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Spinor Decomposition of SU(2) Gauge Potential and Spinor Structures of Chern–Simons and Chern Density*

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Abstract *In this paper, the decomposition of SU(2) gauge potential in terms of Pauli spinor is studied. Using this decomposition, the spinor structures of Chern–Simons form and the Chern density are obtained. Furthermore, the knot quantum number of non-Abelian gauge theory can be expressed by the Chern–Simons spinor structure, and the second Chern number is characterized by the Hopf indices and the Brouwer degrees of ϕ-mapping.*

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Key words: spinor, SU(2) gauge potential decomposition, Chern–Simons, the second Chern class, knot

1 Introduction

In recent years the decomposition theory of gauge potential has played a more and more important role in theoretical physics and mathematics. Since the decomposition theory reveals the inner structure of gauge potential, it inputs the topological and other information to the gauge potential (i.e. the connection of principal bundle), and establishes a direct relationship between differential geometry and topology of gauge field. From this viewpoint much progress has been made by other authors[1,2] and by us, such as the decomposition of U(1) gauge potential and the U(1) topological quantum mechanics, the decomposition of SO(N) spin connection and the structure of GBC topological current, and the decomposition of SU(N) connection and the effective theory of SU(N) QCD, etc.[3−6]

In this paper, the spinor decomposition of SU(2) gauge potential is studied. It is known that the Pauli spinor (the complex vector) is the fundamental representation of SU(2) gauge field which just carries the geometric information of the manifold itself. Therefore compared with other decomposition theories of gauge potential in non-perturbative SU(2) gauge theory,[1,7] the decomposition in terms of spinor is more direct when calculating some topological quantity, such as the Chern–Simons form and the second Chern class.

This paper is arranged as follows. In Sec. 2, the spinor decomposition of SU(2) gauge potential is given. In Sec. 3, we obtain the spinor structure of Chern–Simons form by making use of this decomposition, and furthermore the knot quantum number of non-Abelian gauge theory is expressed by this Chern–Simons spinor structure. In Sec. 4, the spinor structure of SU(2) Chern density is obtained. By making use of the ϕ-mapping topological current theory, the Chern density is expressed as $\delta(\vec{\phi})$. Therefore, the zero points of ϕ field are characterized by the Hopf indices (β_j) and Brouwer degrees (η_j) of ϕ-mapping, and the second Chern number, which is directly related to the Euler characteristic through the top Chern class on 4-dimensional manifold, is characterized by β_j and η_j.

2 Spinor Decomposition of SU(2) Gauge Potential

Let M be a compact oriented 4-dimensional manifold, on which the principal bundle $P(\pi,\, M,\, \mathrm{SU}(2))$ is defined. It is well known that in the SU(2) gauge field theory with spinor representation Ψ, the covariant derivative of Ψ is defined as

$$D_\mu\Psi = \partial_\mu\Psi - \frac{1}{2\mathrm{i}}A^a_\mu\sigma_a\Psi \qquad (\mu = 1,2,3,4)\,, \tag{1}$$

where

$$A = A_\mu\,\mathrm{d}x^\mu = \frac{1}{2\mathrm{i}}A^a_\mu\sigma_a\,\mathrm{d}x^\mu \tag{2}$$

is the SU(2) gauge potential, i.e. the connection of principal bundle P; and $T_a = \sigma_a/2\mathrm{i}$ ($a = 1,2,3$) are the SU(2) generators with σ_a being the Pauli matrices. The complex conjugate of $D_\mu\Psi$ is

$$D^\dagger_\mu\Psi^\dagger = \partial_\mu\Psi^\dagger + \frac{1}{2\mathrm{i}}\Psi^\dagger A^a_\mu\sigma_a\,. \tag{3}$$

The SU(2) gauge field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]\,, \tag{4}$$

where

$$F = \frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,, \quad F_{\mu\nu} = \frac{1}{2\mathrm{i}}F^a_{\mu\nu}\sigma_a\,. \tag{5}$$

