Point defects of a three-dimensional vector order parameter

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From the definition of topological charges of point defects, we obtain the densities of point defects. The evolution of point defects is also studied from the topological properties of a three-dimensional vector order parameter. The point defects are found generating or annihilating at the limit points and encountering, splitting, or merging at the bifurcation points of the three-dimensional vector order parameter.
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I. INTRODUCTION

Defects play an important role in the understanding of a variety of problems in physics. In particular there has been progress in the study the defects associated with an n-component vector order parameter field $\vec{\phi}(\vec{r},t)$. For the scalar case $n=1$, the defects are domain walls that are points for the spatial dimensionality $d=1$, lines for $d=2$, planes for $d=3$, etc. More generally, for $n=d$, one has point defects. This leads to vortices for $n=d=2$ and monopoles for $n=d=3$, etc. In certain cosmological [1] and phase ordering [2] problems key questions involve an understanding of the evolution and correlation among point defects. In studying such objects in field theory questions arise as to how one can define quantities such as the densities of a point defect and the associated point defect velocity field. It is interesting to consider the appropriate form for the point defect densities when expressed in terms of the vector order parameter field $\vec{\phi}(\vec{r},t)$. This has been carried out by Halperin [3] and exploited by Liu and Mazenko [4]: In the case $n=d$, the first ingredient is the rather obvious result

$$\sum_{\alpha} \delta(\vec{r}-\vec{r}_{\alpha}(t))=\delta(\vec{\phi}(\vec{r},t))\left|D\left(\frac{\phi}{x}\right)\right|,$$

where the second factor on the right-hand side is just the Jacobian of the transformation from the variable $\vec{\phi}$ to $\vec{r}$. This is combined with the less obvious result

$$\eta_{\alpha}=\operatorname{sgn}D(\phi/x)|_{\vec{r}_{\alpha}}$$

to give

$$\rho(\vec{r},t)=\sum_{\alpha} \eta_{\alpha}\delta(\vec{r}-\vec{r}_{\alpha}(t))=\delta(\vec{\phi})D(\phi/x). \tag{1}$$

Unfortunately, their analysis is incomplete, which we will discuss in detail.

In this paper we will investigate the evolution of point defects of a three-dimensional vector order parameter by making use of the ϕ-mapping topological current theory [5,6], which is a useful tool in studying the topological invariant and structure of physics systems and has been used to study the topological current of magnetic monopole, topological string theory, the topological characteristics of dislocations and disclinations continuum, the topological structure of the defects of space-time in the early universe as well as its topological bifurcation, the topological structure of the Gauss-Bonnet-Chern theorem, and the topological structure of the London equation in superconductors.

II. DENSITIES AND VELOCITY FIELD OF POINT DEFECTS

We can introduce a three-dimensional unit vector field

$$\vec{n}(\vec{r},t)=\vec{\phi}(\vec{r},t)/\|\phi\|, \quad \|\phi\|^2=\phi^a\phi^a, \tag{2}$$

where

$$\vec{\phi}(\vec{r},t)=(\phi^1,\phi^2,\phi^3) \tag{3}$$

is a vector order parameter. From Eq. (2) it is easy to see that the zeros of the vector order parameter $\vec{\phi}$ are just the singularities of $\vec{n}$. It is known that the zero points of $\vec{\phi}$, i.e.,

$$\begin{aligned}\phi^1(x,y,z,t)&=0,\\ \phi^2(x,y,z,t)&=0,\\ \phi^3(x,y,z,t)&=0,\end{aligned} \tag{4}$$

determine the locations of point defects. If the Jacobian determinant

$$D\left(\frac{\phi}{x}\right)=\frac{\partial(\phi^1,\phi^2,\phi^3)}{\partial(x^1,x^2,x^3)}\neq 0,$$

the solutions of Eq. (4) are generally expressed as

$$x=x_l(t), \quad y=y_l(t), \quad z=z_l(t), \quad l=1,2,\ldots,N, \tag{5}$$

which are the world lines of N point defects $\vec{r}_l(t)$ ($l=1,2,\ldots,N$) and represent N point defects moving in space.

