beta) & -i\cosh\phi(t) \\ i\cosh\phi(t) & \lambda\cosh(\phi(t)+\beta)\end{pmatrix}, \end{aligned} \tag{64}$$

with

$$\phi(t) = \frac{2\mathrm{Im}E_+ t}{\hbar} = -\frac{2\mathrm{Im}E_- t}{\hbar} = \frac{2\sqrt{r^2\sin^2\theta - s_1 s_2}t}{\hbar} \tag{65}$$

From the above results, the inner product defined by $\Upsilon_2(t)$ satisfies the norm conservation condition. However, its structure retains an explicit dependence on the imaginary part of the energy spectrum, which is manifested through the time-dependent function $\phi(t)$. In contrast, the norm balance equation associated with Υ_1 already includes the contribution from the imaginary part of the energy spectrum. As a result, the term $\phi(t)$ does not appear explicitly in the final expression.

In summary, the essential distinction between the two metrics lies in how the effect of the imaginary part of the spectrum is represented. Although Υ_1 contains no explicit compensating factor associated with $\mathrm{Im}\,E_\pm$, its effect remains explicit in the corresponding norm-balance equation. In contrast, $\Upsilon_2(t)$ incorporates this effect into its own time dependence, thus canceling the gain-loss contribution in the norm equation and ensuring norm conservation.

C. Exceptional point

In the exceptional point region $s_1 s_2 = r^2\sin^2\theta$, we consider the branch $r\sin\theta > 0$. The eigenvalues and eigenstates of the system coalesce, and the Hamiltonian becomes nondiagonalizable. In this case, the coalesced eigenstate takes the form $|\tilde{E}_+\rangle = |\tilde{E}_-\rangle = \frac{1}{\sqrt{2r\sin\theta}}\binom{\sqrt{s_2}}{i\sqrt{s_1}}$.

On the one hand, although only one linearly independent eigenstate remains, the eigenstate expressions for Υ_1 and Υ_3 give

$$\Upsilon_1 = \Upsilon_3 = \begin{pmatrix} \sqrt{s_2/s_1} & -i \\ i & \sqrt{s_1/s_2} \end{pmatrix}. \tag{66}$$

Both matrices are singular. However, the defining relation $H^\dagger\Upsilon_3 = \Upsilon_3 H$ admits

$$\Upsilon_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{67}$$

Υ_3 can remain invertible at the exceptional point. On the other hand, since $\Upsilon_2(t)$ is constructed from the dynamical compatibility condition, we choose another complete initial basis. The system is initially prepared in the quantum states $|\mathbf{1}(0)\rangle = \binom{1}{0}$ and $|\mathbf{2}(0)\rangle = \binom{0}{1}$, which evolve in time as

$$\begin{aligned} |\mathbf{1}(t)\rangle &= e^{\frac{-ir\cos\theta\cdot t}{\hbar}}\begin{pmatrix} 1+\frac{r\sin\theta\cdot t}{\hbar} \\ \frac{-is_2 t}{\hbar} \end{pmatrix}, \\ |\mathbf{2}(t)\rangle &= e^{\frac{-ir\cos\theta\cdot t}{\hbar}}\begin{pmatrix} \frac{-is_1 t}{\hbar} \\ 1-\frac{r\sin\theta\cdot t}{\hbar} \end{pmatrix}, \end{aligned} \tag{68}$$

From Eq. (36), one can further obtain

$$\begin{aligned} \Upsilon_2(t) &= \begin{pmatrix} (1+q)^2+\lambda^2 q^2 & i[(\lambda^{-1}-\lambda)q+(\lambda+\lambda^{-1})q^2] \\ -i[(\lambda^{-1}-\lambda)q+(\lambda+\lambda^{-1})q^2] & (1-q)^2+\lambda^{-2}q^2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (1-q)^2+\lambda^{-2}q^2 & i[(\lambda-\lambda^{-1})q-(\lambda+\lambda^{-1})q^2] \\ -i[(\lambda-\lambda^{-1})q-(\lambda+\lambda^{-1})q^2] & (1+q)^2+\lambda^2 q^2 \end{pmatrix}, \end{aligned} \tag{69}$$

where $q = \frac{r\sin\theta\cdot t}{\hbar}$.

