Majorana's stellar representation of single-particle reduced density matrix for completely symmetric states

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(Received 26 July 2022; accepted 11 May 2023; published 30 May 2023)

The completely symmetric states play an essential role in quantum physics. In this paper, we calculate the reduced density matrix (RDM) for a single particle of the completely symmetric system coupled by N spin-$\frac{1}{2}$ particles, because it helps to investigate the evolution of expectation value for the observable and to calculate the entanglement between the subsystems. Furthermore, we use Majorana's stellar representation (MSR) to represent the results because it provides an intuitive geometric perspective to comprehend the quantum states in the high-dimensional Hilbert space with distributions and trajectories of the Majorana stars on a Bloch sphere. With the operation properties of the generalized many-body anticommutator, we get the general MSR form of a single-qubit RDM. As the application and verification, we calculate the single-qubit RDM for the Dicke states with the results. Similarly, we further solve the RDM of the spin-$\frac{N}{2}$ state in a uniform magnetic field and study the systems with symmetric structures on the Bloch sphere. The results exhibit the relations between the composite systems and the subsystems, and provide a new idea for the numerical solution of multiqubit systems.

DOI: 10.1103/PhysRevA.107.052219

I. INTRODUCTION

The completely symmetric state is fundamental in quantum physics since all elementary particles are either bosons or fermions indistinguishable from one another in the same class. As a simple but representative example, $N = 2J$ spin-$\frac{1}{2}$ particles (such as electrons, protons, or neutrons) can form a spin-J particle, which is a composite boson system described by the completely symmetric state. This model can be further extended when replacing the spin-$\frac{1}{2}$ particles to arbitrary two-level systems or the so-called pseudospins. Over the past few decades, the model has gotten lots of attention since every spin-$\frac{1}{2}$ particle (or pseudospin) can be seen as a qubit in the field of quantum information. Several recent experiments have been performed on this kind of system [1–7]. However, these experiments all focus on few-qubit systems. When N increases, the composite system is not only hard to build in the experiment but also hard to process in numerical simulation due to the exponential growth of the dimensional of the Hilbert space [8–10]. Because the space where the whole system is located is 2^N dimensional, many computing resources will be consumed during numerical calculation, even beyond the scope of computing power. In this case, it is necessary to find further analytical solutions to this model.

The traditional way to describe quantum states is using the state vectors of Hilbert space. In 1927, the density matrix was introduced to describe the statistical concepts in quantum mechanics by von Neumann [11,12]. As a generalization of the state vectors, the density matrix is widely used since it can represent not only pure states but also mixed states.

However, when it comes to a composite system, we often focus on the observables in the subsystems. At this time, the description of the measurements should use the reduced density matrix (RDM), which is obtained by partially tracing the density matrix over the unmeasured subsystems. The RDM was introduced by Paul Dirac in 1930 when dealing with atoms involving many electrons [13]. It is widely used in various fields, including laser physics [14,15], quantum information [16–21], and quantum chemistry [22]. In the field of quantum information, it plays an important role in calculating the entanglement between two partitions of the system, which is quantified by the von Neumann entropy of the RDM [18,19], and so does it for the generalized entanglement in a many-body system [20,23–25]. For those occasions, this article studies the completely symmetric states by focusing on the RDM for a single particle.

As we all know, the evolution of a two-level quantum system can be described by the trajectory of a point on the Bloch sphere. Similarly, in 1932, Italian theoretical physicist Ettore Majorana proposed a method to describe the high-dimensional system with multiple points called the stars on the Bloch sphere, that is, Majorana's stellar representation (MSR) [26], which can represent the superposition of multiple quantum states of particles on the Bloch sphere. Majorana believes that in the case of quantum spin, the phase space is defined by the coordinates of $2J$ Majorana stars on the Bloch sphere. We can easily visualize the quantum space by diffusing the