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The generalized winding number W_l of $\vec{\phi}$ at one of zero points $\vec{r}_l$ is defined by the Gauss map n: $\partial\Omega_l \to S^2$ [7],

$$W_l = \frac{1}{8\pi}\int_{\partial\Omega_l} n^*(\epsilon_{abc} n^a dn^b \wedge dn^c), \tag{6}$$

where $\partial\Omega_l$ is the boundary of a neighborhood Ω_l of $\vec{r}_l$ with $\vec{r}_l \notin \partial\Omega_l, \Omega_l \cap \Omega_m = \emptyset$. Topologically this means that when the point $\vec{r}$ covers $\partial\Omega_l$ once, the unit vector $\vec{n}$ will cover S^2, or $\vec{\phi}$ covers the corresponding region W_l times, which is a topological invariant and is also called the degree of the Gauss map [8,9]. The winding number W_l is also called the topological charge of the lth point defect locating at the lth zero points of the vector order parameter $\vec{\phi}$. Using the Stokes theorem in the exterior differential form, one can deduce that

$$W_l = \frac{1}{8\pi}\int_{\Omega_l} \epsilon_{abc}\epsilon^{ijk}\partial_i n^a \partial_j n^b \partial_k n^c d^3 r. \tag{7}$$

So it is clear that the densities of point defect are just

$$\rho = \frac{1}{8\pi}\epsilon_{abc}\epsilon^{ijk}\partial_i n^a \partial_j n^b \partial_k n^c, \tag{8}$$

which are the time components of the topological current of the three-dimensional vector order parameter [10]

$$K^\mu = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda\rho}\epsilon_{abc}\partial_\nu n^a \partial_\lambda n^b \partial_\rho n^c, \quad \mu = 0,1,2,3. \tag{9}$$

Obviously, the current (9) is identically conserved,

$$\partial_\mu K^\mu = 0. \tag{10}$$

Following the ϕ-mapping topological current theory it can be rigorously proved that

$$K^\mu = \delta^3(\vec{\phi}) D^\mu\left(\frac{\phi}{x}\right), \tag{11}$$

where the Jacobian $D^\mu(\phi/x)$ is defined as

$$\epsilon^{abc} D^\mu\left(\frac{\phi}{x}\right) = \epsilon^{\mu\nu\lambda\rho}\partial_\nu\phi^a \partial_\lambda\phi^b \partial_\rho\phi^c \tag{12}$$

in which the usual three-dimensional Jacobian

$$D\left(\frac{\phi}{x}\right) = D^0\left(\frac{\phi}{x}\right).$$

Now the densities of point defects are expressed in terms of the vector order parameter field $\vec{\phi}(\vec{r},t)$,

$$\rho = \delta^3(\vec{\phi}) D\left(\frac{\phi}{x}\right). \tag{13}$$

Here one can see that the densities of point defects (13) are obtained directly from the definition of the topological charge of point defects (winding numbers of zero points), which is more general than usually considered.

According to the δ-function theory [11] and the ϕ-mapping topological current theory, one can prove that

$$\delta^3(\vec{\phi}) = \sum_{l=1}^{N} \frac{\beta_l}{|D(\phi/x)_{\vec{r}_l}|}\delta^3(\vec{r} - \vec{r}_l), \tag{14}$$

where the positive integer β_l is called the Hopf index [6] of the map $x \to \phi$. The meaning of β_l is that when the point $\vec{r}$ covers the neighborhood of the zero $\vec{r}_l$ once, the vector field $\vec{\phi}$ covers the corresponding region β_l times. Submitting Eq. (14) into Eq. (8), we obtain the densities of point defects

$$\rho = \sum_{l=1}^{N} \beta_l \eta_l \delta^3(\vec{r} - \vec{r}_l), \tag{15}$$

where η_l is the Brouwer degree [6]

$$\eta_l = \operatorname{sgn} D(\phi/x)_{\vec{r}_l} = \pm 1 . \tag{16}$$

One can find the relation between the Hopf index β_l, the Brouwer degree η_l, and the winding number W_l,

$$W_l = \beta_l \eta_l ,$$

from Eqs. (7) and (15).

Here we see that the result (1) obtained by Halperin, Liu, and Mazenko is not complete. They did not considering the case $\beta_l \neq 1$. Furthermore, they did not discuss what will happen when $D(\phi/x) = 0$, i.e., η_l is indefinite, which we will discuss in following sections.