D. Time-dependent case

We extend the preceding model to the time-dependent Hamiltonian

$$H(t) = \begin{pmatrix} re^{i\theta} & se^{-i\omega t} \\ se^{i\omega t} & re^{-i\theta} \end{pmatrix}. \tag{70}$$

Considering the transformation

$$|\psi_r(t)\rangle = D(t)|\chi_r(t)\rangle, \quad \langle\psi_l(t)| = \langle\chi_l(t)|D^\dagger(t), \tag{71}$$

with

$$D(t) = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}. \tag{72}$$

Then, the evolution problem governed by $H(t)$ is equivalent to solving

$$\begin{aligned} i\hbar\frac{d}{dt}|\chi_r(t)\rangle &= H_{\mathrm{eff}}|\chi_r(t)\rangle, \\ i\hbar\frac{d}{dt}|\chi_l(t)\rangle &= H_{\mathrm{eff}}^\dagger|\chi_l(t)\rangle, \end{aligned} \tag{73}$$

with the time-independent effective Hamiltonian

$$\begin{aligned} H_{\mathrm{eff}} &= D^\dagger(t)H(t)D(t) - i\hbar D^\dagger(t)\dot{D}(t) \\ &= \begin{pmatrix} re^{i\theta}-\hbar\omega/2 & s \\ s & re^{-i\theta}+\hbar\omega/2 \end{pmatrix}. \end{aligned} \tag{74}$$

Since $D(0) = I$, the evolution operators are

$$\begin{aligned} U_r(t,0) &= \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_0^t H(\tau)\,d\tau\right) = D(t)e^{-iH_{\mathrm{eff}}t/\hbar}, \\ U_l(t,0) &= \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_0^t H^\dagger(\tau)\,d\tau\right) = D(t)e^{-iH_{\mathrm{eff}}^\dagger t/\hbar}. \end{aligned} \tag{75}$$

The instantaneous eigenvalues of $H(t)$ are

$$E_\pm = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta}. \tag{76}$$

Using $H(t) = D(t)H(0)D^\dagger(t)$, the corresponding biorthonormal eigenstates are

$$
\begin{aligned}
|E_+(t)\rangle &= \frac{D(t)}{\sqrt{2\cos\alpha}}\begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2}\end{pmatrix}, \\
|E_-(t)\rangle &= \frac{D(t)}{\sqrt{2\cos\alpha}}\begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2}\end{pmatrix}, \\
\langle\widetilde{E}_+(t)| &= \frac{1}{\sqrt{2\cos\alpha}}(e^{i\alpha/2} \quad e^{-i\alpha/2})D^\dagger(t), \\
\langle\widetilde{E}_-(t)| &= \frac{1}{\sqrt{2\cos\alpha}}(e^{-i\alpha/2} \quad -e^{i\alpha/2})D^\dagger(t),
\end{aligned} \tag{77}
$$

where

$$
\cos\alpha = \sqrt{1-\frac{r^2}{s_1 s_2}\sin^2\theta} = \sqrt{1-\frac{r^2}{s^2}\sin^2\theta}. \tag{78}
$$

In this model, the time dependence does not affect the $\mathcal{PT}$ symmetry and the instantaneous spectrum. Therefore, we consider the same three cases as above: the $\mathcal{PT}$ unbroken region, the $\mathcal{PT}$ broken region, and the exceptional point.