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stars on the Bloch sphere. The MSR builds a bridge between the high-dimensional Hilbert space and the two-dimensional Bloch sphere. So it is widely used in many fields, such as spin Bose gas [27–33], the Lipkin-Meshkov-Glick model [34,35], quantum entanglement [36–45], Bose-Einstein condensation [27,30,33,46–48], and geometric phase [45,49–53]. The MSR and recent related applications show that the evolution of a high-spin state can be visually displayed through loops of the Majorana stars on the Bloch sphere. For high-spin condensates, spin-orbit coupling drives the Majorana stars to move periodically on the Bloch sphere, forming the so-called "Majorana spin helix." In addition to the above research, the arrangements and movements of Majorana stars have become a powerful tool for studying physical problems related to symmetry, such as the classification of entanglement classes [54]. The pair correlations between Majorana stars are naturally related to the quantum entanglement of particles [52]. In this respect, it provides an intuitive way of studying the measurement and classification of multiparticle entanglement of N particles. The topological structure of the trajectory of Majorana stars is closely related to the parity of the system, which is determined by the properties of Bloch states at two high-symmetry points [53]. In this article, the MSR of the system is clear and intuitive when we study the high-dimensional completely symmetric state coupled by N qubits. We use MSR to represent RDM because it helps to simplify the calculation of the RDM by visualizing the quantum states of the systems as the corresponding configurations of the Majorana stars on the Bloch sphere, thus simplifying the calculation of the expectation values of the observables in the subsystem and von Neumann entropy. In addition, by studying the dynamic evolution of Majorana stars on the Bloch sphere driven by a nonlinear Hamiltonian, we can calculate the values of the RDM in the evolution process, which can well characterize the dynamic evolution of von Neumann entropy, that is, the entanglement between subsystems.

Specifically, for an arbitrary single qubit, the normalized state vector can be expressed as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{1}$$

and the corresponding density matrix is

$$\rho = \tfrac{1}{2}(\mathbb{I} + \boldsymbol{u}\cdot\boldsymbol{\sigma}), \tag{2}$$

where $\mathbb{I}$ is the identity matrix and the coefficient vector $\boldsymbol{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is called Bloch vector. Then for the completely symmetric state coupled by N spin-$\frac{1}{2}$ particles (or pseudospin), if the N normalized states of subsystems are given as $|\psi_i\rangle$, $i = 1, 2, \ldots, N$, the wave function of the system can be expressed as

$$|\Psi_{(N)}\rangle = \frac{1}{\mathcal{N}_N}\sum_{P\in S_N}\bigotimes_{i=1}^{N}|\psi_{P(i)}\rangle, \tag{3}$$

and the corresponding density matrix is

$$\rho_{(N)} = \frac{1}{\mathcal{N}_N^2}\sum_{P,\tilde{P}\in S_N}\bigotimes_{i=1}^{N}|\psi_{P(i)}\rangle\langle\psi_{\tilde{P}(i)}|, \tag{4}$$

where $\mathcal{N}_N$ is the normalization constant, S_N denotes the Nth permutation group, $\sum_{P\in S_N}$ sums all the $N!$ terms with $P = \begin{pmatrix} 1 & 2 & \cdots & N \\ P(1) & P(2) & \cdots & P(N)\end{pmatrix} \in S_N$ [55], and $\sum_{P,\tilde{P}\in S_N}$ sums all the $N! \times N!$ terms with $P, \tilde{P} \in S_N$. Similarly, every state of subsystem $|\psi_i\rangle$ is corresponding with a Bloch vector $\boldsymbol{u}_i$. Just like Eq. (2), we can use the vectors $\{\boldsymbol{u}_i\}$ to characterize the composite system, i.e., give the MSR of the system.

Furthermore, since the qubits are identical, the expectation value of the observable in the composite system can be represented by the expectation value of the corresponding observable in the subsystems as $\langle\boldsymbol{F}_{\text{total}}\rangle = N\langle\boldsymbol{F}\rangle$. For the observable $\boldsymbol{F}$ in the subsystem of a single qubit, the expectation value is

$$\langle\boldsymbol{F}\rangle = \mathrm{Tr}(\boldsymbol{F}\varrho^{[N]}), \tag{5}$$

where $\varrho^{[N]}$ is the RDM of a single qubit and can be obtained by partially tracing the density matrix $\rho_{(N)}$ from the second subsystem to the Nth subsystem as

$$\begin{aligned}\varrho^{[N]} &= \frac{1}{\mathcal{N}_N^2}\sum_{P,\tilde{P}\in S_N}|\psi_{P(1)}\rangle\langle\psi_{\tilde{P}(1)}|\prod_{i=2}^{N}\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle \\ &= \frac{1}{\mathcal{N}_N^2}\sum_{P,\tilde{P}\in S_N}|\psi_{P(1)}\rangle\prod_{i=1}^{N-1}\langle\psi_{\tilde{P}(i)}|\psi_{P(i+1)}\rangle\langle\psi_{\tilde{P}(N)}|. \end{aligned}\tag{6}$$

Obviously, the same result can be obtained by tracing arbitrary $N-1$ subsystems in $\rho_{(N)}$, which presents the complete symmetry of the system in form. So the result of $\varrho^{[N]}$ expressed by Eq. (6) is the RDM of an arbitrary qubit in our system.