Following our theory, we can also get the velocity of the lth point defect

$$v_l^i = \frac{dx_l^i}{dt} = \frac{D^i(\phi/x)}{D(\phi/x)}\bigg|_{\vec{r}_l}, \quad i = 1,2,3,$$

from which one can identify the point defects velocity field as

$$v^i = \frac{D^i(\phi/x)}{D(\phi/x)}, \quad i = 1,2,3, \tag{17}$$

where it is assumed that the velocity field is used in expressions multiplied by the point defects locating the δ function. The expressions given by Eq. (17) for the velocity of point defects are useful because they avoid the problem of having to specify the position of point defects explicitly. The positions are implicitly determined by the zeros of the order parameter field.

The current densities of point defects (N point defects with topological charge $\beta_l \eta_l$ moving in space) can be written as the same form as the current densities in hydrodynamics:

$$J^i = \sum_{l=1}^{N} \beta_l \eta_l \delta^3(\vec{r} - \vec{r}_l(t))\frac{dx_l^i}{dt}. \tag{18}$$

From Eqs. (11) and (14) the current densities of point defects can be written as the concise forms

$$J^i = K^i = \delta^3(\vec{\phi}) D^i\left(\frac{\phi}{x}\right) \tag{19}$$

or

$$J^i = \frac{1}{8\pi} \epsilon^{i\nu\lambda\rho} \epsilon_{abc} \partial_\nu n^a \partial_\lambda n^b \partial_\rho n^c. \tag{20}$$

According to Eq. (10), the topological charges of point defects are conserved:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0, \tag{21}$$

which is only the topological property of vector order parameter.

III. GENERATION AND ANNIHILATION OF POINT DEFECTS

As being discussed before, the zeros of the smooth vector $\vec{\phi}$ play an important roles in studying the point defects of a three-dimensional vector order parameter. Now we begin studying the properties of the zero points (locations of point defects), in other words, the properties of the solutions of Eq. (4). As we knew before, if the Jacobian

$$D\left(\frac{\phi}{x}\right) = \frac{\partial(\phi^1, \phi^2, \phi^3)}{\partial(x^1, x^2, x^3)} \neq 0, \tag{22}$$

we will have the isolated solutions (5) of Eq. (4). The isolated solutions are called regular points. However, when the condition (22) fails, the usual implicit function theorem is of no use. The above results (5) will change in some way and will lead to the branch process. We denote one of the zero points as $(t^*, \vec{z}_i)$. Mazenko [12] also obtained the velocity field of point defects (17), but did not discuss the case when $D(\phi/x) = 0$, i.e., η_l is indefinite. If the Jacobian

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*, \vec{z}_i)} \neq 0, \tag{23}$$

we can use the Jacobian $D^1(\phi/x)$ instead of $D(\phi/x)$ for the purpose of using the implicit function theorem [13]. Then we have a unique solution of Eqs. (4) in the neighborhood of the points $(t^*, \vec{z}_i)$,

$$t = t(x^1), \tag{24}$$

$$x^i = x^i(x^1), \quad i = 2,3,$$

with $t^* = t(z_i^1)$. We call the critical points $(t^*, \vec{z}_i)$ the limit points. In the present case, we know that

$$\frac{dx^1}{dt}\bigg|_{(t^*, \vec{z}_i)} = \frac{D^1(\phi/x)|_{(t^*, \vec{z}_i)}}{D(\phi/x)|_{(t^*, \vec{z}_i)}} = \infty, \tag{25}$$

i.e.,

$$\frac{dt}{dx^1}\bigg|_{(t^*, \vec{z}_i)} = 0.$$

The Taylor expansion of the solution of Eq. (24) at the limit point $(t^*, \vec{z}_i)$ is [5]

$$t - t^* = \frac{1}{2} \frac{d^2 t}{(dx^1)^2}\bigg|_{(t^*, \vec{z}_i)} (x^1 - z_i^1)^2, \tag{26}$$

which is a parabola in the x^1-t plane. From Eq. (26) we can obtain the two solutions $x_1^1(t)$ and $x_2^1(t)$, which give two branch solutions (world lines of point defects) of Eqs. (4). If $[d^2 t/(dx^1)^2]|_{(t^*, \vec{z}_i)} > 0$, we have the branch solutions for $t > t^*$ [see Fig. 1(a)]; otherwise, we have the branch solutions for $t < t^*$ [see Fig. 1(b)]. These two cases are related to the origin and annihilation of point defects.