For a complete instantaneous eigenbasis, the metric $\Upsilon_1(t)$ is

$$
\begin{aligned}
\Upsilon_1(t) &= \sum_n |\tilde{E}_n(t)\rangle\langle\tilde{E}_n(t)| \\
&= \frac{1}{|\cos\alpha|}\begin{pmatrix} \cosh(\mathrm{Im}\,\alpha) & -i\sin(\mathrm{Re}\,\alpha)e^{-i\omega t} \\ i\sin(\mathrm{Re}\,\alpha)e^{i\omega t} & \cosh(\mathrm{Im}\,\alpha)\end{pmatrix}.
\end{aligned} \tag{79}
$$

Choosing the instantaneous eigenstates at $t = 0$ as the complete initial basis, the dynamical metrics with the same initial condition can be written as

$$
\Upsilon(t) = \Upsilon_2(t) = \Upsilon_3(t) = U_l(t,0)\Upsilon_1(0)U_l^\dagger(t,0). \tag{80}
$$

To evaluate this expression, we write

$$
H_{\mathrm{eff}} = r\cos\theta\, I + M, \quad M = \begin{pmatrix} z & s \\ s & -z\end{pmatrix}, \tag{81}
$$

where $z = ir\sin\theta - \hbar\omega/2$.

The direct calculation gives

$$
M^2 = (s^2+z^2)I = \kappa^2 I, \quad \kappa = \sqrt{s^2+z^2}. \tag{82}
$$

Therefore, the matrix satisfies the following equation:

$$
e^{-iMt/\hbar} = \cos\frac{\kappa t}{\hbar}\, I - i\frac{\sin(\kappa t/\hbar)}{\kappa}M. \tag{83}
$$

Substituting this result into $U_l(t,0) = D(t)e^{-iH_{\mathrm{eff}}^\dagger t/\hbar}$, we obtain

$$
U_l(t,0) = e^{-ir\cos\theta\, t/\hbar}D(t)\big(f^*I - ig^*M^\dagger\big), \tag{84}
$$

where $f(t) = \cos\frac{\kappa t}{\hbar}$ and $g(t) = \frac{\sin(\kappa t/\hbar)}{\kappa}$. Consequently,

$$
\begin{aligned}
\Upsilon(t) &= D(t)(f^*I - ig^*M^\dagger)\Upsilon_1(0)(fI + igM)D^\dagger(t) \\
&= \frac{1}{|\cos\alpha|}\begin{pmatrix} F(t)+C(t) & e^{-i\omega t}B(t) \\ e^{i\omega t}B^*(t) & F(t)-C(t)\end{pmatrix},
\end{aligned} \tag{85}
$$

with

$$
\begin{aligned}
F(t) =\ & \cosh(\mathrm{Im}\,\alpha)[|f|^2 + (s^2+|z|^2)|g|^2] \\
& - 2sr\sin\theta\sin(\mathrm{Re}\,\alpha)|g|^2, \\
B(t) =\ & \hbar\omega\sin(\mathrm{Re}\,\alpha)\,\mathrm{Re}(f^*g) \\
& + 2[r\sin\theta\sin(\mathrm{Re}\,\alpha) - s\cosh(\mathrm{Im}\,\alpha)]\,\mathrm{Im}(f^*g) \\
& + i\sin(\mathrm{Re}\,\alpha)\big[(s^2+|z|^2)|g|^2 - |f|^2\big] \\
& - 2isr\sin\theta\cosh(\mathrm{Im}\,\alpha)|g|^2, \\
C(t) =\ & 2[s\sin(\mathrm{Re}\,\alpha) - r\sin\theta\cosh(\mathrm{Im}\,\alpha)]\,\mathrm{Re}(f^*g) \\
& + \hbar\omega\cosh(\mathrm{Im}\,\alpha)\,\mathrm{Im}(f^*g).
\end{aligned} \tag{86}
$$

To examine the properties of the norms defined by different metrics, we expand the quantum state as

$$
|\psi(t)\rangle = \sum_{n=\pm} c_n(t)|E_n(t)\rangle, \quad c_n(t) = \langle\tilde{E}_n(t)|\psi(t)\rangle. \tag{87}
$$