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To complete the decomposition of SU(2) gauge potential, multiplying Eq. (1) with $\Psi^\dagger\sigma_b$ and Eq. (3) with $\sigma_b\Psi$, respectively, and using

$$\sigma_a\sigma_b+\sigma_b\sigma_a=2\delta_{ab}I\,, \tag{6}$$

one can easily find

$$A^a_\mu=\frac{\mathrm{i}}{\Psi^\dagger\Psi}(\Psi^\dagger\sigma_a\partial_\mu\Psi-\partial_\mu\Psi^\dagger\sigma_a\Psi)-\frac{\mathrm{i}}{\Psi^\dagger\Psi}(\Psi^\dagger\sigma_aD_\mu\Psi-D^\dagger_\mu\Psi^\dagger\sigma_a\Psi)\,. \tag{7}$$

Since any 2×2 Hermitian matrix X can be expressed in terms of Clifford basis $(I,\ \vec\sigma)$,

$$X=\frac12\mathrm{Tr}(X)I+\frac12\mathrm{Tr}(X\sigma_a)\sigma_a\,, \tag{8}$$

from Eqs. (2), (7), and (8) we can obtain

$$A_\mu=a_\mu+b_\mu\,, \tag{9}$$

where

$$a_\mu=\frac{1}{\Psi^\dagger\Psi}(\partial_\mu\Psi\Psi^\dagger-\Psi\partial_\mu\Psi^\dagger)-\frac{1}{2\Psi^\dagger\Psi}\mathrm{Tr}(\partial_\mu\Psi\Psi^\dagger-\Psi\partial_\mu\Psi^\dagger)I\,, \tag{10}$$

$$b_\mu=-\Big[\frac{1}{\Psi^\dagger\Psi}(D_\mu\Psi\Psi^\dagger-\Psi D^\dagger_\mu\Psi^\dagger)-\frac{1}{2\Psi^\dagger\Psi}\mathrm{Tr}(D_\mu\Psi\Psi^\dagger-\Psi D^\dagger_\mu\Psi^\dagger)I\Big]\,. \tag{11}$$

The gauge potential A_μ should satisfy the SU(2) gauge transformation[8]

$$A'_\mu=SA_\mu S^\dagger+\partial_\mu SS^\dagger\,, \tag{12}$$

where S is the transformation matrix, $S^\dagger=S^{-1}$ ($S\in$ SU(2)). In the following it can be proved that a_μ and b_μ satisfy the gauge transformation and the vectorial transformation, respectively,

$$a'_\mu=Sa_\mu S^\dagger+\partial_\mu SS^\dagger\,, \tag{13}$$

$$b'_\mu=Sb_\mu S^\dagger\,. \tag{14}$$

Noticing $D'_\mu\Psi'=SD_\mu\Psi$ and $\Psi'=S\Psi$, from Eq. (11) the transformation (14) can be proved. Defining the projection operator

$$\hat P=\frac{\Psi\Psi^\dagger}{\Psi^\dagger\Psi}=\frac12(I+\hat n)\,, \tag{15}$$

where

$$\hat n=n^a\sigma_a,\qquad n^a=\frac{\Psi\sigma_a\Psi^\dagger}{\Psi^\dagger\Psi}\,, \tag{16}$$

we can prove

$$a'_\mu=[Sa_\mu S^\dagger+\partial_\mu SS^\dagger]+\frac12\Big[L_\mu-\frac12\mathrm{Tr}(L_\mu)\Big]\,, \tag{17}$$

where

$$L_\mu=(\partial_\mu S)\hat nS^\dagger-S\hat n(\partial_\mu S^\dagger)\,. \tag{18}$$

But we have

$$L_\mu-\frac12\mathrm{Tr}(L_\mu)=\frac12\mathrm{Tr}[(L_\mu)\sigma_a]\sigma_a=0\,, \tag{19}$$

so the transformation formula (13) is proved to be true. Therefore the transformation (12) is satisfied, and we obtain the result that the expression (9) with Eqs. (10) and (11) is just the spinor decomposition of SU(2) gauge potential. Compared with other gauge potential decompositions,[17] the expression (9) is more direct when calculating some SU(2) topological quantity, such as the Chern–Simons form and the second Chern class. This will be detailed in Secs. 3 and 4.

