Communications in Theoretical Physics

PAPER

Probability balance equation, inner product and Loschmidt echo for non-Hermitian systems

To cite this article: Libin Fu 2026 *Commun. Theor. Phys.* **78** 085102

Communications in Theoretical Physics

Commun. Theor. Phys. **78** (2026) 085102 (7pp)

https://doi.org/10.1088/1572-9494/ae6a7c
CSTR: 32041.14.CTPae6a7c

Editor's Suggestion

Probability balance equation, inner product and Loschmidt echo for non-Hermitian systems

Libin Fu*

Graduate School of China Academy of Engineering Physics, Beijing 100193, China

E-mail: lbfu@gscaep.ac.cn

Received 11 April 2026, revised 8 May 2026
Accepted for publication 8 May 2026
Published 3 June 2026

Abstract

Loschmidt echo (LE) is a powerful tool for characterizing quantum dynamical properties, yet it lacks a unified definition in non-Hermitian systems due to their intrinsic biorthogonal structure. To address this issue, we start with time-independent non-Hermitian quantum systems and rigorously derive the probability balance equation from the Schrödinger equation. On this foundational basis, we construct a cross inner product that is consistent with quantum probability interpretation and computationally convenient. We then extend this theoretical framework to time-dependent non-Hermitian systems and further derive the corresponding probability balance equation for such systems. By means of the proposed cross inner product, we generalize the Hermitian definition of LE to non-Hermitian systems and obtain its analytical expression in the biorthogonal basis under first-order perturbation theory. We show that the non-Hermitian LE naturally reduces to the standard Hermitian form in the Hermitian limit, which verifies the self-consistency and rationality of our framework. We also compare the proposed cross inner product with the *G*-inner product for non-Hermitian systems, noting their differences and consistent Hermitian limit. This work establishes a unified definition of LE for general non-Hermitian systems, and provides a consistent theoretical tool for investigating dynamical quantum phase transitions, quantum state stability, and non-equilibrium critical behaviors in both PT-symmetric and non-PT-symmetric non-Hermitian systems.

Keywords: non-Hermitian system, Loschmidt echo, cross inner product, biorthogonal basis

1. Introduction

There has been a great deal of interest in the study of non-Hermitian systems [1–5]. The systems with non-Hermitian terms have been realized in many fields, such as optical systems [6–9], microwave systems [10, 11], and electronic systems [12–14]. And recently, non-Hermitian quantum physics has emerged as a core research direction in condensed matter physics [15, 16]. In the research on condensed matter physics, the Loschmidt echo (LE) serves as a key physical quantity describing the correlation between the initial quantum state and the time-evolved state. It is a core tool for uncovering dynamical quantum phase transitions, quantum state stability, and non-equilibrium critical behaviors [17–20]. However, unlike in Hermitian systems where the LE has a unified and self-consistent definition, the biorthogonal nature of non-Hermitian systems has completely disrupted this traditional framework, resulting in the LE being defined in a fragmented manner for a long time.

In two recent studies [21, 22], researchers have proposed their respective definitions of the LE tailored to the specific properties of non-Hermitian systems, and these definitions all

[*] Author to whom any correspondence should be addressed.

stem from clear physical considerations. In [21], they focus on general non-Hermitian systems with complex eigenvalues (not restricted by parity-time (PT) symmetry). To address the issues of complex probabilities and the loss of topological information caused by the traditional Hermitian inner product, it introduces the concept of 'associated state' and constructs a biorthogonal LE with automatic normalization through the biorthogonal inner product. Its core consideration is to ensure that the definition satisfies the probability interpretation in any non-Hermitian scenario while fully preserving the biorthogonal topological properties. On the other hand, the study of [22] targets non-Hermitian systems with real eigenvalues in the PT-symmetric regime. Leveraging the simplified orthogonal relationship between left eigenstates and right eigenstates, it defines the LE as the product of the inner products of the initial left (right) ground states and the evolved left (right) ground states. Its physical consideration is to adapt to the needs of finite-size scaling analysis and critical exponent extraction, thereby reducing the complexity of numerical calculations. Although the definitions in both documents can effectively solve physical problems in specific non-Hermitian scenarios, they lack a systematic unified framework and horizontal comparison.

In view of the above discussion, the present work aims to establish a unified and physically consistent framework for the LE in non-Hermitian systems. We start from the biorthogonal eigenvalue equations, the Schrödinger evolution equation, and the probability balance condition that govern non-Hermitian quantum systems. Based on these fundamental physical requirements, we rigorously derive a proper cross inner product that is compatible with quantum probability continuity. With this cross inner product, we further construct a consistent definition of the LE that follows the definition for Hermitian systems [17–19], and we calculate its analytical expression under the biorthogonal perturbation theory. This approach enables us to unify the previously fragmented definitions of the LE and to lay a solid foundation for the study of quantum dynamical processes in general non-Hermitian systems. Additionally, we compare the newly proposed cross inner product with the G-inner product within the framework of non-Hermitian systems, emphasizing their differences and unified behavior at the Hermitian limit.