The norm equation associated with $\Upsilon_1(t)$ satisfies

$$
\begin{aligned}
\frac{dN_{\Upsilon_1}}{dt} &= \frac{2}{\hbar}\sum_{n=\pm}|c_n|^2\,\mathrm{Im}\,E_n + 2\sum_{m,n=\pm}\mathrm{Re}\,(c_m^* c_n A_{m\tilde{n}}) \\
&= \frac{2}{\hbar}\sum_{n=\pm}|c_n|^2\,\mathrm{Im}\,E_n + \omega\,\mathrm{Re}\,[\tan(\alpha^*)]\big(|c_-|^2 - |c_+|^2\big) \\
&\quad + 2\omega\,\mathrm{Im}\left[\frac{1}{\cos(\alpha^*)}\right]\mathrm{Re}(c_+^* c_-).
\end{aligned} \tag{88}
$$

The first term arises from the imaginary parts of the eigenvalues. The remaining two terms are the diagonal and off-diagonal geometric contributions from the evolving eigenbasis, respectively.

Similarly, the norm equation associated with $\Upsilon(t) = U_l(t,0)\Upsilon_1(0)U_l^\dagger(t,0)$ satisfies

$$
\begin{aligned}
\frac{dN_\Upsilon}{dt} &= \frac{i}{\hbar}\langle\psi|H^\dagger\Upsilon - \Upsilon H|\psi\rangle + \langle\psi|\dot{\Upsilon}|\psi\rangle \\
&= \frac{i}{\hbar}\langle\psi|H^\dagger\Upsilon - \Upsilon H|\psi\rangle + \frac{i}{\hbar}\langle\psi|\Upsilon H - H^\dagger\Upsilon|\psi\rangle \\
&= 0.
\end{aligned} \tag{89}
$$

The time dependence of $\Upsilon(t)$ exactly compensates for the norm variation generated by the state evolution.

In the $\mathcal{PT}$ unbroken region, $s^2 > r^2\sin^2\theta$, α is real. The metrics become

$$
\Upsilon_1(t) = \frac{1}{\cos\alpha}\begin{pmatrix} 1 & -i\sin\alpha\, e^{-i\omega t} \\ i\sin\alpha\, e^{i\omega t} & 1\end{pmatrix}, \tag{90}
$$

and

$$
\Upsilon(t) = \frac{1}{\cos\alpha}\begin{pmatrix} F(t)+C(t) & e^{-i\omega t}B(t) \\ e^{i\omega t}B^*(t) & F(t)-C(t)\end{pmatrix}, \tag{91}
$$

where the coefficients reduce to

$$
F(t) = |f|^2 + \left(s^2\cos^2\alpha + \frac{\hbar^2\omega^2}{4}\right)|g|^2,
$$

$$B(t) = \hbar\omega \sin\alpha \ \mathrm{Re}(f^* g) - 2s\cos^2\alpha \ \mathrm{Im}(f^* g) - i\sin\alpha\left[|f|^2 + \left(s^2\cos^2\alpha - \frac{\hbar^2\omega^2}{4}\right)|g|^2\right],$$
$$C(t) = \hbar\omega \ \mathrm{Im}(f^* g). \quad (92)$$

Since $\mathrm{Im}\, E_\pm = 0$, the norm equation gives

$$\frac{dN_{\Upsilon_1}}{dt} = \omega\tan\alpha[|c_-(t)|^2 - |c_+(t)|^2]. \quad (93)$$

Therefore, the norm variation $\frac{dN_{\Upsilon_1}}{dt}$ in the real spectrum region is entirely due to the geometric contribution from the evolving eigenbasis.