3 Spinor Structures of Chern–Simons Form and Knot Quantum Number

The Chern–Simons 3-form is defined as[8−10]

$$\Omega=\frac{1}{8\pi^2}\mathrm{Tr}\Big(A\wedge\mathrm{d}A-\frac23A\wedge A\wedge A\Big)\,, \tag{20}$$

i.e.

$$\Omega=-\frac{1}{16\pi^2}\epsilon^{\mu\nu\lambda}\Big[A^a_\mu\partial_\nu A^a_\lambda-\frac13\epsilon^{abc}A^a_\mu A^b_\nu A^c_\lambda\Big]\mathrm{d}^3x\,. \tag{21}$$

The traditional decomposition theory of gauge potential (including Riemann geometry) always uses the parallel field condition[1,4,11]

$$D_\mu\Psi=0\,, \tag{22}$$

so the solution is then $b_\mu=0$, and A^a_μ can be solved in terms of Ψ

$$A^a_\mu=a^a_\mu=\frac{\mathrm{i}}{\Psi^\dagger\Psi}(\Psi^\dagger\sigma_a\partial_\mu\Psi-\partial_\mu\Psi^\dagger\sigma_a\Psi)\,. \tag{23}$$

Since A_μ has been solved as Eq. (23), the parallel field condition (22) can certainly be taken.

In the above text the spinor Ψ is a 2×1 matrix

$$\Psi=\begin{pmatrix}\phi^0+\mathrm{i}\phi^1\\ \phi^2+\mathrm{i}\phi^3\end{pmatrix}\,, \tag{24}$$

where ϕ^a $(a=0,1,2,3)$ are real functions, $\phi^a\phi^a=|\phi|^2=\Psi^\dagger\Psi$. For simplicity, we introduce a unit vector n^a $(a=0,1,2,3)$,

$$n^a=\frac{\phi^a}{|\phi|},\quad n^an^a=1\,. \tag{25}$$

Obviously the zero points of ϕ^a are just the singular points of n^a. We also introduce a normalized spinor Ψ_n,

$$\Psi_n=\frac{1}{\sqrt{\Psi^\dagger\Psi}}\Psi=\begin{pmatrix}n^0+\mathrm{i}n^1\\ n^2+\mathrm{i}n^3\end{pmatrix}\,. \tag{26}$$

In the following, without danger of confusion, we can still use the symbol "Ψ" instead of "Ψ_n" to denote the normalized spinor. Thus equation (23) becomes

$$A^a_\mu=\mathrm{i}(\Psi^\dagger\sigma_a\partial_\mu\Psi-\partial_\mu\Psi^\dagger\sigma_a\Psi)\,. \tag{27}$$

Then we can use Eqs. (21) and (27) to study the spinor structure of Ω. Noticing that the Pauli matrix elements satisfy the formulas

$$\sigma^{\alpha\beta}_a\sigma^{\alpha'\beta'}_a=2\delta_{\alpha\beta'}\delta_{\alpha'\beta}-\delta_{\alpha\beta}\delta_{\alpha'\beta'}\,, \tag{28}$$

$$\epsilon_{abc}\sigma^{\alpha\beta}_a\sigma^{\alpha'\beta'}_b\sigma^{\alpha''\beta''}_c$$

$$= -2\mathrm{i}(\delta_{\alpha\beta'}\delta_{\alpha'\beta''}\delta_{\alpha''\beta} - \delta_{\alpha\beta''}\delta_{\alpha''\beta'}\delta_{\alpha'\beta})\,, \tag{29}$$

we arrive at

$$\Omega = -\frac{1}{4\pi^2}\Psi^\dagger \mathrm{d}\Psi \wedge \mathrm{d}\Psi^\dagger \wedge \mathrm{d}\Psi\,. \tag{30}$$

This is just the spinor structure of Chern–Simons 3-form of SU(2) gauge field theory.

