Sharing tripartite nonlocality sequentially by arbitrarily many independent observers

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Bipartite entangled states whose violations of the Clauser-Horne-Shimony-Holt Bell inequality can be observed by a single Alice and arbitrarily many sequential Bobs exist [Brown and Colbeck, Phys. Rev. Lett. **125**, 090401 (2020)]. Here we consider their analogs for tripartite systems: a tripartite entangled state is shared among Alice, Bob, and multiple Charlies. The first Charlie measures his qubit and then passes his qubit to the next Charlie, who measures again with other measurements, and so on. The goal is to maximize the number of Charlies that can observe some kind of nonlocality with the single Alice and Bob. It has been shown that at most two Charlies can share genuine nonlocality of the Greenberger-Horne-Zeilinger state via the violation of the Svetlichny inequality with Alice and Bob [S. Saha *et al.*, Quantum Inf. Process. **18**, 42 (2019); Zhang and Fei, Phys. Rev. A **103**, 032216 (2021)]. In this work, we show that arbitrarily many Charlies can have standard nonlocality (via violations of the Mermin inequality) and some other kind of genuine nonlocality (which is known as genuinely nonsignal nonlocality) with a single Alice and single Bob.

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I. INTRODUCTION

Quantum nonlocality is one of the most striking features of quantum physics. Through the quantum violation of a suitable set of inequalities, Bell [1] demonstrated that the predictions of quantum mechanics are in contradiction to the classical causal relations. Beyond its importance in quantum foundations, quantum nonlocality is also the key resource for device-independent quantum information processing, such as building quantum protocols to decrease communication complexity [2,3] and providing secure quantum communication [4,5].

Recently, the question of whether a single entangled pair can generate a long sequence of nonlocal correlations has gained extensive attention. For bipartite quantum systems, in [6,7], the authors showed that at most two Bobs can achieve an expected Clauser-Horne-Shimony-Holt violation with a single Alice when each Bob performs different measurements with equal probability and sharpness and the measurements of each Bob are independent of the choices of measurement settings and outcomes of the previous Bobs. In [8], with unequal sharpness for each Bob's measurements, the authors proved that arbitrarily many independent Bobs can share the nonlocality of the Bell state with a single Alice. This result was soon extended to higher-dimensional bipartite systems (see Ref. [9]). There has also been some progress on the setting where multiple Alices and Bobs are considered [10,11].

It is natural to ask whether a similar property holds in the multipartite setting. For tripartite quantum systems, the authors of [12] showed that via the violation of the Mermin inequality, at most six Charlies can simultaneously demonstrate standard tripartite nonlocality with a single Alice and a single Bob. On the other hand, based on the Svetlichny inequality [13], at most two Charlies can simultaneously share genuine tripartite nonlocality with a single Alice and a single Bob [9,12]. As quantum nonlocality can be observed by violating different kinds of inequalities, it is interesting to ask whether there is some kind of nonlocality that can be detected sequentially by arbitrarily many Charlies with a single Alice and a single Bob via some related inequality. We will give affirmative answers for the settings where the nonlocality can be obtained via the violation of the Mermin inequality or the genuinely nonsignal nonlocality defined in Ref. [14]. Contrary to the result that at most six Charlies can simultaneously observe the violations of the Mermin inequality with a single Alice and Bob, here we find that arbitrarily many independent Charlies can observe this violation with the single Alice and Bob. This might be possible because we choose a measurement strategy for Alice, Bob, and multiple Charlies that is different from the one in Ref. [12].

The rest of this paper is organized as follows. In Sec. II, we review some definitions of the tripartite nonlocality. In Sec. III, we introduce the scenario of sharing tripartite nonlocality sequentially with multiple Charlies and a single Alice and a single Bob. In Sec. IV, we give a constructive measurement strategy which enables arbitrarily many independent Charlies to observe the violation of the Mermin inequality with a single Alice and Bob. In Sec. V, we provide a specific measurement strategy which enables arbitrarily many independent Charlies to observe the genuinely nonsignal nonlocality with a single Alice and Bob. Finally, we draw our conclusions in Sec. VI.

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II. TRIPARTITE QUANTUM NONLOCALITY

Different from the ones in bipartite systems, quantum states in tripartite systems can be not only entangled or nonlocally correlated but also genuinely entangled or genuinely nonlocally correlated. Quantum nonlocality can be revealed via violations of various Bell inequalities. For tripartite quantum systems, except for the well-known Svetlichny inequalities [13], in [14] other three-qubit genuine nonlocality and three-way nonlocal correlations were studied.

Now we consider a tripartite scenario where each of three spatially separated parties, Alice, Bob, and Charlie, performs the measurements X_i, Y_j, and Z_k on their subsystems, respectively, with outcomes A, B, and C, $i, j, k \in \{0, 1\}, A, B, C \in \{0, 1\}$. Let $P(ABC|X_iY_jZ_k)$ denote the joint outcome probabilities where Alice measures her system by X_i with outcome A (with similar notation for Bob and Charlie). First, if the probability correlations $P(ABC|X_iY_jZ_k)$ among the measurement outcomes can be written as

$$P(ABC|X_iY_jZ_k) = \sum_\lambda q_\lambda P_\lambda(A|X_i)P_\lambda(B|Y_j)P_\lambda(C|Z_k),$$

with $0 \leqslant q_\lambda \leqslant 1$ and $\sum_\lambda q_\lambda = 1$, then they are called fully local. If $P(ABC|X_iY_jZ_k)$ is not fully local, we say $P(ABC|X_iY_jZ_k)$ exhibits *standard tripartite nonlocality*. In particular, it can be detected by violations of the Mermin inequalities [15], which have the following form:

$$\langle X_1Y_0Z_0\rangle + \langle X_0Y_1Z_0\rangle + \langle X_0Y_0Z_1\rangle - \langle X_1Y_1Z_1\rangle \leqslant 2, \quad (1)$$

where $\langle X_iY_jZ_k\rangle = \sum_{ABC}(-1)^{A+B+C}P(ABC|X_iY_jZ_k)$. As pointed out by Svetlichny [13], if the correlation can be written in the form

$$\begin{aligned}P(ABC|X_iY_jZ_k) = &\sum_\lambda q_\lambda P_\lambda(AB|X_iY_j)P_\lambda(C|Z_k)\\ &+ \sum_\mu q_\mu P_\mu(AC|X_iZ_k)P_\mu(B|Y_j)\\ &+ \sum_\nu q_\nu P_\nu(BC|Y_jZ_k)P_\nu(A|X_i), \quad (2)\end{aligned}$$

where $0 \leqslant q_\lambda, q_\mu, q_\nu \leqslant 1$, and $\sum_\lambda q_\lambda + \sum_\mu q_\mu + \sum_\nu q_\nu = 1$, then $P(ABC|X_iY_jZ_k)$ is called S_2 local. Otherwise, it is called *three-way genuine nonlocality*, which is also known as Svetlichny nonlocality. Svetlichny found that the three-way genuine nonlocality can be observed by violations of Svetlichny inequality, which is defined as

$$\begin{aligned}&\langle X_0Y_0Z_0\rangle + \langle X_0Y_1Z_0\rangle + \langle X_0Y_0Z_1\rangle - \langle X_0Y_1Z_1\rangle\\ &+ \langle X_1Y_0Z_0\rangle - \langle X_1Y_1Z_0\rangle - \langle X_1Y_0Z_1\rangle + \langle X_1Y_1Z_1\rangle \leqslant 4.\end{aligned}$$