When $\omega = 0$, we recover the time-independent Hamiltonian H with $s_1 = s_2 = s$. Then $F(t) = 1, C(t) = 0$ and $B(t) = -i\sin\alpha$, so Eq. (91) reduces to

$$\Upsilon(t) = \Upsilon_1 = \frac{1}{\cos\alpha}\begin{pmatrix} 1 & -i\sin\alpha \\ i\sin\alpha & 1 \end{pmatrix}. \quad (94)$$

In the $\mathcal{PT}$ broken region, we use the parametrization $\alpha = \pi/2 - i\beta$ as in the time-independent case, giving

$$z = is\cosh\beta - \frac{\hbar\omega}{2},$$
$$\kappa^2 = -s^2\sinh^2\beta + \frac{\hbar^2\omega^2}{4} - i\hbar\omega s\cosh\beta. \quad (95)$$

The functions $f(t)$ and $g(t)$ retain their previous definitions with this value of κ. The metrics become

$$\Upsilon_1(t) = \frac{1}{\sinh\beta}\begin{pmatrix} \cosh\beta & -ie^{-i\omega t} \\ ie^{i\omega t} & \cosh\beta \end{pmatrix}, \quad (96)$$

and

$$\Upsilon(t) = \frac{1}{\sinh\beta}\begin{pmatrix} F(t) + C(t) & e^{-i\omega t}B(t) \\ e^{i\omega t}B^*(t) & F(t) - C(t) \end{pmatrix}, \quad (97)$$

where

$$F(t) = \cosh\beta\left[|f|^2 + \left(s^2\sinh^2\beta + \frac{\hbar^2\omega^2}{4}\right)|g|^2\right],$$
$$B(t) = \hbar\omega \ \mathrm{Re}(f^* g) - i\left[|f|^2 + \left(s^2\sinh^2\beta - \frac{\hbar^2\omega^2}{4}\right)|g|^2\right],$$
$$C(t) = -2s\sinh^2\beta \ \mathrm{Re}(f^* g) + \hbar\omega\cosh\beta \ \mathrm{Im}(f^* g). \quad (98)$$

Since $\mathrm{Im}\, E_\pm \neq 0$, the norm equation gives

$$\frac{dN_{\Upsilon_1}}{dt} = \frac{2\ \mathrm{Im}\, E_+}{\hbar}[|c_+(t)|^2 - |c_-(t)|^2] + \frac{2\omega}{\sinh\beta}\mathrm{Re}[c_+^*(t)c_-(t)] \quad (99)$$

The first term arises from the imaginary parts of the eigenvalues, whereas the second is the geometric contribution from the evolving eigenbasis.

When $\omega = 0$, Eq. (97) likewise reduces to

$$\Upsilon(t) = \frac{1}{\sinh\beta}\begin{pmatrix} \cosh(\phi - \beta) & -i\cosh\phi \\ i\cosh\phi & \cosh(\phi + \beta) \end{pmatrix}, \quad (100)$$

where $\phi = 2\sqrt{r^2\sin^2\theta - s^2}\, t/\hbar$.

At the exceptional point, $s^2 = r^2\sin^2\theta$ with $s \neq 0$, the instantaneous energy eigenbasis is incomplete. The metric $\Upsilon_1(t)$ constructed from the energy eigenstates can be written as

$$\Upsilon_1^{\mathrm{EP}}(t) = \begin{pmatrix} 1 & -ie^{-i\omega t} \\ ie^{i\omega t} & 1 \end{pmatrix}. \quad (101)$$

Since $\det \Upsilon_1^{\mathrm{EP}}(t) = 0$, this metric is singular and not positive definite.

For $\Upsilon_2(t)$, if the initial states are chosen as the instantaneous energy eigenstates at the exceptional point, the resulting metric is likewise not positive definite. However, since $\Upsilon_2(t)$ is constructed from the dynamical compatibility condition, another complete initial basis can be chosen, for example,

$$|\mathbf{1}(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\mathbf{2}(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (102)$$