The rest of the paper is organized as follows. We first review the properties of Hilbert space for non-Hermitian system in section 2. Then we proceed to construct the cross inner product from the probability continuity equation in section 3. In section 4, the LE of non-Hermitian system will be obtained based on the definition of cross inner product. Finally, we give a conclusion of our paper in section 5.

2. Properties of Hilbert space for non-Hermitian quantum systems

For non-Hermitian systems, the properties of Hilbert space have been well studied [23, 24]. In this section, we firstly review these properties. Consider a non-Hermitian system with Hamiltonian H, and it then has

$$H = H_R + \mathrm{i}H_I, \tag{1}$$

where

$$H_R = \frac{1}{2}\left(H + H^\dagger\right), \tag{2}$$

$$H_I = \frac{1}{2\mathrm{i}}\left(H - H^\dagger\right), \tag{3}$$

are two Hermitian operators. For such non-Hermitian systems, the orthogonality and completeness properties are described by the following biorthogonal bases. Generally, the eigenstates of H and $H^\dagger$ are given by

$$H|E_n\rangle = E_n|E_n\rangle, \quad \langle E_n|H^\dagger = E_n^*\langle E_n|, \tag{4}$$

$$H^\dagger|\tilde{E}_n\rangle = E_n^*|\tilde{E}_n\rangle, \quad \langle\tilde{E}_n|H = E_n\langle\tilde{E}_n|. \tag{5}$$

The above bases form a biorthogonal set,

$$\langle\tilde{E}_n|E_m\rangle = \delta_{nm}, \tag{6}$$

then the completeness is described by

$$\sum_n |E_n\rangle\langle\tilde{E}_n| = I. \tag{7}$$

Here we have assumed the normalization of the eigenstates.

Let us define $\varepsilon_n = \mathrm{Re}E_n$ and $\Gamma_n = \mathrm{Im}E_n$ From the eigen equations above and equations (2) and (3), we can easily get

$$\varepsilon_n = \mathrm{Re}E_n = \mathrm{Re}\langle\tilde{E}_n|H|E_n\rangle, \tag{8}$$

$$\Gamma_n = \mathrm{Im}E_n = \mathrm{Im}\langle\tilde{E}_n|H|E_n\rangle. \tag{9}$$

3. Probability balance equation and cross inner product definition for non-Hermitian systems

In the above section we review the properties of Hilbert space for non-Hermitian systems. We know that Hilbert spaces do not have a unique inner product, and any mapping satisfying conjugate symmetry, linearity, and positive definiteness is a valid inner product. For any physical system, the selection of an inner product ought to be grounded in its intrinsic physical meaning. For Hermitian quantum systems, the inner product is defined so that the squared norm of a state agrees with the Born rule, i.e. for a state $|\psi\rangle = \sum_n c_n|E_n\rangle$, its squared norm is

$$M = \langle\psi|\psi\rangle = \sum_n |c_n|^2. \tag{10}$$

However, for non-Hermitian systems, from the completeness relation (7) we find that the above expression becomes

$$\langle\psi|\psi\rangle = \sum_n |c_n|^2\langle E_n|E_n\rangle + \sum_{m\neq n} c_m^* c_n\langle E_m|E_n\rangle. \tag{11}$$

This expression thus no longer has a well-defined physical meaning, as expected.

In fact, the definition of the inner product is constructed based on a state and its corresponding state in the dual space. For a Hermitian system, the dual space of the state space is simply the space spanned by the complex conjugates of the states. However, owing to the biorthogonal basis, the dual space of the state space for a non-Hermitian system is not merely composed of the conjugate states, but is instead constituted by the dual orthogonal basis [25–28]. A transformation is thus required between the state space and its dual space [29] which is denoted as D . Specifically, the transformation D should be linear and satisfy the following condition,

$$D:|\phi\rangle\to|\tilde{\phi}\rangle, \tag{12}$$

where $|\phi\rangle=\sum_n b_n|E_n\rangle$ and $|\tilde{\phi}\rangle=\sum_n b_n|\tilde{E}_n\rangle$. By these considerations, the inner product is defined as follows [29],

$$(\phi,\psi)=\langle\tilde{\phi}|\psi\rangle=\sum_n b_n^* c_n, \tag{13}$$

in which $|\psi\rangle=\sum_n c_n|E_n\rangle$. It is necessary to point out that the definition in [21] is identical to this one. Hence, the squared norm for the state is just

$$M=\langle\tilde{\psi}|\psi\rangle=\sum_n|c_n|^2. \tag{14}$$

This yields the same result as equation (10) with the same physical meaning, namely the total probability.