In particular, even if $P(ABC|X_iY_jZ_k)$ violates the Mermin equality, it does not necessarily demonstrate that the correlation exhibits Svetlichny nonlocality. In [14], the authors introduced two alternative definitions of three-way nonlocality that are strictly weaker than Svetlichny nonlocality. Here apart from Svetlichny nonlocality, we mainly study Definition 1 in [14] regarding the genuine nonlocality.

We assume that the probabilities $P(ABC|X_iY_jZ_k)$ satisfy Eq. (2). Moreover, for any possible A, B, C, and C' and X_i, Y_j, Z_k, and Z'_k, the equalities

$$\sum_B P_\lambda(AB|X_iY_j) = \sum_{B'} P_\lambda(AB'|X_iY'_j)$$

are satisfied, and other equalities can also be obtained from permutations of the parties; then the correlations are called nonsignal local (denoted NS local). Otherwise, we call them *genuinely nonsignal nonlocal* (i.e., genuinely NS nonlocal).

To detect the NS genuine nonlocality, we consider the following inequality (denoted as the NS inequality):

$$\langle Y_0Z_0\rangle + \langle X_0Z_0\rangle + \langle X_1Y_0\rangle - \langle X_0Y_1Z_1\rangle + \langle X_1Y_1Z_1\rangle \leqslant 3, \quad (3)$$

where $\langle X_iY_j\rangle = \sum_{AB}(-1)^{A+B}P(AB|X_iY_j)$.

By definition, the relations among standard nonlocality, Svetlichny nonlocality, and genuinely nonsignal nonlocality can be seen in Fig. 1.

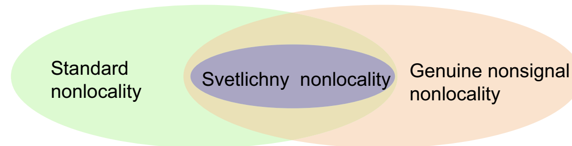

FIG. 1. The relations among standard nonlocality, Svetlichny nonlocality, and genuinely nonsignal nonlocality.

III. SCENARIO OF SHARING OF TRIPARTITE NONLOCALITY BY MULTIPLE CHARLIES

The Pauli operators are denoted σ_i for $i \in \{1, 2, 3\}$. Throughout this paper, we use two-outcome positive operator-valued measurements (POVMs) $\{E, \mathbb{I} - E\}$, where E has the form $E = \frac{\mathbb{I}+\gamma\sigma_{\vec{r}}}{2}$, $\gamma \in [0, 1]$ is the sharpness of the measurement, $\vec{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$, $\| \vec{r} \| = 1$, and $\sigma_{\vec{r}} = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3$, for example, $\{A_{0|0}, A_{1|0}\}$, where $A_{0|0} = \frac{1}{2}(\mathbb{I} + \sigma_1)$ and $A_{1|0} = \mathbb{I} - A_{0|0} = \frac{1}{2}(\mathbb{I} - \sigma_1)$. Therefore, to define a two-outcome measurement, it is enough to define one measurement element.

Now we introduce the scenario of sharing tripartite nonlocality sequentially with multiple Charlies and a single Alice and a single Bob. The corresponding measurement scenario illustrated in Fig. 2 is considered.

Three particles are prepared in the state $\rho_{ABC^{(1)}} = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$, where the Greenberger-Horne-Zeilinger state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. These three particles are spatially separated and shared between Alice, Bob, and multiple Charlies (i.e., Charlie$^{(1)}$, Charlie$^{(2)}$, Charlie$^{(3)}$, ..., Charlie$^{(n)}$). Alice performs measurement X on the first particle and gets outcome A. Bob performs measurement Y on the second particle and gets outcome B. And multiple Charlies perform measurements $Z^{(k)}$ on the third particle and get outcomes $C^{(k)}$ sequentially. In particular, Charlie$^{(1)}$ performs measurements on the third particle, and after doing measurements, he passes the particle to Charlie$^{(2)}$. Then Charlie$^{(2)}$ delivers the particle to Charlie$^{(3)}$ after doing measurements and so on. Moreover, each Charlie performs measurements independent of the measurement choices and outcomes of the previous Charlies in this sequence. Here we consider the

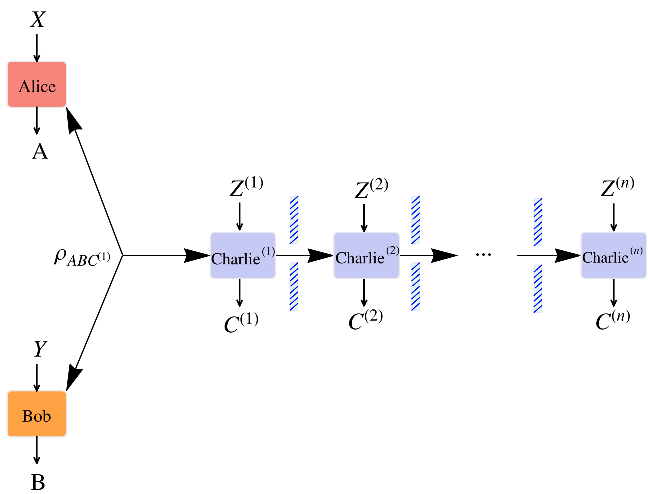

FIG. 2. Sharing the genuine tripartite nonlocality with multiple Charlies: A quantum state $\rho_{ABC^{(1)}}$ is initially distributed between Alice, Bob, and Charlie$^{(1)}$. After Charlie$^{(1)}$ performs his randomly selected measurement and records the outcomes, he passes the postmeasurement quantum state to Charlie$^{(2)}$, who then repeats the process. In particular, the measurement choices and outcomes of each Charlie are not conveyed.

unbiased input scenario; that is, all possible measurement settings of each Charlie are uniformly distributed.