One of the result of Eq. (25), that the velocity of point defects is infinite when they are annihilating, agrees with that obtained by Bray [14] who has a scaling argument associated

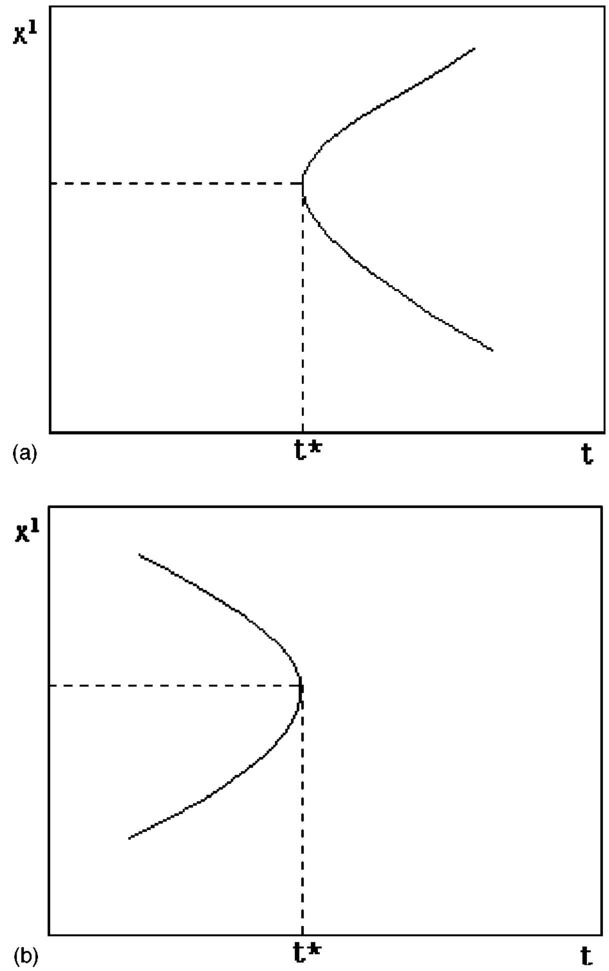

FIG. 1. Projecting the world lines of point defects onto the (x^1-t) plane. (a) Branch solutions for Eq. (26) when $[d^2t/(dx^1)^2]|_{(t^*, \vec{z}_i)} > 0$, i.e., a pair of point defects with opposite charges is generated at the limit point, i.e., the origin of point defects. (b) Branch solutions for Eq. (26) when $[d^2t/(dx^1)^2]|_{(t^*, \vec{z}_i)} < 0$, i.e., a pair of point defects with opposite charges are annihilated at the limit point.

with the point defect final annihilation, which leads to large velocity tail. From Eq. (25) we also get the result that the velocity of the point defects is infinite when they are generating, which is gained only from the topology of the three-dimensional vector order parameter.

Since the topological charge of point defect is identically conserved (21), the topological charge of these two point defects must be opposite at the limit point, i.e.,

$$\beta_{i_1}\eta_{i_1} = -\beta_{i_2}\eta_{i_2}, \tag{27}$$

which shows that $\beta_{i_1}=\beta_{i_2}$ and $\eta_{i_1}=-\eta_{i_2}$. One can see that the fact the Brouwer degree η is indefinite at the limit points implies that it can change discontinuously at limit points along the world lines of point defects (from ± 1 to ∓ 1). It is easy to see from Fig. 1 that when $x^1 > z_l^1$, $\eta_{i_1}=\pm 1$ and when $x^1 < z_l^1$, $\eta_{i_2}=\mp 1$.

For a limit point it is also required that $D^1(\phi/x)|_{(t^*,\vec{z}_i)} \neq 0$. As to a bifurcation point [15], it must satisfy a more complex condition. This case will be discussed in the following section in detail.