This choice gives $\Upsilon_2(0) = I$ and the resulting metric is

$$\Upsilon_2(t) = U_l(t,0)U_l^\dagger(t,0) = \begin{pmatrix} F_{\mathrm{EP}}(t) + C_{\mathrm{EP}}(t) & e^{-i\omega t}B_{\mathrm{EP}}(t) \\ e^{i\omega t}B_{\mathrm{EP}}^*(t) & F_{\mathrm{EP}}(t) - C_{\mathrm{EP}}(t) \end{pmatrix}, \quad (103)$$

where

$$F_{\mathrm{EP}}(t) = |f|^2 + \left(2s^2 + \frac{\hbar^2\omega^2}{4}\right)|g|^2,$$
$$C_{\mathrm{EP}}(t) = -2r\sin\theta \ \mathrm{Re}(f^* g) + \hbar\omega \ \mathrm{Im}(f^* g),$$
$$B_{\mathrm{EP}}(t) = -2s[\mathrm{Im}(f^* g) + ir\sin\theta|g|^2]. \quad (104)$$

Since $U_l(t,0)$ is invertible, the metric $\Upsilon_2(t) = U_l(t,0)U_l^\dagger(t,0)$ is positive definite. Thus, even at an exceptional point, the positive definiteness of $\Upsilon_2(t)$ can be ensured by choosing another complete initial basis.

IV. CONCLUSION

In this work, we compared three metric operators for finite-dimensional non-Hermitian systems with complex spectra: the positive-definite metric Υ_1 associated with the cross inner product, the dynamical metric $\Upsilon_2(t)$, and the pseudo-Hermitian metric Υ_3. Their time dependence, positive definiteness, norm equations, and generalized completeness relations have been analyzed.

For time-independent Hamiltonians, the norm associated with Υ_1 obeys a balance equation determined by the imaginary parts of the eigenvalues. In the long-time limit, it is dominated by the contribution associated with the largest imaginary part among those with nonzero initial weights. By contrast, $\Upsilon_2(t)$ incorporates the same spectral contribution into the time dependence of the metric and preserves the norm. When the initial basis associated with $\Upsilon_2(t)$ is chosen as the energy eigenbasis, one can choose $\Upsilon_2(t_0) = \Upsilon_1$. In particular, for an entirely real spectrum and under the same normalization convention, the three metrics reduce to the same pseudo-Hermitian metric

$$\Upsilon_2(t) = \Upsilon_1 = \Upsilon_3 = \eta. \quad (105)$$

At exceptional points, the energy eigenstates are incomplete and Υ_1 is singular and positive semidefinite. Choosing

$\Upsilon_2(t_0) = \Upsilon_1$ likewise yields a singular positive-semidefinite $\Upsilon_2(t)$. However, another complete initial basis can be chosen to obtain a positive-definite dynamical metric. Similarly, although the energy-eigenstate expression for Υ_3 yields a singular matrix, its defining pseudo-Hermitian relation can still admit Hermitian invertible solutions.

For time-dependent Hamiltonians, the norm-balance equation for $\Upsilon_1(t)$ contains both the spectral gain-loss contribution and an additional geometric contribution arising from the evolution of the instantaneous biorthogonal basis. For $\Upsilon_3(t)$, the instantaneous pseudo-Hermitian relation alone is insufficient to ensure norm conservation. Therefore, $\Upsilon_3(t)$ must satisfy the same metric evolution equation as $\Upsilon_2(t)$. When the same initial metric is chosen, they coincide throughout the evolution and can be denoted by $\Upsilon(t) = \Upsilon_2(t) = \Upsilon_3(t)$. If the initial basis is chosen as the instantaneous energy eigenbasis of $H(t_0)$, then $\Upsilon(t_0) = \Upsilon_1(t_0)$, although $\Upsilon(t)$ and $\Upsilon_1(t)$ are generally unequal at later times. At instantaneous exceptional points, $\Upsilon_1(t)$ is likewise singular and positive semidefinite. In contrast, the dynamical metric $\Upsilon(t)$ remains positive definite when constructed using a complete initial basis.

Finally, the concrete model illustrates how the spectral and geometric contributions enter the norm balance equation for Υ_1, while $\Upsilon_2(t)$ preserves the norm by satisfying the dynamical compatibility condition. The properties of these metrics at exceptional points are also illustrated. These results clarify the distinctions and direct relations among the three metric constructions.