The goal is to maximize the number of Charlies that can observe some kind of nonlocality with a single Alice and Bob. Therefore, it is crucial to find out what the state $\rho_{ABC^{(k)}}$ shared by Alice, Bob, and Charlie$^{(k)}$ is after Charlie$^{(k-1)}$ performs his measurements. In fact, suppose Charlie$^{(k-1)}$ performed the measurement according to $Z^{(k-1)} = z$ and received the outcome $C^{(k-1)} = c$; the postmeasurement state can be described by the Lüders rule:

$$\rho_{ABC^{(k)}} = \frac{1}{2}\sum_{c,z}\left(\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{C_{c|z}^{(k-1)}}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{C_{c|z}^{(k-1)}}\right). \tag{4}$$

IV. SHARING OF TRIPARTITE NONLOCALITY BY MULTIPLE CHARLIES VIA MERMIN INEQUALITY

First, we consider how many Charlies can simultaneously demonstrate tripartite nonlocality via Mermin inequality (1) with a single Alice and Bob. Therefore, it is important to find out the Mermin value $\mathbf{I}_M^{(k)}$ among Alice, Bob, and Charlie$^{(k)}$, which is defined by

$$\mathrm{Tr}\left[\rho_{ABC^{(k)}}\left(X_1Y_0Z_0^{(k)} + X_0Y_1Z_0^{(k)} + X_0Y_0Z_1^{(k)} - X_1Y_1Z_1^{(k)}\right)\right]. \tag{5}$$

Here $X_0, X_1, Y_0, Y_1, Z_0^{(k)}$, and $Z_1^{(k)}$ are the observables corresponding to their measurements that will be defined in the following.

To explain how we can define a sequence of pairs of POVMs for Alice, Bob, Charlie$^{(k)}$ such that $\mathbf{I}_M^{(k)} > 2$, $k \in \{1, 2, \cdots, n\}$, we give the following measurement strategy for Alice, Bob, and Charlie$^{(k)}$. In this measurement strategy, Alice's POVMs are defined by

$$A_{0|0} = \frac{\mathbb{I}+\sigma_1}{2}, \quad A_{0|1} = \frac{\mathbb{I}+\sigma_2}{2}; \tag{6}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I}-\theta\sigma_2}{2}, \quad B_{0|1} = \frac{\mathbb{I}+\theta\sigma_1}{2} \tag{7}$$

for $\theta \in (0, 1)$. For each $k = 1, 2, \dots, n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I}+\sigma_1}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I}+\gamma_k\sigma_2}{2}, \tag{8}$$

So the observables are given by $X_i = A_{0|i} - A_{1|i}$, $Y_i = B_{0|i} - B_{1|i}$, and $Z_i^{(k)} = C_{0|i}^{(k)} - C_{1|i}^{(k)}$, with $i = 0, 1$. Under these measurements and the initial state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, we can calculate the expected Mermin value for Alice, Bob, and Charlie$^{(k)}$ as follows (see Appendix A for the detailed calculation):

$$\mathbf{I}_M^{(k)} = 2^{2-k}\theta\left[\gamma_k + \prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)\right]. \tag{9}$$

The inequality $\mathbf{I}_M^{(k)} > 2$ implies Alice, Bob, and Charlie$^{(k)}$ can observe the standard nonlocality. To ensure arbitrarily many Charlies can share the standard nonlocality with a single Alice and Bob, it is sufficient to prove that for any $n \in \mathbb{N}$, some (θ, γ_k) exists such that $\mathbf{I}_M^{(k)} > 2$ holds for all $k = 1, 2, \dots, n$. From Eq. (9), we have

$$\mathbf{I}_M^{(k)} > 2 \Leftrightarrow \gamma_k > \frac{2^{k-1}}{\theta} - \prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right), \tag{10}$$

which motivates us to find a sequence $\{\gamma_k(\theta)\}$ such that for all $k \in \{1, 2, \dots, n\}$, $\gamma_k(\theta) \in [0, 1]$, and $\gamma_k > \frac{2^{k-1}}{\theta} - \prod_{j=1}^{k-1}(1 + \sqrt{1-\gamma_j^2})$.

To achieve this, we will give a specific sequence and prove that this sequence satisfies the above conditions. In fact, set $\epsilon > 0$, $\gamma_1(\theta) := (1+\epsilon)(\frac{1}{\theta} - 1)$, and, for $k \geqslant 2$,

$$\gamma_k(\theta) = \begin{cases} (1+\epsilon)\left(\frac{2^{k-1}}{\theta} - P_k\right), & 0 \leqslant \gamma_{k-1}(\theta) \leqslant 1, \\ \infty, & \text{otherwise}, \end{cases} \tag{11}$$

where $P_k = \prod_{j=1}^{k-1}(1 + \sqrt{1-\gamma_j^2})$. Then we have the following statement, which is sufficient to deduce that arbitrarily many Charlies can share the standard nonlocality of $\rho_{ABC^{(1)}}$ with a single Alice and Bob.

Theorem 1. For each $n \in \mathbb{N}$, a sequence $\{\gamma_k(\theta)\}_{k=1}^n$ and $\theta_n \in (0, 1)$ such that $\mathbf{I}_M^{(k)} > 2$ and $0 < \gamma_k(\theta) < 1, \theta \in (\theta_n, 1)$ for $k = 1, 2, \dots, n$ exists.

The proof of Theorem 1 is given in Appendix C. Theorem 1 shows that by initially sharing the GHZ state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, the number of Charlies that violate the Mermin inequality with a single Alice and Bob is unbounded.

We note that the authors of [12] showed that at most six Charlies can simultaneously demonstrate standard tripartite nonlocality with a single Alice and Bob, where they assumed that the measurements of both Alice and Bob are projective measurements. In contrast to the results from [12], with Theorem 1 and Remark 1, we show that using different measurement strategies for Alice, Bob, and Charlies enables more Charlies to share the standard tripartite nonlocality.

Remark 1. (1) We state that when the initial state is $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, even if both Alice and Bob perform the projective measurements, i.e., $\theta = 1$, unbounded numbers of Charlies also exist such that $\rho_{ABC^{(k)}}$ can violate the Mermin inequality with a single Alice and Bob. However, the measurement strategies defined by Eqs. (6), (7), and (8) should be slightly changed as follows: $\gamma_1(\varphi) = (1+\epsilon)\varphi$, and for $k \geqslant 2$,

$$\gamma_k(\varphi) = \begin{cases} (1+\epsilon)[2^{k-1} - P_k(\varphi)], & 0 \leqslant \gamma_{k-1}(\varphi) \leqslant 1, \\ \infty, & \text{otherwise}, \end{cases} \tag{12}$$

where $P_k(\varphi) = \prod_{j=1}^{k-1}[1 + \sqrt{1-\gamma_j^2(\varphi)}]$. In fact, in the case of $\theta = 1$, Eq. (10) can be changed into

$$\mathbf{I}_M^{(k)} > 2 \Leftrightarrow \gamma_k > 2^{k-1} - \prod_{j=1}^{k-1}\left(1 + \sqrt{1-\gamma_j^2}\right). \tag{13}$$

Therefore, to ensure the first Charlie obtains a violation of Mermin inequality it is sufficient to make sure $0 < \gamma_1 < 1$, which can be satisfied by choosing a small $\varphi > 0$ in our setting. With arguments similar to those in Appendix C, one could deduce that $\lim_{\varphi\to 0^+}\gamma_k(\varphi) = 0$ for all k, which is sufficient to yield our statement.