IV. BIFURCATION OF THE POINT DEFECT VELOCITY FIELD

In this section we have the restrictions of Eqs. (4) at the bifurcation point $(t^*,\vec{z}_i)$,

$$D\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{z}_i)} = 0, \qquad D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{z}_i)} = 0, \tag{28}$$

which will lead to an important fact that the function relationship between t and x^1 is not unique in the neighborhood of the bifurcation point $(\vec{z}_i, t^*)$. It is easy to see that

$$\frac{dx^1}{dt}\bigg|_{(t^*,\vec{z}_i)} = \frac{D^1(\phi/x)|_{(t^*,\vec{z}_i)}}{D(\phi/x)|_{(t^*,\vec{z}_i)}}, \tag{29}$$

which under the constraint (28) directly shows that the direction of the integral curve of Eq. (29) is indefinite, i.e., the velocity field of point defects is indefinite at the point $(\vec{z}_i, t^*)$. This is why the very point $(\vec{z}_i, t^*)$ is called a bifurcation point of the three-dimensional vector order parameter $\vec{\phi}$.

Next we will find a simple way to search for the different directions of all branch curves (or velocity field of the point defect) at the bifurcation point. Assume that the bifurcation point $(\vec{z}_i, t^*)$ has been found from Eqs. (4) and (28). We know that, at the bifurcation point $(\vec{z}_i, t^*)$, the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is smaller than 3 [for $D(\phi/x)|_{(t^*,\vec{z}_i)}=0$]. First, we suppose that the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is 2 (the case of a smaller rank will be discussed later). In addition, according to the ϕ-mapping topological current theory, the Taylor expansion

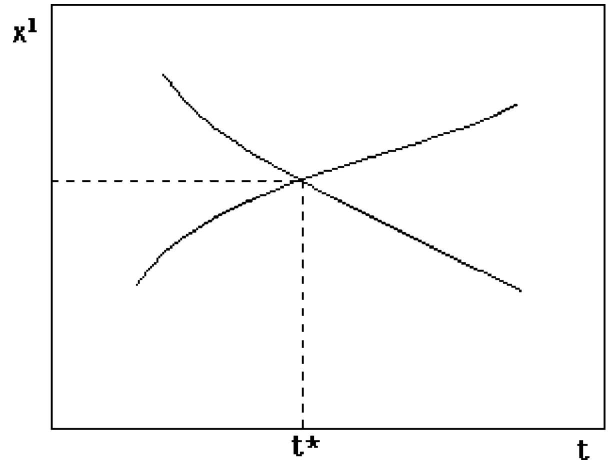

FIG. 2. Projecting the world lines of point defects onto the $(x^1$-$t)$-plane. Two world lines intersect with different directions at the bifurcation point, i.e., two point defects are encountered at the bifurcation point.

of the solution of Eq. (4) in the neighborhood of the bifurcation point $(\vec{z}_i, t^*)$ can be expressed as [5]

$$A(x^1-z_i^1)^2 + 2B(x^1-z_i^1)(t-t^*) + C(t-t^*)^2 = 0, \tag{30}$$

which leads to

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + C = 0, \tag{31}$$

and

$$C\left(\frac{dt}{dx^1}\right)^2 + 2B\frac{dt}{dx^1} + A = 0, \tag{32}$$

where A, B, and C are three constants. The solutions of Eq. (31) or (32) give different directions of the branch curves (world lines of point defects) at the bifurcation point. There are four possible cases that will show the physical meanings of the bifurcation points.

Case 1 $(A\neq 0)$. For $\Delta = 4(B^2-AC) > 0$ from Eq. (31) we get two different directions of the velocity field of point defects

$$\frac{dx^1}{dt}\bigg|_{1,2} = \frac{-B \pm \sqrt{B^2-AC}}{A}, \tag{33}$$

which are shown in Fig. 2, where two world lines of two point defects intersect with different directions at the bifurcation point. This shows that two point defects are encountered at and then depart from the bifurcation point.