In this article, we study the MSR of the single-qubit RDM on the basis of Eq. (6). However, the normalization constant $\mathcal{N}_N$ has been obtained in our previous work [56], so we just need to find the MSR of the summation part of Eq. (6), and we can mark it as

$$\tilde{\varrho}^{[N]} = \sum_{P,\tilde{P}\in S_N}|\psi_{P(1)}\rangle\prod_{i=1}^{N-1}\langle\psi_{\tilde{P}(i)}|\psi_{P(i+1)}\rangle\langle\psi_{\tilde{P}(N)}|, \tag{7}$$

which can be treated as the unnormalized RDM. But on the other hand, when we get the MSR of $\tilde{\varrho}^{[N]}$, since $\tilde{\varrho}^{[N]} = \mathcal{N}_N^2\varrho^{[N]}$ and the RDM is normalized as $\mathrm{Tr}\varrho^{[N]} = 1$, the normalization constant $\mathcal{N}_N$ can be obtained as

$$\mathcal{N}_N = \sqrt{\mathrm{Tr}\tilde{\varrho}^{[N]}}. \tag{8}$$

In this way, we reproduce the MSR of the normalization constant $\mathcal{N}_N$ in this article.

The structure of this paper is as follows. In Sec. I (this section), we introduce the basic concepts and the problem we study. In Sec. II, we introduce the mathematical correspondence and physical significance of MSR. In Sec. III, we first take some few-qubit states as the examples and then calculate the MSR of the RDM for the composite system of N-qubit with the properties of the generalized many-body anticommutators. Furthermore, we calculate the RDM of the Dicke state in MSR to check the correctness of the results. In Sec. IV, we use the results to get the RDM of the spin-$\frac{N}{2}$ state in a uniform magnetic field and some symmetric structures on the Bloch sphere, respectively. In Sec. V, we provide a summary and outlook.

II. MAJORANA'S STELLAR REPRESENTATION

A. Mathematical correspondence of MSR

In the Introduction, we discussed the correspondence between the quantum states of two-level systems and Bloch vectors, while Ettore Majorana proposed MSR based on Bloch spheres and Bloch vectors. Next, let us introduce the mathematical correspondence of MSR.

In introducing the Bloch vector ($|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \alpha|0\rangle + \beta|1\rangle$), we know that the coefficient ratio of the two states is $\zeta = \frac{\beta}{\alpha} = \tan\frac{\theta}{2}e^{i\phi}$ and that the coefficients α, β, and ζ can form the characteristic equation $\alpha\zeta - \beta = 0$. The coefficients α and β of the original wave function correspond to the coefficient of the first-order term and constant term of the linear equation with one unknown with ζ as the variable. On the contrary, if the value of ζ is known, combined with the normalization relation, we can also get the probabilities $P_\alpha = \cos^2\frac{\theta}{2} = \frac{1}{1+|\zeta|^2}$ and $P_\beta = \sin^2\frac{\theta}{2} = \frac{|\zeta|^2}{1+|\zeta|^2}$ of the wave function onto each state after the collapse.

Majorana proposed that for a higher-dimensional space, such as $N+1$-dimensional wave function $|\Psi\rangle = \sum_{i=0}^{N} C_i|i\rangle$, we can also use an equation of degree N with one unknown to describe it, that is,

$$\sum_{i=0}^{N} a_i\zeta^i = 0, \tag{9}$$

where the coefficient in Eq. (9) is $a_i = f(i)C_i$, and the setting of $f(i)$ is related to the specific model system. The simplest one is that when we choose $f(i) = 1$, we can get $a_i = C_i$. For the N roots $\{\zeta_i\}$ of Eq. (9), we can also find N groups of (θ_i, ϕ_i) such that $\zeta_i = \tan\frac{\theta_i}{2}e^{i\phi_i}$. Similarly, if $\{(\theta_i, \phi_i)\}$, i.e., $\{\zeta_i\}$, is known, we can also deduce $\{a_i\}$ and $\{C_i\}$, and these N groups of (θ_i, ϕ_i) are called the Majorana stars used to describe the state $|\Psi\rangle$.

B. Physical significance of MSR

We discussed the mathematical correspondence of MSR above, and now we will explain the physical significance of this representation.

Like the Schwinger representation, MSR is also proposed to describe systems coupled by multiple spin-$\frac{1}{2}$ particles.