(2) We demonstrate that if Alice performs the sharp measurements and one of Bob's measurements is sharp, then unbounded numbers of Charlies also exist such that $\rho_{ABC^{(k)}}$ can violate the Mermin inequality for any time with a single Alice and Bob when the initial state is $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ based on the following measurement strategy: Alice's POVMs are defined by

$$A_{0|0} = \frac{\mathbb{I} + \sigma_1}{2}, \quad A_{0|1} = \frac{\mathbb{I} + \sigma_2}{2}. \tag{14}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I} - \theta_1\sigma_2}{2}, \quad B_{0|1} = \frac{\mathbb{I} + \theta_2\sigma_1}{2} \tag{15}$$

for $\theta_1, \theta_2 \in [0, 1]$. For each $k = 1, 2, \ldots, n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I} + \sigma_1}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I} + \gamma_k\sigma_2}{2}. \tag{16}$$

Here we assume $\theta_1 = 1, \theta_2 \neq 1$; under these measurements and the initial state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, we can get

$$\mathbf{I}_M^{(k)} = 2^{1-k}(1+\theta_2)\left[\gamma_k + \prod_{j=1}^{k-1}\left(1 + \sqrt{1-\gamma_j^2}\right)\right]. \tag{17}$$

In order to observe $\mathbf{I}_M^{(k)} > 2$, we will need

$$\mathbf{I}_M^{(k)} > 2 \Leftrightarrow \gamma_k > \frac{2^k}{1+\theta_2} - \prod_{j=1}^{k-1}\left(1 + \sqrt{1-\gamma_j^2}\right). \tag{18}$$

Next, we can define $\{\gamma_k(1,\theta_2)\}$ for some fixed $\epsilon > 0$,

$$\gamma_k(1,\theta_2) = \begin{cases} (1+\epsilon)\left(\frac{2}{1+\theta_2} - 1\right), & k = 1, \\ (1+\epsilon)\left(\frac{2^k}{1+\theta_2} - P_k\right), & 0 \leqslant \gamma_{k-1}(1,\theta_2) \leqslant 1, \\ \infty, & \text{otherwise}, \end{cases} \tag{19}$$

where $P_k = \prod_{j=1}^{k-1}(1 + \sqrt{1-\gamma_j^2})$.

With methods similar to those in Appendix C, one can deduce that $\lim_{\theta_2\to 1^-}\gamma_k(1,\theta_2) = 0$ for all k and further demonstrate our statement.

Remark 2. When starting with the W state $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, we prove that at most two Charlies can demonstrate standard nonlocality through the violation of Mermin inequality with a single Alice and Bob using the following measurement strategy: Alice's POVMs are defined by

$$A_{0|0} = \frac{\mathbb{I} + \cos(\theta_1)\sigma_3 - \sin(\theta_1)\sigma_1}{2}, \tag{20}$$

$$A_{0|1} = \frac{\mathbb{I} + \sin(\theta_1)\sigma_3 + \cos(\theta_1)\sigma_1}{2}. \tag{21}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I} + \cos(\theta_2)\sigma_3 - \sin(\theta_2)\sigma_1}{2}, \tag{22}$$

$$B_{0|1} = \frac{\mathbb{I} + \sin(\theta_2)\sigma_3 + \cos(\theta_2)\sigma_1}{2} \tag{23}$$

for $\theta_i \in [0, \frac{\pi}{2}]$, $i \in \{1, 2\}$. For each $k = 1, 2, \ldots, n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I} - \sigma_3}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I} - \gamma_k\sigma_1}{2}. \tag{24}$$

The proof of Remark 2 is given in Appendix D. Note that when $\theta_1 + \theta_2 = \frac{\pi}{2}$ and $\gamma_1 = 1$, Alice, Bob, and the first Charlie could yield a maximal violation (with a value of 3) of Mermin inequality under the assumption that the initial state is the W state. However, some other measurement strategy with a larger number of Charlies may exist that could yield a violation of Mermin inequality with a single Alice and Bob.

V. SHARING OF GENUINE TRIPARTITE NONLOCALITY BY MULTIPLE ALICES VIA NS INEQUALITY

Second, we consider how many Charlies can simultaneously demonstrate tripartite genuinely nonsignal nonlocality through the NS inequality (3) with a single Alice and Bob. Therefore, we need to calculate the NS value between Alice, Bob, and Charlie$^{(k)}$. The NS value is defined as

$$\begin{aligned} \mathbf{I}_{\mathrm{NS}}^{(k)} \equiv \mathrm{Tr}\big[\rho_{ABC^{(k)}}\big(&Y_0Z_0^{(k)} + X_0Z_0^{(k)} \\ &+ X_1Y_0 - X_0Y_1Z_1^{(k)} + X_1Y_1Z_1^{(k)}\big)\big]. \end{aligned} \tag{25}$$

To explain how we can define a sequence of pairs of POVMs for Alice, Bob, and Charlie$^{(k)}$ such that $\mathbf{I}_{\mathrm{NS}}^{(k)} > 3$, $k \in \{1, 2, \ldots, n\}$, we give the following measurement strategy for Alice, Bob, and Charlie$^{(k)}$. In this measurement strategy, Alice's POVMs are defined by

$$A_{0|0} = \frac{\mathbb{I} + \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1}{2}, \tag{26}$$

$$A_{0|1} = \frac{\mathbb{I} + \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1}{2} \tag{27}$$

for $\theta \in [0, \frac{\pi}{2}]$. Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I} + \sigma_3}{2}, \quad B_{0|1} = \frac{\mathbb{I} + \sigma_1}{2} \tag{28}$$

for each $k = 1, 2, \ldots, n$. Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I} + \sigma_3}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I} + \gamma_k\sigma_1}{2}. \tag{29}$$