In quantum mechanics, the squared norm M in equation (10) is just total probability, and is conserved, which can be proved from Schrödinger equation. Indeed, one of the most compelling manifestations of the Born rule and the choice of inner product in quantum mechanics is the continuity equation and conservation of probability, which is fundamentally grounded in the Schrödinger equation. This point has generally been overlooked in previous discussions for non-Hermitian systems.

In the following, we will explore the issue of inner product choice in non-Hermitian systems starting from the evolution equations.

3.1. Probability balance equation and cross inner product

For the non-Hermitian system with Hamiltonian given in the above section, its evolution is still satisfied the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle=H|\psi(t)\rangle, \tag{15}$$

and its complex conjugate is

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|=-\langle\psi(t)|H^\dagger. \tag{16}$$

Because of the completeness property discussed in section 2, the state of system can be expressed as follows,

$$|\psi(t)\rangle=\sum_n c_n|E_n\rangle,\quad c_n=\langle\tilde{E}_n|\psi(t)\rangle. \tag{17}$$

Let us define

$$\Upsilon=\sum_n|\tilde{E}_n\rangle\langle\tilde{E}_n|. \tag{18}$$

By left-multiplying equation (15) by Υ and $\langle\psi(t)|$ in sequence, and right-multiplying equation (16) by Υ and $|\psi(t)\rangle$ in sequence, we obtain the following,

$$\mathrm{i}\langle\psi(t)|\Upsilon\left(\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\right)=\langle\psi(t)|\Upsilon H|\psi(t)\rangle, \tag{19}$$

and

$$\mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\right)\Upsilon|\psi(t)\rangle=-\langle\psi(t)|H^\dagger\Upsilon|\psi(t)\rangle. \tag{20}$$

Adding the above two equations and utilizing the biorthonormal relation (6), we arrive at the dynamical relation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\Upsilon|\psi(t)\rangle=\mathrm{i}\langle\psi(t)|\left(\Upsilon H-H^\dagger\Upsilon\right)|\psi(t)\rangle. \tag{21}$$

To further simplify the right-hand side, we substitute the biorthonormal eigenstate expansion and adopt the eigen-equations of non-Hermitian systems. The two expectation terms can be explicitly evaluated as

$$\langle\psi|\Upsilon H|\psi\rangle=\sum_{m,n,k}c_m^*c_n\langle E_m|\tilde{E}_k\rangle\langle\tilde{E}_k|H|E_n\rangle=\sum_n|c_n|^2E_n, \tag{22}$$

$$\langle\psi|H^\dagger\Upsilon|\psi\rangle=\sum_{m,n,k}c_m^*c_n\langle E_m|H^\dagger|\tilde{E}_k\rangle\langle\tilde{E}_k|E_n\rangle=\sum_n|c_n|^2E_n^*. \tag{23}$$

Substituting the above results and employing the definition of the decay/gain rate in (9), $\Gamma_n=\mathrm{Im}(E_n)$, we obtain the final evolution equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\Upsilon|\psi(t)\rangle=2\sum_n|c_n|^2\mathrm{Im}E_n=2\sum_n|c_n|^2\Gamma_n. \tag{24}$$

The right-hand side of this equation represents the total gain (loss) rate of the system, which quantifies $\Gamma_n=\mathrm{Im}E_n$ the gain (loss) strength of the nth eigen-channel.

At the same time, one can easily find

$$\langle\psi(t)|\Upsilon|\psi(t)\rangle=\sum_n|c_n|^2=M. \tag{25}$$

It is obvious that equation (24) serves as probability continuity equation of non-Hermitian quantum system, and M in (25) defined above is just the total probability of the non-Hermitian system. For non-Hermitian systems, the total probability is not conserved due to the presence of gain (source) and loss (sink), and this equation should thus be referred to as the probability balance equation. It is also clear that, when the biorthogonal characteristics vanish for Hermitian systems, they will naturally revert to the definition of Hermitian systems.

It is necessary to point out that the total probability of the non-Hermitian system discussed above cannot be simply regarded as a probability in the conventional quantum mechanical sense due to its incorporation of the system's intrinsic gain and loss. Instead, its physical essence corresponds to the total effective population of quantum states coupled with the intrinsic gain and loss of the non-Hermitian system, whose core physical meaning is to quantitatively characterize the overall evolution intensity of quantum states and the open dynamical features dominated by gain and loss in such systems [30, 31]. Nevertheless, for the convenience of physical discussion and consistent terminology with existing non-Hermitian literatures, we still adopt the term 'probability' to refer to this effective state weight in the following text.