ACKNOWLEDGMENTS

This work was supported by Science Challenge Project (No. TZ2025017) and the National Natural Science Foundation of China (Grants No. 12575011 and No. 12088101) and the Science and Technology Development Plan Project of Jilin Province, China (Grant No. 20240101321JC).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: METRIC Υ_2

1. Formal solution and redundancy

In Ref. [32], a generalized inner product is defined as $\langle\langle\psi_1(t)|\psi_2(t)\rangle\rangle$. The newly introduced ket $|\psi(t)\rangle\rangle$ is formally identical to the ordinary state vector $|\psi(t)\rangle$, while the corresponding bra is not the conventional Hermitian conjugate, but is instead obtained by the action of a linear map $\Upsilon_2(t)$ on the ordinary bra

$$\langle\langle\psi(t)| = \langle\psi(t)|\,\Upsilon_2(t). \tag{A1}$$

The metric operator $G(t)$ satisfies the evolution equation

$$\partial_t \Upsilon_2(t) = \frac{i}{\hbar}[\Upsilon_2(t)H(t) - H^\dagger(t)\Upsilon_2(t)]. \tag{A2}$$

To simplify the notation, we introduce a superoperator $\mathcal{L}(t)$ whose action is defined by

$$\mathcal{L}(t)\,\Upsilon_2 = \frac{i}{\hbar}\big(\Upsilon_2 H - H^\dagger \Upsilon_2\big), \tag{A3}$$

so that the evolution equation can be written as

$$\partial_t \Upsilon_2(t) = \mathcal{L}(t)\,\Upsilon_2(t). \tag{A4}$$

In a finite-dimensional Hilbert space, the formal solution to this equation reads

$$\begin{aligned}\Upsilon_2(t) &= \mathcal{T}\exp\left\{\int_{t_0}^{t}\mathcal{L}(\tau)\,d\tau\right\}\Upsilon_2(t_0)\\ &= \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}H^\dagger(\tau)d\tau\right)\\ &\quad\times\Upsilon_2(t_0)\,\tilde{\mathcal{T}}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t}H(\tau)d\tau\right)\\ &= U_l(t,t_0)\Upsilon_2(t_0)U_l^\dagger(t,t_0),\end{aligned} \tag{A5}$$

with

$$U_r(t,t_0) = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}H(\tau)d\tau\right), \tag{A6}$$

$$U_l(t,t_0) = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}H^\dagger(\tau)d\tau\right), \tag{A7}$$

where $\mathcal{T}$ denotes the time-ordering operator and $\tilde{\mathcal{T}}$ the anti-time-ordering operator.

It is worth noting that the formal solution possesses a redundancy, manifested in the arbitrariness of $\Upsilon_2(t_0)$. Specifically, consider a transformation matrix T satisfying

$$\partial_t T = \frac{i}{\hbar}(TH - HT), \tag{A8}$$

whose formal solution is given by

$$\begin{aligned}T(t) &= \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}H(\tau)d\tau\right)T(t_0)\\ &\quad\times\tilde{\mathcal{T}}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t}H(\tau)d\tau\right)\\ &= [U_l^\dagger(t,t_0)]^{-1}T(t_0)U_l^\dagger(t,t_0).\end{aligned} \tag{A9}$$

From this expression and its Hermitian conjugate, one obtains

$$\begin{aligned}U_l^\dagger(t,t_0)T(t) &= T(t_0)U_l^\dagger(t,t_0),\\ T^\dagger(t)U_l(t,t_0) &= U_l(t,t_0)T^\dagger(t_0).\end{aligned} \tag{A10}$$

Consequently, the metric

$$\begin{aligned}G(t) &= T^\dagger(t)\Upsilon_2(t)T(t)\\ &= T^\dagger(t)U_l(t,t_0)\Upsilon_2(t_0)U_l^\dagger(t,t_0)T(t)\\ &= U_l(t,t_0)T^\dagger(t_0)\Upsilon_2(t_0)T(t_0)U_l^\dagger(t,t_0)\\ &= U_l(t,t_0)G(t_0)U_l^\dagger(t,t_0),\end{aligned} \tag{A11}$$

also satisfies Eq. (A2). In a definite eigenbasis, $\Upsilon_2(t_0)$ is uniquely determined and redundancy is removed.