For a boson system with a total spin of J, the quantum state can be represented as

$$|\Psi\rangle^J = \sum_{M=-J}^{J} C_M|J, M\rangle, \tag{10}$$

and when expanded using the Schwinger representation, it can be described as

$$|\Psi_{(N)}\rangle = \sum_{M=-\frac{N}{2}}^{\frac{N}{2}} C_M \frac{a^{\dagger(\frac{N}{2}+M)}b^{\dagger(\frac{N}{2}-M)}}{\sqrt{\left(\frac{N}{2}+M\right)!\left(\frac{N}{2}-M\right)!}}|\emptyset\rangle, \tag{11}$$

where $N = 2J$. On the other hand, since the boson system with total spin J is coupled by $N = 2J$ spin-$\frac{1}{2}$ identical particles, its wave function can be expressed as Eq. (3). For each spin-$\frac{1}{2}$ quantum state, we have $|\psi_i\rangle = \cos\frac{\theta_i}{2}|0\rangle + e^{i\phi_i}\sin\frac{\theta_i}{2}|1\rangle$. We introduce operators $\hat{a}^\dagger$ and $\hat{b}^\dagger$ to act on the vacuum state to generate $\hat{a}^\dagger|\emptyset\rangle = |0\rangle$ and $\hat{b}^\dagger|\emptyset\rangle = |1\rangle$, then

$$\begin{aligned}|\psi_i\rangle &= \left(\cos\frac{\theta_i}{2}\hat{a}^\dagger + e^{i\phi_i}\sin\frac{\theta_i}{2}\hat{b}^\dagger\right)|\emptyset\rangle \\ &= \hat{a}^\dagger_{\psi_i}|\emptyset\rangle,\end{aligned} \tag{12}$$

where $\hat{a}^\dagger_{\psi_i} = \cos\frac{\theta_i}{2}\hat{a}^\dagger + e^{i\phi_i}\sin\frac{\theta_i}{2}\hat{b}^\dagger$, and $|\Psi_{(N)}\rangle$ can be written as

$$|\Psi_{(N)}\rangle = \frac{\sqrt{N!}}{\mathcal{N}_N}\prod_{i=1}^{N}\hat{a}^\dagger_{\psi_i}|\emptyset\rangle. \tag{13}$$

The characteristic equation corresponding to the above wave function is

$$\sum_{M=-J}^{J}\frac{(-1)^M C_M}{\sqrt{(J-M)!(J+M)!}}\zeta^{J+M} = 0. \tag{14}$$

Let $i = J - M$, then

$$\sum_{i=0}^{2J}\frac{(-1)^i C_{J-i}}{\sqrt{(2J-i)!i!}}\zeta^{2J-i} = 0, \tag{15}$$

and then we get each complex solution $\zeta_i = \tan\frac{\theta_i}{2}e^{i\phi_i}$, corresponding to Majorana star (θ_i, ϕ_i).

III. REDUCED DENSITY MATRIX IN MSR

A. RDM for few-qubit states in MSR

To introduce and illustrate our work, we first study the few-qubit systems and expound the systems coupled by two and three qubits for examples.

When $N = 2$, the permutation group S_2 has two elements $P = \binom{1\ \ 2}{1\ \ 2}$ and $P = \binom{1\ \ 2}{2\ \ 1}$, that is, there are two index sequences as $P(1)P(2) \in \{12, 21\}$, and the corresponding wave function of the system is expressed as

$$|\Psi_{(2)}\rangle = \frac{1}{\mathcal{N}_2}(|\psi_1\rangle\otimes|\psi_2\rangle + |\psi_2\rangle\otimes|\psi_1\rangle). \tag{16}$$

Accordingly, the density matrix $\rho_{(2)}$ has four items, which can be expanded to

$$\begin{aligned}\rho_{(2)} = \frac{1}{\mathcal{N}_2^2}(&|\psi_1\rangle\langle\psi_1|\otimes|\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_2|\otimes|\psi_2\rangle\langle\psi_1| \\ &+ |\psi_2\rangle\langle\psi_1|\otimes|\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_2|\otimes|\psi_1\rangle\langle\psi_1|).\end{aligned} \tag{17}$$

Next, after tracing the part of the second subsystem, i.e., the factors behind the direct-product notations, we can get the normalized RDM $\varrho^{[2]}$. Then, by ignoring the normalization constant, the unnormalized RDM as Eq. (7) for a single-qubit in the two-qubit system is expressed as

$$\begin{aligned}\tilde{\varrho}^{[2]} = &|\psi_1\rangle\langle\psi_2|\psi_2\rangle\langle\psi_1| + |\psi_1\rangle\langle\psi_1|\psi_2\rangle\langle\psi_2| \\ &+ |\psi_2\rangle\langle\psi_2|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_1|\psi_1\rangle\langle\psi_2|.\end{aligned} \tag{18}$$

Further, introducing the density matrix representation of the subsystem $\rho_i = |\psi_i\rangle\langle\psi_i|$ and the normalization condition