Considering the above discussion, we can propose a definition of inner product, denoted as cross inner product here, that is self-consistent with equation (25) as follows,

$$(\phi, \psi) = \langle\phi|\Upsilon|\psi\rangle = \sum_n b_n^* c_n, \tag{26}$$

in which $|\phi\rangle = \sum_n b_n|E_n\rangle$ and $|\psi\rangle = \sum_n c_n|E_n\rangle$.

We find that the definition of the cross inner product actually introduces the following transformation,

$$\Upsilon : \Upsilon|\psi\rangle = |\tilde{\psi}\rangle\,, \tag{27}$$

where $|\tilde{\psi}\rangle = \sum_n c_n|\tilde{E}_n\rangle$. Then one has

$$\langle\tilde{\phi}|\psi\rangle = \langle\phi|\tilde{\psi}\rangle. \tag{28}$$

Comparing with (13), one can find for biorthogonal system, the inner product has two equivalent definitions. Υ acts as a bridge connecting the state space and its dual space in non-Hermitian systems, which not only connects them to the probability continuity equation but also simplifies the calculation.

Notably, the above definition of the cross inner product bears a formal analogy to the chiral axial vector current in relativistic quantum field theory. The latter is defined as $J^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$, where ψ denotes the Dirac field,$\overline{\psi} = \psi^\dagger\gamma^0$ is the Dirac adjoint, and γ^μ are the Gamma matrices. Since $\gamma^0\gamma^0 = I$ (the identity operator), the corresponding axial charge density reads $\rho = J_0 = \psi^\dagger\gamma_5\psi$. A key feature of such chiral/axial vector systems is the non-conservation of the total axial charge, which arises from the chiral anomaly. In this formal analogy, the metric operator Υ introduced in this work plays a role analogous to the γ_5 matrix in the axial current construction.

3.2. *Probability balance equation for time-dependent system*

The above discussion is restricted to time-independent non-Hermitian systems. In this section, we derive the probability continuity equation for time-dependent systems. For a time-dependent non-Hermitian Hamiltonian $H(t)$, we employ the instantaneous biorthogonal eigenbasis $\{|E_n(t)\rangle, |\tilde{E}_n(t)\rangle\}$ satisfying $H(t)|E_n(t)\rangle = E_n(t)|E_n(t)\rangle$, $\langle\tilde{E}_n(t)|H = E_n(t)\langle\tilde{E}_n(t)|$. At this time, $\Upsilon(t) = \sum_n |\tilde{E}_n(t)\rangle\langle\tilde{E}_n(t)|$. Then, the time derivative of the probability $M(t) = \langle\psi(t)|\Upsilon(t)|\psi(t)\rangle$ still derives directly from the Schrödinger equation and its adjoint. Using the product rule and the explicit time dependence of the metric operator $\Upsilon(t)$, one obtains

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = \mathrm{i}\langle\psi|\left(\Upsilon H - H^\dagger\Upsilon\right)|\psi\rangle + \langle\psi|\frac{\partial\Upsilon(t)}{\partial t}|\psi\rangle. \tag{29}$$

The first term accounts for the non-Hermitian part of the dynamics. To express the second term in a form suitable for analysis, we have

$$\langle\psi|\frac{\partial\Upsilon(t)}{\partial t}|\psi\rangle = \sum_n\left(\langle\psi|\dot{\tilde{E}}_n\rangle c_n + c_n^*\langle\dot{\tilde{E}}_n|\psi\rangle\right). \tag{30}$$

Further expanding

$$\langle\psi|\dot{\tilde{E}}_n\rangle = \sum_m c_m^*\langle E_m|\dot{\tilde{E}}_n\rangle, \quad \langle\dot{\tilde{E}}_n|\psi\rangle = \sum_m c_m\langle\dot{\tilde{E}}_n|E_m\rangle, \tag{31}$$

and relabeling indices in the second sum, we arrive at the compact expression

$$\langle\psi|\frac{\partial\Upsilon(t)}{\partial t}|\psi\rangle = \sum_{m,n} c_m^* c_n\left(\langle E_m|\dot{\tilde{E}}_n\rangle + \langle\dot{\tilde{E}}_m|E_n\rangle\right). \tag{32}$$

Combining the above equations with the decomposition of the first term in the biorthogonal basis, i.e.

$$\mathrm{i}\langle\psi|\left(\Upsilon H - H^\dagger\Upsilon\right)|\psi\rangle = 2\sum_n\Gamma_n|c_n|^2, \tag{33}$$

the total probability evolution reads

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = 2\sum_n\Gamma_n|c_n|^2 + 2\sum_{m,n}\mathrm{Re}\left(c_m^* c_n A_{m\tilde{n}}\right), \tag{34}$$

in which

$$A_{m\tilde{n}} = \langle E_m|\dot{\tilde{E}}_n\rangle. \tag{35}$$

Equation (34) provides a basis-dependent but explicit expression for the rate of change of $M(t)$ in a time-dependent non-Hermitian system, highlighting the contributions from both the imaginary parts of the eigenvalues and the temporal variation of the instantaneous eigenbasis which is just related to the 1-form (Berry connection) of non-Hermitian system [26, 28, 32].