Case 2 $(A\neq 0)$. For $\Delta = 4(B^2-AC) = 0$, from Eq. (31) we get only one direction of the velocity field of point defects

$$\frac{dx^1}{dt}\bigg|_{1,2} = -\frac{B}{A}, \tag{34}$$

which includes three important cases. (a) Two world lines are in contact tangentially, i.e., two point defects are encoun-

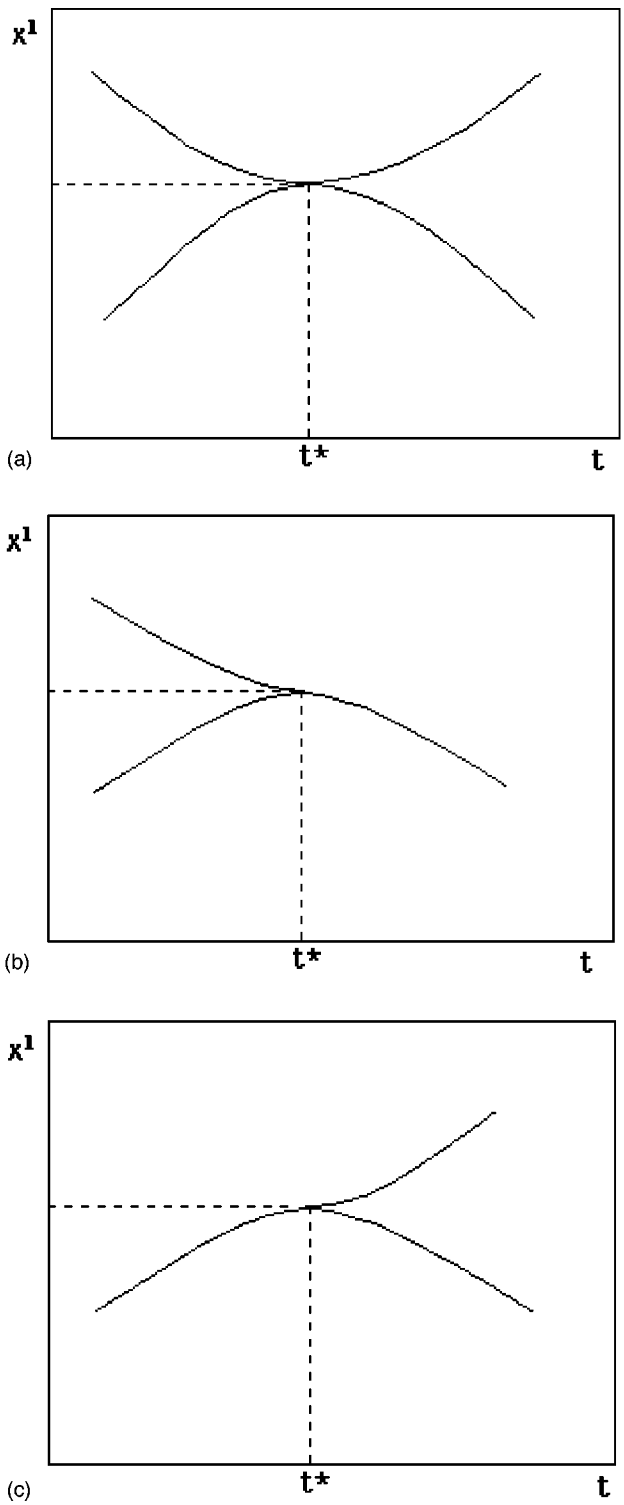

FIG. 3. (a) Two world lines are in contact tangentially, i.e., two point defects are encountered tangentially at the bifurcation point. (b) Two world lines merge into one world line, i.e., two point defects merge into one point defect at the bifurcation point. (c) One world line resolves into two world lines, i.e., one point defect splits into two point defects at the bifurcation point.