$\langle\psi_i|\psi_i\rangle = 1$, Eq. (18) can be simplified as

$$\tilde{\varrho}^{[2]} = \sum_{i=1}^{2} \rho_i + \{\rho_1, \rho_2\}. \tag{19}$$

Then, we substitute the MSR of the subsystems $\rho_i = \frac{1}{2}(\mathbb{I} + \boldsymbol{u}_i \cdot \boldsymbol{\sigma})$, $i = 1, 2$ into Eq. (19) and with the property

$$(\boldsymbol{u}_i \cdot \boldsymbol{\sigma})(\boldsymbol{u}_j \cdot \boldsymbol{\sigma}) = \boldsymbol{u}_i \cdot \boldsymbol{u}_j + i(\boldsymbol{u}_i \times \boldsymbol{u}_j) \cdot \boldsymbol{\sigma}, \tag{20}$$

we can get the MSR of $\tilde{\varrho}^{[2]}$ as

$$\begin{aligned}\tilde{\varrho}^{[2]} &= \left(\frac{3}{2} + \frac{1}{2}\boldsymbol{u}_1 \cdot \boldsymbol{u}_2\right)\mathbb{I} + \sum_{i=1}^{2} \boldsymbol{u}_i \cdot \boldsymbol{\sigma} \\ &= \frac{3}{2} + \frac{1}{2}\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 + \sum_{i=1}^{2} \boldsymbol{u}_i \cdot \boldsymbol{\sigma}.\end{aligned} \tag{21}$$

Here we abbreviate identity matrix $\mathbb{I}$ as 1 to simplify the expression, and we will still use this abbreviation in the calculation processes of the following content. Further, with Eq. (8), we can get the normalization constant as

$$\mathcal{N}_2 = \sqrt{3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}. \tag{22}$$

With the results, the MSR of the normalized RDM $\varrho^{[2]}$ is written as

$$\varrho^{[2]} = \frac{\mathbb{I}}{2} + \frac{\boldsymbol{u}_1 + \boldsymbol{u}_2}{3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2} \cdot \boldsymbol{\sigma}. \tag{23}$$

Next, in the case of $N = 3$, the permutation group S_3 has $3! = 6$ elements. According to Eq. (7), there are $3! \times 3! = 36$ items in the unnormalized RDM, which can be written as

$$\tilde{\varrho}^{[3]} = \sum_{P, \tilde{P} \in S_3} |\psi_{P(1)}\rangle\langle\psi_{\tilde{P}(1)}|\psi_{P(2)}\rangle\langle\psi_{\tilde{P}(2)}|\psi_{P(3)}\rangle\langle\psi_{\tilde{P}(3)}|. \tag{24}$$

Following the previous example, the next step is to find the RDM representation in the form of subsystem density matrix ρ_i, just like Eq. (19). To get the expansion, we first fix the permutation P as $P(1)P(2)P(3) = 123$, and then go through all the elements in $\tilde{P}$ as $\tilde{P}(1)\tilde{P}(2)\tilde{P}(3) \in \{123, 231, 312, 132, 321, 213\}$. With these combinations of the indices, we can get each term as $\rho_1\rho_2\rho_3$, ρ_1, $\rho_1\rho_3\rho_2$, $\rho_1\rho_2$, $\rho_1\mathrm{Tr}(\rho_2\rho_3)$, and $\rho_1\rho_3$, respectively. Further, we can get other terms according to the symmetry of the system and then get

$$\begin{aligned}\tilde{\varrho}^{[3]} = 2\Bigg[&\sum_{i=1}^{3} \rho_i + \{\rho_1, \rho_2\} + \{\rho_1, \rho_3\} + \{\rho_2, \rho_3\} \\ &+ \{\rho_1, \rho_2, \rho_3\} + \frac{1}{2}(\rho_1\mathrm{Tr}\{\rho_2, \rho_3\} + \rho_2\mathrm{Tr}\{\rho_1, \rho_3\} \\ &+ \rho_3\mathrm{Tr}\{\rho_1, \rho_2\})\Bigg],\end{aligned} \tag{25}$$

where $\{\rho_1, \rho_2, \rho_3\} = \sum_{P \in S_3} \rho_{P(1)}\rho_{P(2)}\rho_{P(3)}$ operates as the generalized many-body anticommutator introduced in our previous article [56].