4. LE of non-Hermitian quantum system

The LE, also called quantum fidelity in specific contexts, is a pivotal quantity in quantum physics for quantifying perturbation sensitivity of quantum dynamics and evolution irreversibility. Rooted in the Loschmidt–Boltzmann debate on microscopic-macroscopic reversibility tension, it is widely used in quantum chaos, quantum information, and many-body physics, essential for probing quantum state stability theoretically and practically [18, 19]. The LE denotes the probability of a quantum system reverting to its initial state after 'forward-backward' evolution: forward under an unperturbed Hamiltonian, backward under a perturbed one. Let $\hat{H}$ denote the unperturbed Hamiltonian and $\hat{H}' = \hat{H} + \hat{V}$ represent the perturbed Hamiltonian, where $\hat{V}$ is a small perturbation. For a quantum system with initial state $|\psi_0\rangle$, the time evolution for $\hat{H}$ is

$$|\psi(t)\rangle = \sum_n d_n \mathrm{e}^{-\mathrm{i}E_n t}|E_n\rangle, \quad d_n = \langle E_n|\psi_0\rangle, \tag{36}$$

and for $\hat{H}'$ is

$$|\psi^V(t)\rangle = \sum_n d_n' \mathrm{e}^{-\mathrm{i}E_n' t}|E_n'\rangle, \quad d_n' = \langle E_n'|\psi_0\rangle. \tag{37}$$

The definition of the LE, for a Hermitian quantum system, is then given by the squared modulus of the overlap between the above states. Mathematically, this is expressed as,

$$L(t) = \left|\langle\psi^V(t)|\psi(t)\rangle\right|^2. \tag{38}$$

By employing the perturbation approach and omitting higher-order terms, we arrive at (after some algebra)

$$\begin{aligned} L(t) = \Bigg| &\sum_n \left[|d_n|^2 + \sum_{m\neq n} \frac{V_{mn}}{E_n - E_m}(d_n^* d_m) \right] \mathrm{e}^{\mathrm{i}V_{nn}t} \\ &+ \sum_{n,m\neq n} \frac{V_{mn}}{E_n - E_m} d_m^* d_n \mathrm{e}^{\mathrm{i}(E_n + V_{nn} - E_m)t} \Bigg|^2, \end{aligned} \tag{39}$$

where $V_{mn} = \langle E_m|V|E_n\rangle$. Comparing $\mathrm{e}^{\mathrm{i}V_{nn}t}$ and $\mathrm{e}^{\mathrm{i}(E_n+V_{nn}-E_m)t}$, the latter is a rapidly oscillating term, and thus the trend of the LE is dominated primarily by the term corresponding to $\mathrm{e}^{\mathrm{i}V_{nn}t}$. It can be seen from the above equation that if no degeneracy occurs in the system energy levels, the first perturbation terms can be neglected, so that $L(t) = \left|\sum_n |d_n|^2 \mathrm{e}^{\mathrm{i}V_{nn}t}\right|^2$ [18]. However, if degeneracy occurs, for example, during a phase transition, critical behavior will arise [19].

The system discussed above is Hermitian. For a non-Hermitian system, according to the definition of the cross inner product given in equation (26), the LE should be given by the following expression,

$$L(t) = \left|\langle\psi^V(t)|\Upsilon|\psi(t)\rangle\right|^2. \tag{40}$$

With the perturbative results retained to first order for a non-Hermitian system [23], and replacing $d_n = \langle\tilde{E}_n|\psi_0\rangle$ and $d_n' = \langle\tilde{E}_n'|\psi_0\rangle$ respectively, we have

$$|E_n'\rangle = |E_n\rangle + \sum_{m\neq n} \frac{V_{\tilde{m}n}}{E_n - E_m}|E_m\rangle, \tag{41}$$

$$|\tilde{E}_n'\rangle = |\tilde{E}_n\rangle + \sum_{m\neq n} \frac{V^\dagger_{m\tilde{n}}}{E_n^* - E_m^*}|\tilde{E}_m\rangle, \tag{42}$$

$$E_n' = E_n + V_{\tilde{n}n}, \tag{43}$$

$$d_n' = d_n + \sum_{m\neq n} \frac{\left(V^\dagger_{m\tilde{n}}\right)^*}{E_n - E_m} d_m. \tag{44}$$