The above spinor structure of Chern–Simons form can be applied in studying the knot quantum number of non-Abelian gauge theory. The quantum number of Faddeev–Niemi knot is given by the integration in 3-dimension[12,13]

$$Q_{FN} = \frac{1}{32\pi^2}\int \epsilon_{ijk}C_i H_{jk}\mathrm{d}^3x \quad (i,j,k=1,2,3)\,, \tag{31}$$

where H_{ij} is an Abelian gauge field tensor

$$H_{ij} = -\vec{m}\cdot(\partial_i\vec{m}\times\partial_j\vec{m}) = \partial_i C_j - \partial_j C_i \quad (\vec{m}\cdot\vec{m}=1)\,, \tag{32}$$

and $\vec{m}$ is the non-linear σ-model field, in SU(2) gauge theory $m^a = \Psi^\dagger\sigma^a\Psi$ $(a=1,2,3)$. Here $Q_{FN}\in\pi_3(S^2)$ $(\pi_3(S^2)=Z)$, so it is the Hopf invariant. It is known that[14]

$$\frac{1}{4}\epsilon_{ijk}C_iH_{jk} = \epsilon_{ijk}\mathrm{Tr}\Big(A_i\partial_jA_k - \frac{2}{3}A_iA_jA_k\Big)\,, \tag{33}$$

therefore equation (31) is

$$Q_{FN} = \int\Omega = \frac{1}{8\pi^2}\int\epsilon_{ijk}\mathrm{Tr}\Big(A_i\partial_jA_k - \frac{2}{3}A_iA_jA_k\Big)\,. \tag{34}$$

In this paper, using the spinor expression of Chern–Simons form (30), from Eq. (34) the knot quantum number can be directly expressed as

$$Q_{FN} = \int\Omega = -\frac{1}{4\pi^2}\int\epsilon_{ijk}\Psi^\dagger\partial_i\Psi\partial_j\Psi^\dagger\partial_k\Psi\mathrm{d}^3x\,. \tag{35}$$

Since the complex vector field Ψ is the fundamental field on manifold and just describes the topological property of the manifold itself, the expression (35) is obviously more direct than Eq. (34) in the study of Faddeev–Niemi knot in non-Abelian SU(2) gauge field theory.

4 Spinor Structure of Chern Density and Inner Structure of the Second Chern Number

The definition of Chern–Simons form (20) leads to the second Chern class[15]

$$c_2(P) = \mathrm{d}\Omega\,, \tag{36}$$

$$c_2(P) = \frac{1}{8\pi^2}\mathrm{Tr}(F\wedge F) = \rho(x)\mathrm{d}^4x\,, \tag{37}$$

where $\rho(x)$ is the SU(2) Chern density. From Eqs. (36) and (30) we can obtain the spinor structure of the second Chern class,

$$c_2(P) = -\frac{1}{4\pi^2}\mathrm{d}\Psi^\dagger\wedge\mathrm{d}\Psi\wedge\mathrm{d}\Psi^\dagger\wedge\mathrm{d}\Psi\,, \tag{38}$$

and the spinor structure of SU(2) Chern density,

$$\rho(x) = -\frac{1}{4\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu\Psi^\dagger\partial_\nu\Psi\partial_\lambda\Psi^\dagger\partial_\rho\Psi\,. \tag{39}$$

In terms of the unit vector n^a, the Chern density $\rho(x)$ in Eq. (39) can be expressed as[6]

$$\rho(x) = \frac{1}{12\pi^2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{abcd}\partial_\mu n^a\partial_\nu n^b\partial_\lambda n^c\partial_\rho n^d\,. \tag{40}$$