The observables are given by $X_i = A_{0|i} - A_{1|i}$, $Y_i = B_{0|i} - B_{1|i}$, and $Z_i^{(k)} = C_{0|i}^{(k)} - C_{1|i}^{(k)}$, with $i = 0, 1$. For these measurements and the initial state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, we can calculate the expected NS value for Alice, Bob, and Charlie$^{(k)}$ as follows (see Appendix B for the detailed calculation):

$$\mathbf{I}_{\mathrm{NS}}^{(k)} = (\cos\theta + 1)\frac{\prod_{j=1}^{k-1}\left(1 + \sqrt{1-\gamma_j^2}\right)}{2^{k-1}} + \cos\theta + 2^{2-k}\gamma_k\sin\theta. \tag{30}$$

The inequality $\mathbf{I}_{\mathrm{NS}}^{(k)} > 3$ implies Alice, Bob, and Charlie$^{(k)}$ can observe the genuinely nonsignal nonlocality. To ensure arbitrarily many Charlies can share the genuinely nonsignal nonlocality with a single Alice and Bob, it is sufficient to prove that for any $n \in \mathbb{N}$, some θ exists such that $\mathbf{I}_{\mathrm{NS}}^{(k)} > 3$ holds for all $k = 1, 2, \ldots, n$. From Eq. (30), we have

$$\mathbf{I}_{\mathrm{NS}}^{(k)} > 3 \Leftrightarrow \gamma_k > \frac{3 - \cos\theta - (1+\cos\theta)\frac{\prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})}{2^{k-1}}}{2^{2-k}\sin\theta}, \tag{31}$$

which motivates us to find a sequence $\{\gamma_k(\theta)\}$ such that for all $k \in \{1, 2, \ldots, n\}$, $\gamma_k(\theta) \in [0, 1]$ and

$$\gamma_k > \frac{3 - \cos(\theta) - [1+\cos(\theta)]\frac{\prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})}{2^{k-1}}}{2^{2-k}\sin(\theta)}.$$

To achieve this, we give a specific sequence and prove that this sequence will satisfy the above conditions. In fact, set $\epsilon > 0$, $\gamma_1(\theta) := (1+\epsilon)\frac{1-\cos(\theta)}{\sin(\theta)}$, and, for $k \geqslant 2$,

$$\gamma_k(\theta) = \begin{cases} (1+\epsilon)\frac{3-\cos(\theta)-(1+\cos(\theta))\frac{P_k}{2^{k-1}}}{2^{2-k}\sin(\theta)}, & 0 \leqslant \gamma_{k-1}(\theta) \leqslant 1, \\ \infty, & \text{otherwise,} \end{cases} \tag{32}$$

where $P_k = \prod_{j=1}^{k-1}(1 + \sqrt{1-\gamma_j^2})$.

Then we have the following statement, which is sufficient to deduce that arbitrarily many Charlies can share the genuinely nonsignal nonlocality of $\rho_{ABC^{(1)}}$ with a single Alice and Bob.

Theorem 2. For each $n \in \mathbb{N}$, a sequence $\{\gamma_k(\theta)\}_{k=1}^n$ and a $\theta_n \in (0, 1)$ exist such that $\mathbf{I}_{\mathrm{NS}}^{(k)} > 3$ and $0 < \gamma_k(\theta) < 1, \theta \in (0, \theta_n)$ for $k = 1, 2, \ldots, n$.

The proof of Theorem 2 is given in Appendix E. Theorem 2 shows that by initially sharing the maximally entangled state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, the number of Charlies that violate the NS inequality with a single Alice and Bob is unbounded.

Remark 3. When starting with the W state $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, we prove that at most one Charlie can demonstrate the tripartite genuinely nonsignal nonlocality through NS inequality with a single Alice and Bob using the following measurement strategy: Alice's POVMs are defined by

$$A_{0|0} = \frac{\mathbb{I} + \sigma_3}{2}, \quad A_{0|1} = \frac{\mathbb{I} + \sigma_1}{2}. \tag{33}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I} - \sigma_3}{2}, \quad B_{0|1} = \frac{\mathbb{I} + \sigma_1}{2}. \tag{34}$$

For each $k = 1, 2, \ldots, n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I} + \sigma_3}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I} + \gamma_k\sigma_1}{2}. \tag{35}$$

Moreover, the corresponding NS inequality is chosen to be

$$\langle X_1Y_1\rangle + \langle Y_0Z_0\rangle + \langle X_1Z_1\rangle + \langle X_0Y_0Z_0\rangle - \langle X_1Y_0Z_1\rangle \leqslant 3. \tag{36}$$

The proof of Remark 3 is given in Appendix F. Note that when $\gamma_1 = 1$, Alice, Bob, and the first Charlie could yield a maximal violation (with a value of $\frac{10}{3}$) of NS inequality under the assumption that the initial state is the W state. However, some other measurement strategy with a larger number of Charlies may exist that could yield a violation of NS inequality with a single Alice and Bob.

VI. CONCLUSIONS AND DISCUSSION

In this work, we considered the sequential detection of quantum standard and genuinely nonsignal nonlocality in tripartite quantum systems. Just like for the bipartite settings, using different measurement strategies will enable more observers to share the tripartite nonlocality. In this way, we deduced that arbitrarily many independent Charlies could observe the standard tripartite nonlocality or the genuinely nonsignal one of $|\mathrm{GHZ}\rangle$ with a single Alice and Bob. Moreover, we also proved that when the initial state is $|W\rangle$, for the standard nonlocality, at most two Charlies can share the nonlocality with a single Alice and Bob through the Mermin inequality; for the genuinely nonsignal nonlocality, at most one Charlie can demonstrate this genuine nonlocality with a single Alice and Bob with the NS inequality.

One of the most important applications based on these nonlocal correlations in quantum protocols is device-independent random number generation. The amount of randomness from the measurement outcomes in quantum systems is quantified by the guessing probability and can generally be bounded numerically or analytically. The quantitative relationship between nonlocality and maximum certifiable randomness is difficult to exploit. In bipartite systems [16], for the standard Bell scenario where each party performed a single measurement on his or her subsystem, only a finite amount of randomness can be certified. In [17], the authors proved one could certify any number of random bits from a pair of qubits in a pure state when sequences of measurements were applied to each local system. Moreover, for tripartite systems, the authors of [18,19] studied the randomness using the Mermin-Ardehali-Belinskii-Klyshko inequality and gave upper bounds on the amount of the randomness. However, based on sequential measurement scenarios, whether unbounded certifiable randomness can be obtained from a tripartite genuinely entangled state is not known. So our current work represents a step towards a better understanding of the limitations on how much device-independent randomness can be robustly generated by the multipartite entangled states.