Here, $V_{\tilde{m}n} = \langle\tilde{E}_m|V|E_n\rangle$ and $V^\dagger_{m\tilde{n}} = \langle E_m|V^\dagger|\tilde{E}_n\rangle$. Then to the first order, we can obtain the LE for non-Hermitian system as follows,

$$\begin{aligned} L(t) = \Bigg| &\left[\sum_n |d_n|^2 + \sum_{n,m\neq n} \frac{V^\dagger_{m\tilde{n}}}{E_n^* - E_m^*} d_m^* d_n \right] \mathrm{e}^{\mathrm{i}V^*_{\tilde{n}n}t} \mathrm{e}^{2\Gamma_n t} \\ &+ \sum_{n,m\neq n} \frac{V^*_{\tilde{m}n}}{E_n^* - E_m^*} d_n^* d_m \mathrm{e}^{\mathrm{i}(E_n^* + V^*_{\tilde{n}n} - E_m)t} \Bigg|^2. \end{aligned} \tag{45}$$

If no degeneracy occurs in the system energy levels, the first perturbation terms can be neglected, so that,

$$L(t) = \left|\sum_n |d_n|^2 \mathrm{e}^{\mathrm{i}V^*_{\tilde{n}n}t} \mathrm{e}^{2\Gamma_n t}\right|^2. \tag{46}$$

Compared with Hermitian systems [18], the LE of non-Hermitian systems is found to additionally contain decay terms for each energy channel. It is quite interesting that the LE of non-Hermitian systems obtained under the cross inner product naturally reduces to the Hermitian case. This also demonstrates the rationality of the cross inner product from this perspective.

We would like to point out that the LE defined here does not actually lie in the range $[0, 1]$, as it incorporates the gain and loss inherent to non-Hermitian systems. If such gain and loss behavior dominates, which may obscure some subtle dynamical behaviors, or if this variation is not the focus of the research, it is necessary to introduce a normalized LE, namely,

$$L_N(t) = \frac{\left|\langle\psi^V(t)|\Upsilon|\psi(t)\rangle\right|^2}{\langle\psi^V(t)|\Upsilon|\psi^V(t)\rangle\langle\psi(t)|\Upsilon|\psi(t)\rangle}. \tag{47}$$

The normalization also constitutes a standard approach in the study of non-Hermitian systems.

The concept of the LE has recently been extended by researchers to the study of dynamical phase transitions and other related issues. Defined as the fidelity between the evolved and initial states in these works [21, 33], the LE should be expressed as follows in this sense,

$$L(t) = |\langle\psi(0)|\Upsilon|\psi(t)\rangle|^2, \tag{48}$$

and the normalized LE is correspondingly given by,

$$L_N(t) = \frac{|\langle\psi(0)|\Upsilon|\psi(t)\rangle|^2}{\langle\psi(0)|\Upsilon|\psi(0)\rangle\langle\psi(t)|\Upsilon|\psi(t)\rangle}. \tag{49}$$

Compared with the automatically normalized LE in [21] and the PT-symmetric LE in [22], the above normalized LE is applicable to general non-Hermitian systems and retains the dynamical information of gain (loss) more directly.

5. Comparison with the G-inner product of non-Hermitian formalisms

The cross inner product proposed in this work addresses the fundamental challenge of biorthogonal state spaces in non-Hermitian quantum mechanics, a problem also tackled by the widely used G-inner product (e.g. Ju *et al* [34], Tzeng *et al* [35]). While both frameworks share the core goal of constructing a physically meaningful inner product for non-Hermitian systems, they diverge substantially in motivation and derivation logic, constraint requirements, and targeted applications.

The G-inner product is rooted in the geometric interpretation of Hilbert spaces for non-Hermitian systems. Ju *et al* [34] introduce the metric operator G to ensure the covariant derivative consistency of the Schrödinger equation, since G's time evolution constrained by the dynamic equation:

$$\partial_t G = \mathrm{i}\left(GH - H^{\dagger}G\right). \tag{50}$$

The G-inner product then is defined as

$$\langle\psi(t)|\psi(t)\rangle_G = \langle\psi(t)|G|\psi(t)\rangle. \tag{51}$$

This is derived to preserve the norm of quantum states over time (i.e. probability conservation), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\psi(t)\rangle_G = 0, \tag{52}$$

making the G-inner product a 'constructed metric' tailored for theoretical self-consistency.