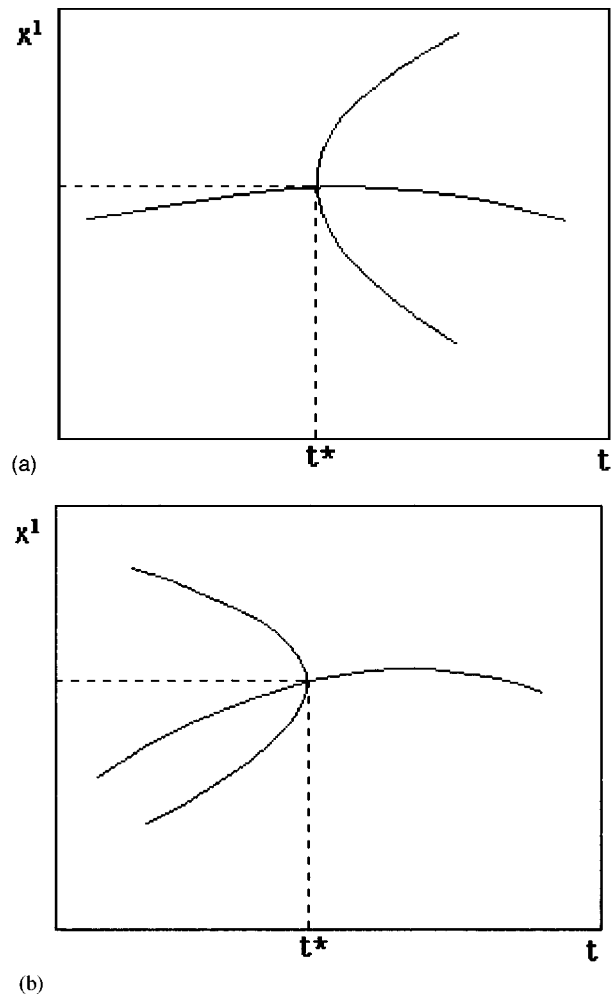

FIG. 4. Two important cases of Eq. (35). (a) One world line resolves into three world lines, i.e., one point defect splits into three point defects at the bifurcation point. (b) Three world lines merge into one world line, i.e., three point defects merge into one point defect at the bifurcation point.

tered tangentially at the bifurcation point [see Fig. 3(a)]. (b) Two world lines merge into one world line, i.e., two point defects merge into one point defect at the bifurcation point [see Fig. 3(b)]. (c) One world line resolves into two world lines, i.e., one point defect splits into two point defects at the bifurcation point [see Fig. 3(c)].

Case 3 $(A=0, C\neq 0)$. For $\Delta=4(B^2-AC)=0$, from Eq. (32) we have

$$\left.\frac{dt}{dx^1}\right|_{1,2}=\frac{-B\pm\sqrt{B^2-AC}}{C}=0, \quad -\frac{2B}{C}. \tag{35}$$

There are two important cases. (a) One world line resolves into three world lines, i.e., one point defect splits into three point defects at the bifurcation point [see Fig. 4(a)]. (b) Three world lines merge into one world line, i.e., three point defects merge into one point defect at the bifurcation point [see Fig. 4(b)].

Case 4 $(A=C=0)$. Equations (31) and (32) give, respectively,

$$\frac{dx^1}{dt}=0, \quad \frac{dt}{dx^1}=0. \tag{36}$$

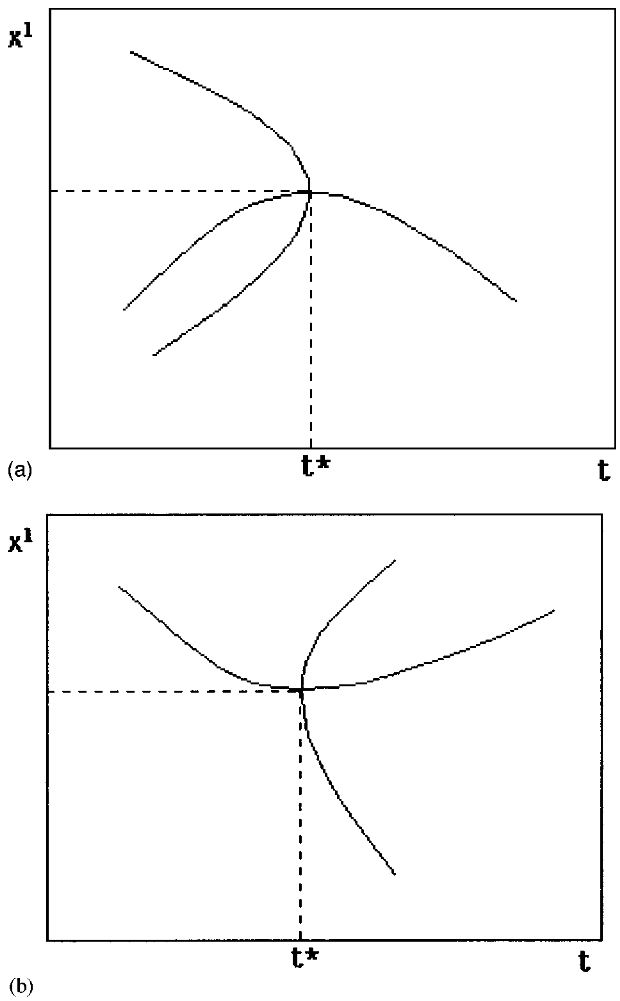

FIG. 5. Two world lines intersect normally at the bifurcation point. This case is similar to Fig. 4. (a) Three point defects merge into one point defect at the bifurcation point. (b) One point defect splits into three point defects at the bifurcation point.