2. Equivalence between the metric dynamics and the conservation of the biorthogonal inner product

Nevertheless, consider the left and right quantum states satisfying the Schrödinger equations for a non-Hermitian Hamiltonian H:

$$i\hbar\frac{\partial}{\partial t}|\psi_r(t)\rangle = H|\psi_r(t)\rangle, \tag{A12}$$

$$i\hbar\frac{\partial}{\partial t}|\psi_l(t)\rangle = H^\dagger|\psi_l(t)\rangle. \tag{A13}$$

The initial conditions are chosen as

$$\langle\psi_l(t_0)| = \langle\psi_r(t_0)|\Upsilon_2(t_0), \tag{A14}$$

$$\langle\psi_l(t_0)|\psi_r(t_0)\rangle = \langle\psi_r(t_0)|\Upsilon_2(t_0)|\psi_r(t_0)\rangle = 1, \tag{A15}$$

we have

$$\begin{aligned}|\psi_r(t)\rangle &= U_r(t,t_0)|\psi_r(t_0)\rangle,\\ \langle\psi_l(t)| &= \langle\psi_l(t_0)|U_l^\dagger(t,t_0).\end{aligned} \tag{A16}$$

If we require the biorthogonal inner product to be preserved

$$\langle\psi_l(t)|\psi_r(t)\rangle = 1, \quad \forall t, \tag{A17}$$

then the evolution operators must satisfy

$$U_l^\dagger(t,t_0)U_r(t,t_0) = I. \tag{A18}$$

Under this condition, the time evolution of the biorthogonal inner product can be expanded as follows:

$$\begin{aligned}\langle\psi_l(t)|\psi_r(t)\rangle &= \langle\psi_l(t_0)|U_l^\dagger(t,t_0)U_r(t,t_0)|\psi_r(t_0)\rangle\\ &= \langle\psi_r(t_0)|\Upsilon_2(t_0)U_l^\dagger(t,t_0)U_r(t,t_0)|\psi_r(t_0)\rangle\\ &= \langle\psi_r(t_0)|U_r^\dagger(t,t_0)[U_r^\dagger(t,t_0)]^{-1}\Upsilon_2(t_0)\\ &\quad U_l^\dagger(t,t_0)U_r(t,t_0)|\psi_r(t_0)\rangle\\ &= \langle\psi_r(t)|[U_r^\dagger(t,t_0)]^{-1}\Upsilon_2(t_0)U_l^\dagger(t,t_0)|\psi_r(t)\rangle\\ &= \langle\psi_r(t)|\Upsilon_2(t)|\psi_r(t)\rangle = 1,\end{aligned} \tag{A19}$$

with

$$\Upsilon_2(t) = [U_r^\dagger(t,t_0)]^{-1}\Upsilon_2(t_0)\,U_l^\dagger(t,t_0). \tag{A20}$$

Substituting Eq. (A6) into Eq. (A20), we obtain

$$\begin{aligned}\Upsilon_2(t) = &\mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}H^\dagger(\tau)d\tau\right)\\ &\times\Upsilon_2(t_0)\,\tilde{\mathcal{T}}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t}H(\tau)d\tau\right),\end{aligned} \tag{A21}$$

which is fully consistent with the formal solution obtained directly from the superoperator equation, Eq. (A5). Using the relation $U_l^\dagger U_r = I$, i.e., $[U_r^\dagger]^{-1} = U_l$, Eq. (A21) can be further simplified to

$$\Upsilon_2(t) = U_l(t,t_0)\,\Upsilon_2(t_0)\,U_l^\dagger(t,t_0). \tag{A22}$$

Therefore, the differential equation satisfied by the metric $\Upsilon_2(t)$ and the condition for the conservation of the biorthogonal inner product are indeed equivalent.

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