It is also interesting to consider a similar problem in a setting with multiple Alices, Bobs, and Charlies. Moreover, one finds that our method does not help when considering the setting of Svetlichny nonlocality. Maybe some state and measurement strategies exist such that more than two Charlies can share the Svetlichny nonlocality with a single Alice and Bob. Moreover, it is unknown whether can we obtain unbounded sequential violations of Mermin inequality or NS inequality with the initial state being $|W\rangle$ as we consider only some special measurement strategies in our paper.

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APPENDIX A: THE CALCULATION OF $I_M^{(k)}$

Now we derive the Mermin value for the given measurement strategy. Let the measurement strategy of Alice be defined by the POVM effects

$$A_{0|0} = \frac{\mathbb{I} + \sigma_1}{2}, \quad A_{0|1} = \frac{\mathbb{I} + \sigma_2}{2}. \tag{A1}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I} - \theta\sigma_2}{2}, \quad B_{0|1} = \frac{\mathbb{I} + \theta\sigma_1}{2} \tag{A2}$$

for $\theta \in [0, 1]$. For each $k = 1, 2, \dots, n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I} + \sigma_1}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I} + \gamma_k\sigma_2}{2}. \tag{A3}$$

The observables are given by $X_i = A_{0|i} - A_{1|i}$, $Y_i = B_{0|i} - B_{1|i}$, and $Z_i^{(k)} = C_{0|i}^{(k)} - C_{1|i}^{(k)}$, with $i = 0, 1$.

Let $\rho_{ABC^{(k-1)}}$ be shared by Alice, Bob, and Charlie$^{(k-1)}$ prior to Charlie$^{(k-1)}$'s measurements. Using the Lüders rule, the state sent to Charlie$^{(k)}$ is

$$\begin{aligned}
\rho_{ABC^{(k)}} &= \frac{1}{2}\sum_{c,z}\left(\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{C_{c|z}^{(k-1)}}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{C_{c|z}^{(k-1)}}\right)\\
&= \frac{1}{2}\Bigg(\mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}+\sigma_1}{2}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}+\sigma_1}{2} + \mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}-\sigma_1}{2}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}-\sigma_1}{2}\\
&\quad + \mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}+\gamma_{k-1}\sigma_2}{2}}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}+\gamma_{k-1}\sigma_2}{2}} + \mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}-\gamma_{k-1}\sigma_2}{2}}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}-\gamma_{k-1}\sigma_2}{2}}\Bigg)\\
&= \frac{2+\sqrt{1-\gamma_{k-1}^2}}{4}\rho_{ABC^{(k-1)}} + \frac{1}{4}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_1)\rho_{ABC^{(k-1)}}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_1)\\
&\quad + \frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_2)\rho_{ABC^{(k-1)}}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_2),
\end{aligned}$$

where we use the identity for the final calculation

$$\sqrt{\frac{\mathbb{I}\pm\gamma_k\sigma_{\vec{r}}}{2}} = \frac{(\sqrt{1+\gamma_k}+\sqrt{1-\gamma_k})\mathbb{I}\pm(\sqrt{1+\gamma_k}-\sqrt{1-\gamma_k})\sigma_{\vec{r}}}{2\sqrt{2}}. \tag{A4}$$

Then we consider the Mermin value of $\rho_{ABC^{(k)}}$:

$$\begin{aligned}\mathbf{I}_M^{(k)} &= \mathrm{Tr}\big[\rho_{ABC^{(k)}}\big(X_1Y_0Z_0^{(k)} + X_0Y_1Z_0^{(k)} + X_0Y_0Z_1^{(k)} - X_1Y_1Z_1^{(k)}\big)\big] \\ &= \mathrm{Tr}[\rho_{ABC^{(k)}}(-\theta\sigma_2\otimes\sigma_2\otimes\sigma_1 + \theta\sigma_1\otimes\sigma_1\otimes\sigma_1 - \theta\gamma_k\sigma_1\otimes\sigma_2\otimes\sigma_2 - \theta\gamma_k\sigma_2\otimes\sigma_1\otimes\sigma_2)] \\ &= 2^{2-k}\theta\left[\gamma_k + \prod_{j=1}^{k-1}\big(1+\sqrt{1-\gamma_j^2}\big)\right].\end{aligned}$$

In particular, $\mathbf{I}_M^{(1)} = 2\theta(1+\gamma_1)$.

APPENDIX B: THE CALCULATION OF $\mathbf{I}_{\mathrm{NS}}^{(k)}$

Now we derive the NS value for the given measurement strategy in the main text. Let the measurement strategy of Alice be defined by the POVM effects

$$A_{0|0} = \frac{\mathbb{I} + \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1}{2}, \quad A_{0|1} = \frac{\mathbb{I} + \cos(\theta)\sigma_3 + \sin(\theta)\sigma_1}{2}. \tag{B1}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I}+\sigma_3}{2}, \quad B_{0|1} = \frac{\mathbb{I}+\sigma_1}{2}. \tag{B2}$$

For each $k = 1, 2, \dots, n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I}+\sigma_3}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I}+\gamma_k\sigma_1}{2}. \tag{B3}$$

The observables are given by $X_i = A_{0|i} - A_{1|i}$, $Y_i = B_{0|i} - B_{1|i}$, and $Z_i^{(k)} = C_{0|i}^{(k)} - C_{1|i}^{(k)}$, with $i = 0, 1$.

Let $\rho_{ABC^{(k-1)}}$ be shared by Alice, Bob, and Charlie$^{(k-1)}$ prior to Charlie$^{(k-1)}$'s measurements. Using the Lüders rule, the state sent to Charlie$^{(k)}$ is

$$\begin{aligned}\rho_{ABC^{(k)}} &= \frac{1}{2}\sum_{c,z}\Big(\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{C_{c|z}^{(k-1)}}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{C_{c|z}^{(k-1)}}\Big) \\ &= \frac{1}{2}\Bigg(\mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}+\sigma_3}{2}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}+\sigma_3}{2} + \mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}-\sigma_3}{2}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\frac{\mathbb{I}-\sigma_3}{2} \\ &\quad + \mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}+\gamma_{k-1}\sigma_1}{2}}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}+\gamma_{k-1}\sigma_1}{2}} + \mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}-\gamma_{k-1}\sigma_1}{2}}\rho_{ABC^{(k-1)}}\mathbb{I}\otimes\mathbb{I}\otimes\sqrt{\frac{\mathbb{I}-\gamma_{k-1}\sigma_1}{2}}\Bigg) \\ &= \frac{2+\sqrt{1-\gamma_{k-1}^2}}{4}\rho_{ABC^{(k-1)}} + \frac{1}{4}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_3)\rho_{ABC^{(k-1)}}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_3) \\ &\quad + \frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_1)\rho_{ABC^{(k-1)}}(\mathbb{I}\otimes\mathbb{I}\otimes\sigma_1),\end{aligned}$$

where we use the identity for the final calculation

$$\sqrt{\frac{\mathbb{I}\pm\gamma_k\sigma_{\vec{r}}}{2}} = \frac{(\sqrt{1+\gamma_k}+\sqrt{1-\gamma_k})\mathbb{I}\pm(\sqrt{1+\gamma_k}-\sqrt{1-\gamma_k})\sigma_{\vec{r}}}{2\sqrt{2}}. \tag{B4}$$