For further calculation, we review the MSR of the anticommutator of the density matrices for qubits, and it tells [56]

$$\begin{aligned}&\{\rho_1, \rho_2, \dots, \rho_N\} \\ &= \frac{N!}{2^N}\left[\sum_{k=0}^{[\frac{N}{2}]} \frac{D_{T_N}^{(k)}}{(2k-1)!!} + \sum_{k=0}^{[\frac{N-1}{2}]} \frac{\boldsymbol{V}_{T_N}^{(k)}}{(2k+1)!!} \cdot \boldsymbol{\sigma}\right],\end{aligned} \tag{26}$$

where

$$D_{T_N}^{(k)} = \begin{cases} \sum_{T_N''} (\boldsymbol{u}_{i_1} \cdot \boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2} \cdot \boldsymbol{u}_{j_2}) \dots (\boldsymbol{u}_{i_k} \cdot \boldsymbol{u}_{j_k}), & k > 0, \\ 1, & k = 0, \end{cases} \tag{27}$$

and

$$\boldsymbol{V}_{T_N}^{(k)} = \sum_{T_N''} (\boldsymbol{u}_{i_1} \cdot \boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2} \cdot \boldsymbol{u}_{j_2}) \dots (\boldsymbol{u}_{i_k} \cdot \boldsymbol{u}_{j_k})\boldsymbol{u}_{j_{k+1}}. \tag{28}$$

In $D_{T_N}^{(k)}$ and $\boldsymbol{V}_{T_N}^{(k)}$ above, $i_1, j_1, \dots, i_k, j_k, j_{k+1} \in T_N$ are different indices in the index set $T_N = \{1, 2, \dots, N\}$, and $\sum_{T_N''}$ represents the sum of all cases in which the ordering relations $i_1 < i_2 < \dots < i_k$ and $i_m < j_m$ are satisfied.

According to Eq. (26), we can get some further results as

$$\{\rho_i, \rho_j\} = \tfrac{1}{2}[1 + \boldsymbol{u}_i \cdot \boldsymbol{u}_j + (\boldsymbol{u}_i + \boldsymbol{u}_j) \cdot \boldsymbol{\sigma}], \tag{29}$$

$$\tfrac{1}{2}\rho_k\mathrm{Tr}\{\rho_i, \rho_j\} = \tfrac{1}{4}(1 + \boldsymbol{u}_i \cdot \boldsymbol{u}_j)(1 + \boldsymbol{u}_k \cdot \boldsymbol{\sigma}), \tag{30}$$

and

$$\{\rho_1, \rho_2, \rho_3\} = \tfrac{3}{4}\left[1 + D_{T_3}^{(1)} + \left(\boldsymbol{V}_{T_3}^{(0)} + \tfrac{1}{3}\boldsymbol{V}_{T_3}^{(1)}\right) \cdot \boldsymbol{\sigma}\right], \tag{31}$$

where $D_{T_3}^{(1)} = \boldsymbol{u}_1 \cdot \boldsymbol{u}_2 + \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_3$, $\boldsymbol{V}_{T_3}^{(0)} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$ and $\boldsymbol{V}_{T_3}^{(1)} = (\boldsymbol{u}_1 \cdot \boldsymbol{u}_2)\boldsymbol{u}_3 + (\boldsymbol{u}_2 \cdot \boldsymbol{u}_3)\boldsymbol{u}_1 + (\boldsymbol{u}_1 \cdot \boldsymbol{u}_3)\boldsymbol{u}_2$.

Substituting Eqs. (29)–(31) into Eq. (25), we can get the MSR of the unnormalized RDM $\tilde{\varrho}^{[3]}$ as

$$\tilde{\varrho}^{[3]} = 9 + 3D_{T_3}^{(1)} + \left(5\boldsymbol{V}_{T_3}^{(0)} + \boldsymbol{V}_{T_3}^{(1)}\right) \cdot \boldsymbol{\sigma}, \tag{32}$$

and substituting it into Eq. (8), the normalization constant is obtained as

$$\mathcal{N}_3 = \sqrt{18 + 6D_{T_3}^{(1)}}. \tag{33}$$

With the results, the MSR of the normalized RDM $\varrho^{[3]}$ is written as

$$\varrho^{[3]} = \frac{\mathbb{I}}{2} + \frac{\left(5\boldsymbol{V}_{T_3}^{(0)} + \boldsymbol{V}_{T_3}^{(1)}\right)}{18 + 6D_{T_3}^{(1)}} \cdot \boldsymbol{\sigma}. \tag{34}$$