This case is obvious in Fig. 5 and is similar to case 3.

The above solutions reveal the evolution of the point defects. Besides the encountering of the point defects, i.e., two point defects are encountered at and then depart from the bifurcation point along different branch curves [see Figs. 2 and 3(a)], it also includes splitting and merging of point defects. When a multicharged point defect moves through the bifurcation point, it may split into several point defects along different branch curves [see Figs. 3(c), 4(a), and 5(b)]. On the contrary, several point defects can merge into one point defect at the bifurcation point [see Figs. 3(b) and 4(b)]. A similar analysis of the topological charge shows that the sum of the topological charge of the final point defect(s) must be equal to that of the initial point defect(s) at the bifurcation point, i.e.,

$$\sum_f \beta_{l_f}\eta_{l_f}=\sum_i \beta_{l_i}\eta_{l_i} \tag{37}$$

for fixed l. Furthermore, from the above studies, we see that the generation, annihilation, and bifurcation of point defects are not gradual changes, but start at a critical value of arguments, i.e., a sudden change.

Finally, we discuss the branch process at a higher degenerated point. We have studied the case that the rank of the Jacobian matrix $[\partial\phi/\partial x]$ of Eqs. (4) is $2=3-1$. Now, we consider the case that the rank of the Jacobian matrix is $1=3-2$. Set $J_2(\phi/x)=\partial\phi^1/\partial x^1$ and suppose that $\det J_2\neq 0$. With the same methods used in obtaining Eq. (30), we can get the solution of Eqs. (4) in the neighborhood of the higher degenerated bifurcation point $(\vec{z}_i, t^*)$ [5],

$$a_0\left(\frac{dx^2}{dt}\right)^4+a_1\left(\frac{dx^2}{dt}\right)^3+a_2\left(\frac{dx^2}{dt}\right)^2+a_3\left(\frac{dx^2}{dt}\right)+a_4=0, \tag{38}$$

where a_0, a_1, a_2, and a_3 are also four constants. Therefore, we get different directions of the world lines of point defects at the higher degenerated bifurcation point. The number of different directions of the world lines is at most 4. Comparing with Eqs. (31) and (32), the above solutions also reveal encountering, spliting, and merging of the point defects along more directions.

V. CONCLUSIONS

First, the densities of point defects (13) and (15) have been obtained directly from the definition of topological charges of point defects (winding numbers of zero points of the vector order parameter), which is more general than usually considered and will be helpful as a complement of the works of point defects done by Mazenko *et al.* Second, we have studied the evolution of the point defects of a three-dimensional vector order parameter by making use of the ϕ mapping topological current theory. We conclude that there exist crucial cases of branch processes in the evolution of the point defects when $D(\phi/x)=0$, i.e., η_l is indefinite. This means that the point defects are generated or annihilated at the limit points and are encountered, split, or merge at the bifurcation points of the three-dimensional vector order parameter, which shows that the point defect system is unstable at these branch points. Third, we found the result that the velocity of point defects is infinite when they are annihilating or generating, which is obtained only from the topological properties of the three-dimensional vector order parameter. Fourth, we obtain two restrictions of the evolution of point defects. One restriction is the conservation of the topological charge of the point defects during the branch process [see Eqs. (27) and (37)]; the other restriction is that the number of different directions of the world lines of point defects is at most 4 at the bifurcation points [see Eqs. (30), (32), and (38)]. The first restriction is already known, but the second is pointed out here for the first time to our knowledge. We hope that it can be verified in the future. Finally, we would like to point out that all the results in this paper have been obtained only from the viewpoint of topology without using any particular models or hypothesis.

ACKNOWLEDGMENT

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