Then we consider the NS value of $\rho_{A^{(k)}BC}$:

$$\begin{aligned}\mathbf{I}_{NS}^{(k)} &= \mathrm{Tr}\big[\rho_{ABC^{(k)}}\big(Y_0Z_0^{(k)} + X_0Z_0^{(k)} + X_1Y_0 - X_0Y_1Z_1^{(k)} + X_1Y_1Z_1^{(k)}\big)\big] \\ &= \mathrm{Tr}\{\rho_{ABC^{(k)}}(\mathbb{I}\otimes\sigma_3\otimes\sigma_3 + [\cos(\theta)\sigma_3 - \sin(\theta)\sigma_1]\otimes\mathbb{I}\otimes\sigma_3 + [\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1]\otimes\sigma_3\otimes\mathbb{I} \\ &\quad - \gamma_k[\cos(\theta)\sigma_3 - \sin(\theta)\sigma_1]\otimes\sigma_1\otimes\sigma_1 + \gamma_k[\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1]\otimes\sigma_1\otimes\sigma_1)\} \\ &= 2^{2-k}\gamma_k\sin(\theta) + \frac{\prod_{j=1}^{k-1}\big(1+\sqrt{1-\gamma_j^2}\big)}{2^{k-1}}[1+\cos(\theta)] + \cos(\theta).\end{aligned}$$

In particular, $\mathbf{I}_{\mathrm{NS}}^{(1)} = 1 + 2\cos(\theta) + 2\gamma_1\sin(\theta)$.

APPENDIX C: THE PROOF OF THEOREM 1

For the given measurements in the main text, in order to observe $\mathbf{I}_M^{(k)} > 2$, we need

$$\mathbf{I}_M^{(k)} > 2 \Leftrightarrow \gamma_k > \frac{2^{k-1}}{\theta} - \prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right). \tag{C1}$$

Next, we can define $\{\gamma_k(\theta)\}$ for some fixed $\epsilon > 0$,

$$\gamma_k(\theta) = \begin{cases} (1+\epsilon)\left(\frac{1}{\theta}-1\right), & k=1,\\ (1+\epsilon)\left(\frac{2^{k-1}}{\theta}-P_k\right), & 0 \leqslant \gamma_{k-1}(\theta) \leqslant 1,\\ \infty, & \text{otherwise,}\end{cases} \tag{C2}$$

where $P_k = \prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})$.

Then we can get

$$\frac{\gamma_k(\theta)}{\gamma_{k-1}(\theta)} > 2 \Leftrightarrow 0 < \gamma_{k-1}(\theta) \leqslant 1. \tag{C3}$$

Here $\gamma_1(\theta) = (1+\epsilon)(\frac{1}{\theta}-1)$, and $\lim\limits_{\theta\to 1^-}\gamma_1(\theta) = 0$.

By induction, we can suppose a θ_{k-1} exists such that in the interval $(\theta_{k-1}, 1)$, all $\gamma_i(\theta) \in (0,1)$ and $\lim_{\theta\to 1^-}\gamma_i(\theta) = 0$ for $i = 1, 2, \ldots, k-1$. Then according to the definition of $\gamma_k(\theta)$, we have

$$\lim_{\theta\to 1^-}\gamma_k(\theta) = \lim_{\theta\to 1^-}(1+\epsilon)\left(\frac{2^{k-1}}{\theta} - P_k\right) = (1+\epsilon)(2^{k-1}-2^{k-1}) = 0,$$

where we use the limit $\lim_{\theta\to 1^-} P_k = 2^{k-1}$, which holds as the induction assumptions $\lim_{\theta\to 1^-}\gamma_i(\theta) = 0$ for $i = 1, 2, \ldots, k-1$. So $\forall\, n \in \mathbb{N}$, we can find a $\theta_n \in (0,1)$ such that $0 < \gamma_1(\theta) < \gamma_2(\theta) < \cdots < \gamma_n(\theta) < 1$ for all $\theta \in (\theta_n, 1)$.

APPENDIX D: THE PROOF OF REMARK 2

Alice's POVMs are defined by

$$A_{0|0} = \frac{\mathbb{I} + \cos(\theta_1)\sigma_3 - \sin(\theta_1)\sigma_1}{2}, \quad A_{0|1} = \frac{\mathbb{I} + \sin(\theta_1)\sigma_3 + \cos(\theta_1)\sigma_1}{2}. \tag{D1}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I} + \cos(\theta_2)\sigma_3 - \sin(\theta_2)\sigma_1}{2}, \quad B_{0|1} = \frac{\mathbb{I} + \sin(\theta_2)\sigma_3 + \cos(\theta_2)\sigma_1}{2} \tag{D2}$$

for $\theta_i \in [0, \frac{\pi}{2}]$, $i \in \{1,2\}$. For each $k = 1, 2, \ldots, n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I} - \sigma_3}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I} - \gamma_k\sigma_1}{2}. \tag{D3}$$

Under these measurements and the initial state $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, we can get

$$\mathbf{I}_M^{(k)} = 2^{1-k}\left[\frac{5}{3}\sin(\theta_1+\theta_2)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right) + \frac{4}{3}\sin(\theta_1+\theta_2)\gamma_k\right]. \tag{D4}$$

Note that

$$\mathbf{I}_M^{(k)} > 2 \Leftrightarrow \gamma_k > \frac{2^k - \frac{5}{3}\sin(\theta_1+\theta_2)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)}{\frac{4}{3}\sin(\theta_1+\theta_2)}. \tag{D5}$$

Next, we can define $\{\gamma_k(\theta_1,\theta_2)\}$ for some fixed $\epsilon > 0$,

$$\gamma_k(\theta_1,\theta_2) = \begin{cases} (1+\epsilon)\frac{2-\frac{5}{3}\sin(\theta_1+\theta_2)}{\frac{4}{3}\sin(\theta_1+\theta_2)}, & k=1,\\ (1+\epsilon)\frac{2^k-\frac{5}{3}\sin(\theta_1+\theta_2)P_k}{\frac{4}{3}\sin(\theta_1+\theta_2)}, & 0 \leqslant \gamma_{k-1}(\theta_1,\theta_2) \leqslant 1,\\ \infty, & \text{otherwise,}\end{cases} \tag{D6}$$

where $P_k = \prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})$. Then we can get

$$\frac{\gamma_k(\theta_1,\theta_2)}{\gamma_{k-1}(\theta_1,\theta_2)} > 2 \Leftrightarrow 0 < \gamma_{k-1}(\theta_1,\theta_2) \leqslant 1. \tag{D7}$$