Similarly, for four qubits, we can get the unnormalized RDM in the form of subsystem density matrix ρ_i as

$$\begin{aligned}\tilde{\varrho}^{[4]} = 6\Bigg(&\sum_{i=1}^{4}\rho_i + \sum_{i_1i_2\in\{12,13,14,23,24,34\}}\{\rho_{i_1},\rho_{i_2}\} + \sum_{i_1i_2i_3\in\{123,124,134,234\}}\{\rho_{i_1},\rho_{i_2},\rho_{i_3}\} + \{\rho_1,\rho_2,\rho_3,\rho_4\}\\ &+\frac{1}{2}\sum_{\substack{i_1i_2i_3\in\{123,124,134,213,214,234,\\312,314,324,412,413,423\}}}\rho_{i_1}\mathrm{Tr}\{\rho_{i_2},\rho_{i_3}\} + \sum_{\substack{i_1i_2i_3i_4\in\{1234,1324,1423,\\2314,2413,3412\}}}\{\rho_{i_1},\rho_{i_2}\}\mathrm{Tr}\{\rho_{i_3},\rho_{i_4}\}\\ &+\frac{1}{3}\sum_{i_1i_2i_3i_4\in\{1234,2134,3124,4123\}}\rho_{i_1}\mathrm{Tr}\{\rho_{i_2},\rho_{i_3},\rho_{i_4}\}\Bigg).\end{aligned} \tag{35}$$

Then with the result in Eq. (26), the MSR of the unnormalized RDM $\tilde{\varrho}^{[4]}$ is obtained as

$$\tilde{\varrho}^{[4]} = 6\big(15 + 5D_{T_4}^{(1)} + D_{T_4}^{(2)}\big) + 9\big(5\boldsymbol{V}_{T_4}^{(0)} + \boldsymbol{V}_{T_4}^{(1)}\big)\cdot\boldsymbol{\sigma}. \tag{36}$$

By substituting it into Eq. (8), the normalization constant is obtained as

$$\mathcal{N}_4 = \sqrt{12\big(15 + 5D_{T_4}^{(1)} + D_{T_4}^{(2)}\big)}, \tag{37}$$

and the MSR of the normalized RDM $\varrho^{[4]}$ is written as

$$\varrho^{[4]} = \frac{\mathbb{I}}{2} + \frac{3\big(5\boldsymbol{V}_{T_4}^{(0)} + \boldsymbol{V}_{T_4}^{(1)}\big)}{4\big(15 + 5D_{T_4}^{(1)} + D_{T_4}^{(2)}\big)}\cdot\boldsymbol{\sigma}. \tag{38}$$

Then when $N = 5$ and $N = 6$, we can get results in the same way, and they are

$$\varrho^{[5]} = \frac{\mathbb{I}}{2} + \frac{35\boldsymbol{V}_{T_5}^{(0)} + 7\boldsymbol{V}_{T_5}^{(1)} + \boldsymbol{V}_{T_5}^{(2)}}{10\big(15 + 5D_{T_5}^{(1)} + D_{T_5}^{(2)}\big)}\cdot\boldsymbol{\sigma}, \tag{39}$$

and

$$\varrho^{[6]} = \frac{\mathbb{I}}{2} + \frac{2\big(35\boldsymbol{V}_{T_6}^{(0)} + 7\boldsymbol{V}_{T_6}^{(1)} + \boldsymbol{V}_{T_6}^{(2)}\big)}{3\big(105 + 35D_{T_6}^{(1)} + 7D_{T_6}^{(2)} + D_{T_6}^{(3)}\big)}\cdot\boldsymbol{\sigma}. \tag{40}$$

B. RDM for N-qubit states in MSR

The above examples of few-qubit systems help us familiarize the procedure to calculate the MSR of the RDM. Next, let us explore the MSR of the RDM for the N-qubit system.

Before that, we need to review the MSR of the normalization constant $\mathcal{N}_N$, which is [56]

$$\mathcal{N}_N^2 = \sum_{P,\tilde{P}\in S_N}\prod_{i=1}^{N}\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle \tag{41}$$

$$= \frac{N!(N+1)!}{2^N}\sum_{J=0}^{[\frac{N}{2}]}\frac{1}{(2J+1)!!}D_{T_N}^{(J)}. \tag{42}$$

And for every fixed pair of P and $\tilde{P}$, the term $\prod_{i=1}^{N}\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle$ can be represented in the form of subsystem density matrix ρ_i as

$$\prod_{i=1}^{N}\langle\psi_{\tilde{P}(i)}|\psi_{P(i)}\rangle = \prod_{j=1}^{q}\mathrm{Tr}\big(\rho_{i_1^{(j)}}\rho_{i_2^{(j)}}\dots\rho_{i_{l_j}^{(j)}}\big), \tag{43}$$

where $1 \leqslant q \leqslant N$.