By making use of our ϕ-mapping topological current theory,[4,6,16–19] we can use Eq. (25) and the Green function relation in ϕ-space

$$\frac{\partial^2}{\partial\phi^a\partial\phi^a}\Big(\frac{1}{||\phi||^2}\Big) = -4\pi^2\delta^4(\vec{\phi}) \tag{41}$$

to reexpress Chern density $\rho(x)$ in a δ-function form,[16,17]

$$\rho(x) = \delta^4(\vec{\phi})D\Big(\frac{\phi}{x}\Big)\,, \tag{42}$$

where $D(\phi/x)$ is the Jacobi determinant,

$$\epsilon^{abcd}D\Big(\frac{\phi}{x}\Big) = \epsilon^{\mu\nu\lambda\rho}\partial_\mu\phi^a\partial_\nu\phi^b\partial_\lambda\phi^c\partial_\rho\phi^d\,. \tag{43}$$

The implicit function theory shows that,[20] under the regular condition[21]

$$D(\phi/x)\neq 0\,, \tag{44}$$

the general solutions of

$$\phi^a(x^0,x^1,x^2,x^3) = 0 \qquad (a=0,1,2,3)\,, \tag{45}$$

can be expressed as N isolated points

$$x^\mu = x^\mu_j \quad (\mu=0,1,2,3;\quad j=1,\ldots,N)\,. \tag{46}$$

In δ-function theory,[22] one can prove

$$\delta^4(\vec{\phi}) = \sum_{j=1}^{N}\frac{\beta_j\delta^4(x^\mu - x^\mu_j)}{|D(\phi/x)|_{x^\mu_j}}\,, \tag{47}$$

where the positive integer β_j is the Hopf index of ϕ-mapping. In topology it means that when the point x^μ covers the neighborhood of the zero point x^μ_j once, the vector field ϕ^a covers the corresponding region in ϕ-space β_j times. Introducing the Brouwer degree of ϕ-mapping

$$\eta_j = \frac{D(\phi/x)}{|D(\phi/x)|_{x^\mu_j}} = \mathrm{sign}[D(\phi/x)]_{x^\mu_j} = \pm 1\,, \tag{48}$$

equation (42) can be expressed as

$$\rho(x) = \sum_{j=1}^{N}\beta_j\eta_j\delta^4(x^\mu - x^\mu_j)\,. \tag{49}$$

Equation (49) directly shows that the Chern density does not vanish only at the N 4-dimensional zero points of ϕ^a, i.e. the singular points of n^a, which are characterized by the Hopf indices β_j and the Brouwer degrees η_j of ϕ-mapping.

Furthermore, when integrating the second Chern class, one obtains the second Chern number,

$$C_2 = \int c_2(P) = \int\rho(x)\mathrm{d}^4x = \sum_{j=1}^{N}\beta_j\eta_j\,. \tag{50}$$

Since the base manifold M is 4-dimensional, the second Chern class $c_2(P)$ is just the top Chern class on P; on the other hand, there is a direct relation between the top Chern class and the Euler class[23]

$$c_2(P) = e(E)\,, \tag{51}$$

where E is a real vector bundle which is the real counterpart of complex vector bundle P, and $e(E)$ is the Euler class on E. Therefore the Euler characteristic, which is just the sum of indices of zero points of vector field ϕ^a on M, is obtained through the Gauss–Bonnet theorem

$$\chi(M) = \int e(E) = \sum_{j=1}^{N} \beta_j \eta_j\,. \tag{52}$$

So the indices of zero points of ϕ^a field can be composed of the topological numbers β_j and η_j.

At last there are three points which should be stressed. Firstly, besides the application in this paper, the spinor decomposition of SU(2) gauge potential can also be applied in studying the U(1) field tensor in SU(2) gauge field. Secondly, when the regular condition (44) fails, the bifurcation processes will occur.[21] Thirdly, when the self-duality condition[8]

$$F^*_{\mu\nu} = F_{\mu\nu} \tag{53}$$

is satisfied (where $F^*_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}/2$ is the dual tensor of $F_{\mu\nu}$), the corresponding zero points of ϕ^a field on R^4 are just the instantons, so their topological numbers can also be studied. These three points will be detailed in our other papers.

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