The strict constraints of G, adherence to the dynamic equation $\partial_t G = \mathrm{i}(GH - H^{\dagger}G)$ [34], it turns out that G is time-dependent in the general case, even for a time-independent Hamiltonian H. Notably, G can be time-independent under specific conditions—e.g. time-independent Hamiltonians in the PT-unbroken phase or pseudo-Hermitian systems, where $GH = H^{\dagger}G$ reduces the dynamic equation to $\partial_t G = 0$ [34, 36]. Even in this time-independent limit, G must satisfy the pseudo-Hermitian constraint to maintain consistency with quantum information principles, imposing non-trivial restrictions on its form.

The Υ operator in our framework inherits Hermiticity and positive definiteness inherently, as a projection operator composed of left eigenstates. For time-independent Hamiltonians, Υ is strictly time-invariant. For time-dependent Hamiltonians, its variation is directly tied to the Berry connection of the instantaneous biorthogonal eigenbasis [26, 28], with no independent dynamic constraints to solve. This simplicity avoids the computational complexity of determining G (e.g. solving the dynamic equation or verifying pseudo-Hermiticity).

Both frameworks maintain consistency with Hermitian quantum mechanics when the non-Hermitian term vanishes. For the G-inner product, G degenerates to the identity matrix, as left and right eigenstates become equivalent. For our cross inner product, Υ also reduces to identity matrix due to the equivalence of left and right eigenstates. This consistency verifies the validity of both formalisms and ensures a smooth transition between Hermitian and non-Hermitian regimes.

6. Conclusion

In summary, we have systematically addressed the problem of fragmented definitions for the LE in non-Hermitian quantum systems. Starting directly from the Schrödinger equation and biorthogonal eigenstate properties, we rigorously derive the probability balance equation for non-Hermitian systems. Based on the probability balance equation and completeness relations, we construct a physically consistent cross inner product, which is linked to the D transformation [29] between the state space and its dual space and offers high computational convenience.

Using this cross inner product, we extend the standard Hermitian LE to non-Hermitian systems and derive its analytical expression under first order perturbation theory adapted to biorthogonal bases. The resulting non-Hermitian LE naturally includes decay terms from each energy channel, which are absent in Hermitian systems. In the Hermitian limit, both the cross inner product and the non-Hermitian LE reduce to the standard quantum mechanical forms, verifying their self consistency and rationality.

This work establishes a unified framework for defining and calculating the LE in general non-Hermitian systems, providing a reliable theoretical foundation for studying dynamical quantum phase transitions, quantum state stability, and non-equilibrium critical behaviors in both PT-symmetric and non-PT-symmetric non-Hermitian systems. The cross inner product derived from fundamental physical principles can be widely extended to various non-Hermitian models. Future research may further explore its applications in realistic physical systems, the geometric properties of non-Hermitian systems, higher order perturbation corrections, and finite-size scaling behaviors of the non-Hermitian LE.

We have compared the proposed cross inner product with the G-inner product in non-Hermitian systems, clarifying their differences and consistent behavior in the Hermitian limit.

Acknowledgments

Supported by Science Challenge Project (No. TZ2025017) and the National Natural Science Foundation of China (Nos. 12088101, U2330401).

References

[1] Bender C M and Boettcher S 1998 Real spectra in non-Hermitian Hamiltonians having PT symmetry *Phys. Rev. Lett.* **80** 5243
[2] Bender C M 2007 Making sense of non-Hermitian Hamiltonians *Rep. Prog. Phys.* **70** 947
[3] Makris K G, El-Ganainy R and Christodoulides D N 2008 Beam dynamics in $\mathcal{PT}$ symmetric optical lattices *Phys. Rev. Lett.* **100** 103904
[4] Doppler J, Mailybaev A A, Böhm J, Kuhl U, Girschik A, Libisch F, Milburn T J, Rabl P, Moiseyev N and Rotter S 2016 Dynamically encircling an exceptional point for asymmetric mode switching *Nature* **537** 76