Note that

$$\gamma_1(\theta_1,\theta_2) = (1+\epsilon)\frac{2-\frac{5}{3}\sin(\theta_1+\theta_2)}{\frac{4}{3}\sin(\theta_1+\theta_2)} = (1+\epsilon)\left[\frac{6}{4\sin(\theta_1+\theta_2)} - \frac{5}{4}\right] \geqslant \frac{1+\epsilon}{4}.$$

From Eq. (D7), we have $\gamma_2(\theta_1,\theta_2) > \frac{1}{2}(1+\epsilon)$ and $\gamma_3(\theta_1,\theta_2) > 1+\epsilon$. Therefore, in the above strategy, at most two Charlies can demonstrate standard nonlocality through the violation of Mermin inequality with a single Alice and Bob.

APPENDIX E: THE PROOF OF THEOREM 2

For the given measurements in the main text, in order to observe $\mathbf{I}_{\rm NS}^{(k)} > 3$, we need

$$\mathbf{I}_{\rm NS}^{(k)} > 3 \Leftrightarrow \gamma_k > \frac{3-\cos(\theta) - [1+\cos(\theta)]\frac{\prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})}{2^{k-1}}}{2^{2-k}\sin(\theta)}. \tag{E1}$$

Next, we can define $\{\gamma_k(\theta)\}$ for some fixed $\epsilon > 0$,

$$\gamma_k(\theta) = \begin{cases} (1+\epsilon)\frac{1-\cos(\theta)}{\sin(\theta)}, & \text{k=1} \\ (1+\epsilon)\frac{3-\cos(\theta)-(1+\cos(\theta))\frac{P_k}{2^{k-1}}}{2^{2-k}\sin(\theta)}, & 0 \leqslant \gamma_{k-1}(\theta) \leqslant 1, \\ \infty, & \text{otherwise,} \end{cases} \tag{E2}$$

where $P_k = \prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})$. Then we can get

$$\frac{\gamma_k(\theta)}{\gamma_{k-1}(\theta)} > 2 \Leftrightarrow 0 < \gamma_{k-1}(\theta) \leqslant 1, \tag{E3}$$

where $\gamma_1(\theta) = (1+\epsilon)\frac{1-\cos(\theta)}{\sin(\theta)}$ and $\lim_{\theta\to 0^+}\gamma_1(\theta) = 0$.

By induction, we can suppose a θ_{k-1} exists such that in the interval $(0,\theta_{k-1})$, all $\gamma_i(\theta) \in (0,1)$ and $\lim_{\theta\to 0^+}\gamma_i(\theta) = 0$ for $i = 1,2,\dots,k-1$. Note that when looking at P_k as a function in the small interval $(0,\theta_{k-1})$, its differential can be calculated as

$$P_k'(\theta) = \sum_{j=1}^{k-1}\left(\frac{-2\gamma_j\gamma_j'}{2\sqrt{1-\gamma_j^2}}\right)\frac{P_k}{1+\sqrt{1-\gamma_j^2}},$$

which tends to zero as $\theta \to 0^+$. Then according to the definition of $\gamma_k(\theta)$, by L'Hôpital's rule, we have

$$\lim_{\theta\to 0^+}\gamma_k(\theta) = \lim_{\theta\to 0^+}(1+\epsilon)\frac{\sin\theta - \frac{-\sin\theta P_k(\theta)+(1+\cos\theta)P_k'(\theta)}{2^{k-1}}}{2^{2-k}\cos(\theta)} = 0.$$

So $\forall n \in \mathbb{N}$, we can find a $\theta_n \in (0,1)$ such that $0 < \gamma_1(\theta) < \gamma_2(\theta) < \cdots < \gamma_n(\theta) < 1$ for all $\theta \in (0,\theta_n)$.

APPENDIX F: THE PROOF OF REMARK 3

Alice's POVMs are defined by

$$A_{0|0} = \frac{\mathbb{I}+\sigma_3}{2}, \quad A_{0|1} = \frac{\mathbb{I}+\sigma_1}{2}. \tag{F1}$$

Bob's POVMs are defined by

$$B_{0|0} = \frac{\mathbb{I}-\sigma_3}{2}, \quad B_{0|1} = \frac{\mathbb{I}+\sigma_1}{2}. \tag{F2}$$

For each $k = 1,2,\dots,n$, Charlie$^{(k)}$'s POVMs are defined by

$$C_{0|0}^{(k)} = \frac{\mathbb{I}+\sigma_3}{2}, \quad C_{0|1}^{(k)} = \frac{\mathbb{I}+\gamma_k\sigma_1}{2}. \tag{F3}$$

Moreover, the corresponding NS inequality is

$$\langle X_1Y_1\rangle + \langle Y_0Z_0\rangle + \langle X_1Z_1\rangle + \langle X_0Y_0Z_0\rangle - \langle X_1Y_0Z_1\rangle \leqslant 3. \tag{F4}$$

With these measurements and the initial state $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, we get

$$\mathbf{I}_{\rm NS}^{(k)} = \frac{2}{3} + \frac{2^{3-k}}{3}\left[\gamma_k + \prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)\right]. \tag{F5}$$

Therefore, we have

$$\mathbf{I}_{\rm NS}^{(k)} > 2 \Leftrightarrow \gamma_k > \frac{7}{2^{3-k}} - \prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right). \tag{F6}$$

Next, we can define $\{\gamma_k\}$ for some fixed $\epsilon > 0$,

$$\gamma_k = \begin{cases} \frac{3}{4}(1+\epsilon), & k=1, \\ (1+\epsilon)\left(\frac{7}{2^{3-k}} - P_k\right), & 0 \leqslant \gamma_{k-1} \leqslant 1, \\ \infty, & \text{otherwise,} \end{cases} \tag{F7}$$

where $P_k = \prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})$.

Then we get

$$\frac{\gamma_k}{\gamma_{k-1}} > 2 \Leftrightarrow 0 < \gamma_{k-1} \leqslant 1. \tag{F8}$$

Here $\gamma_1 = \frac{3}{4}(1+\epsilon)$; then $\gamma_2 > 2\gamma_1 = \frac{3}{2}(1+\epsilon) > 1$. So in our setting, at most one Charlie can demonstrate genuinely nonsignal nonlocality through the violation of the NS inequality with a single Alice and Bob.

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