[5] Ozawa T *et al* 2019 Topological photonics *Rev. Mod. Phys.* **91** 015006
[6] Ramezani H, Kottos T, El-Ganainy R and Christodoulides D N 2010 Unidirectional nonlinear PT-symmetric optical structures *Phys. Rev. A* **82** 043803
[7] El-Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 Theory of coupled optical $\mathcal{PT}$-symmetric structures *Opt. Lett.* **32** 2632
[8] Guo A, Salamo G J, Duchesne D, Morandotti R, Volatier-Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 Observation of *PT*-symmetry breaking in complex optical potentials *Phys. Rev. Lett.* **103** 093902
[9] Rüter C E, Makris K G, El-Ganainy R, Christodoulides D N, Segev M and Kip D 2010 Observation of parity time symmetry in optics *Nat. Phys.* **6** 192
[10] Bittner S, Dietz B, Günther U, Harney H L, Miski-Oglu M, Richter A and Schäfer F 2012 PT symmetry and spontaneous symmetry breaking in a microwave billiard *Phys. Rev. Lett.* **108** 024101
[11] Liu Z-P, Zhang J, Özdemir S K, Peng B, Jing H, Lü X-Y, Li C-W, Yang L, Nori F and Liu Y-X 2016 Metrological phase transitions in a quantum Rabi model with gain and loss *Phys. Rev. Lett.* **117** 110802
[12] Bender N, Factor S, Bodyfelt J D, Ramezani H, Christodoulides D N, Ellis F M and Kottos T 2013 Observation of asymmetric transport in structures with active nonlinearities *Phys. Rev. Lett.* **110** 234101
[13] Assawaworrarit S, Yu X and Fan S 2017 Robust wireless power transfer using a nonlinear parity-time-symmetric circuit *Nature* **546** 387
[14] Choi Y, Hahn C, Yoon J W and Song S H 2018 Observation of an anti-PT-symmetric exceptional point and energy-difference conserving dynamics in electrical circuit resonators *Nat. Commun.* **9** 2182
[15] Yoshida T, Peters R and Kawakami N 2018 Non-Hermitian perspective of the band structure in heavy-fermion systems *Phys. Rev. B* **98** 035141
[16] Shen H and Fu L 2018 Quantum oscillation from in-gap states and a non-Hermitian Landau level problem *Phys. Rev. Lett.* **121** 026403
[17] Jalabert R A and Pastawski H M 2001 Environment-independent decoherence rate in classically chaotic systems *Phys. Rev. Lett.* **86** 2490
[18] Cerruti N R and Tomsovic S 2002 Sensitivity of wave field evolution and manifold stability in chaotic system *Phys. Rev. Lett.* **88** 054103
[19] Quan H T, Song Z, Liu X F, Zanardi P and Sun C P 2006 Decay of Loschmidt echo enhanced by quantum criticality *Phys. Rev. Lett.* **96** 140604
[20] Zeng Y, Zhou B and Chen S 2023 Dynamical singularity of the rate function for quench dynamics in finite-size quantum systems *Phys. Rev. B* **107** 134302
[21] Jing Y, Dong J-J, Zhang Y-Y and Hu Z X 2024 Biorthogonal dynamical quantum phase transitions in non-Hermitian systems *Phys. Rev. Lett.* **132** 220402
[22] Tang J-C, Kou S-P and Sun G 2022 Dynamical scaling of Loschmidt echo in non-Hermitian systems *Europhys. Lett.* **137** 40001
[23] Sternherim M M and Walker J F 1972 Non-Hermitian Hamiltonians, decaying states and perturbation theory *Phys. Rev. C* **6** 114
[24] Brody D C 2014 Biorthogonal quantum mechanics *J. Phys. A: Math. Theor.* **47** 035305
[25] Wong J 1967 Results on certain non-Hermitian Hamiltonians *J. Math. Phys.* **8** 2039
[26] Sun C P 1993 High-order adiabatic approximation for non-Hermitian quantum system and complexification of Berry's phase *Phys. Scr.* **48** 393
[27] Lathouwers L *et al* 1976 *Quantum Science Methods and Structures* ed J-L Calais (Plenum) pp 77–91
[28] Ibàñez S and Muga J G 2014 Adiabaticity condition for non-Hermitian Hamiltonians *Phys. Rev. A* **89** 033403
[29] Leung P T, Suen W M, Sun C P and Yong K 1998 Waves in open systems via a biorthogonal basis *Phys. Rev. E* **57** 6101
[30] Jia L, Xing H and Fu L 2022 Excited-state resonance tunneling in strong-field ionization *Phys. Rev. A* **106** 022814
[31] Jia L, Jiang W-C, Xing H and Fu L 2023 Extracting the central positions and widths of Fano-type tunneling channels in a spectrum from the ionization rate *Phys. Rev. A* **108** 013119
[32] Sun C P 1989 Classical analogues of quantum Berry's Phase *Chin. Phys. Lett.* **6** 481
[33] Liu Y, Jiang X-P, Cao J and Chen S 2020 Non-Hermitian mobility edges in one-dimensional quasicrystals with parity-time symmetry *Phys. Rev. B* **101** 147205
[34] Ju C-Y, Miranowicz A, Chen G-Y and Nori F 2019 Non-Hermitian Hamiltonians and no-go theorems in quantum information *Phys. Rev. A* **100** 062118
[35] Tzeng Y-C, Ju C-Y, Chen G-Y and Huang W-M 2021 Hunting for the non-Hermitian exceptional points with fidelity susceptibility *Phys. Rev. Res.* **3** 013015
[36] Hajong G, Modak R and Mandal B P 2024 Hellmann-Feynman theorem in non-Hermitian systems *Phys. Rev. A* **109** 022227