Similarly, the terms of the unnormalized RDM in Eq. (7) can be represented in the form of the subsystem density matrix ρ_i as

$$|\psi_{P(1)}\rangle\prod_{i=1}^{N-1}\langle\psi_{\tilde{P}(i)}|\psi_{P(i+1)}\rangle\langle\psi_{\tilde{P}(N)}|$$

$$= \big(\rho_{i_1^{(0)}}\rho_{i_2^{(0)}}\dots\rho_{i_k^{(0)}}\big)\prod_{j=1}^{q}\mathrm{Tr}\big(\rho_{i_1^{(j)}}\rho_{i_2^{(j)}}\dots\rho_{i_{l_j}^{(j)}}\big), \tag{44}$$

where $0 \leqslant q \leqslant N-1$, and when $q = 0$, we define the result as $\rho_{i_1^{(0)}}\rho_{i_2^{(0)}}\dots\rho_{i_N^{(0)}}$.

Then we divide the index set T_N into two parts. The first part is $I_0 = \{i_1^{(0)},\dots,i_k^{(0)}\}$, which corresponds to the first factor in Eq. (44). The second part is $T_N - I_0$, corresponding to the following traced factors, and we can define this part as a constant C, which satisfies

$$C = \prod_{i=1}^{N-k}\langle\psi_{\tilde{P}_i}|\psi_{P_i}\rangle, \tag{45}$$

and $\{P_1,\dots,P_{N-k}\} = \{\tilde{P}_1,\dots,\tilde{P}_{N-k}\} = T_N - I_0$. Then to Eq. (7), we sum all pairs of P and $\tilde{P}$ that have the same factor $\rho_{i_1^{(0)}}\rho_{i_2^{(0)}}\dots\rho_{i_k^{(0)}}$. Since there are totally $\frac{(N-1)!}{(N-k)!}$ of them, the sum can be written as

$$\begin{aligned}&C_{I_0}\big(\rho_{i_1^{(0)}}\rho_{i_2^{(0)}}\dots\rho_{i_k^{(0)}}\big)\\ &\quad= \frac{(N-1)!}{(N-k)!}\Bigg(\sum_{P,\tilde{P}}C\Bigg)\big(\rho_{i_1^{(0)}}\rho_{i_2^{(0)}}\dots\rho_{i_k^{(0)}}\big).\end{aligned} \tag{46}$$

According to Eqs. (41), (42), and (45), we get

$$\sum_{P,\tilde{P}}C = \frac{(N-k)!(N-k+1)!}{2^{N-k}}\sum_{i=0}^{[\frac{N-k}{2}]}\frac{1}{(2i+1)!!}D_{T_N-I_0}^{(i)}, \tag{47}$$

and then

$$\begin{aligned}C_{I_0} &= \frac{(N-1)!}{(N-k)!}\Bigg(\sum_{P,\tilde{P}}C\Bigg)\\ &= \frac{(N-1)!(N-k+1)!}{2^{N-k}}\sum_{i=0}^{[\frac{N-k}{2}]}\frac{1}{(2i+1)!!}D_{T_N-I_0}^{(i)}.\end{aligned} \tag{48}$$

Next, the sum of the terms in Eq. (7) for fixed C_{I_0} is

$$S_{I_0} = C_{I_0}\left\{\rho_{i_1^{(0)}}, \rho_{i_2^{(0)}}, \dots, \rho_{i_k^{(0)}}\right\}. \tag{49}$$

As $\sum_{I_0}$ goes through all cases of I_0, the unnormalized RDM is obtained as

$$\tilde{\varrho}^{[N]} = \sum_{I_0} S_{I_0}. \tag{50}$$

The calculation of $\sum_{I_0} S_{I_0}$ is in Appendix A, and it tells

$$\tilde{\varrho}^{[N]} = \frac{(N-1)!(N+1)!}{2^{N+1}}\left[N\sum_{J=0}^{[\frac{N}{2}]}\frac{1}{(2J+1)!!}D_{T_N}^{(J)} + (N+2)\sum_{J=0}^{[\frac{N-1}{2}]}\frac{1}{(2J+3)!!}\boldsymbol{V}_{T_N}^{(J)}\cdot\boldsymbol{\sigma}\right]. \tag{51}$$

Then by substituting the result into Eq. (8), the normalization constant is obtained as shown in Eq. (42), with which the final result of the single-particle RDM for the N-qubit system in MSR is

$$\varrho^{[N]} = \frac{\mathbb{I}}{2} + \left(\frac{1}{2}+\frac{1}{N}\right)\left[\sum_{J=0}^{[\frac{N}{2}]}\frac{D_{T_N}^{(J)}}{(2J+1)!!}\right]^{-1} \times \sum_{J=0}^{[\frac{N-1}{2}]}\frac{\boldsymbol{V}_{T_N}^{(J)}}{(2J+3)!!}\cdot\boldsymbol{\sigma}. \tag{52}$$

C. Verification of RDM in MSR with Dicke state

In this subsection, we take the Dicke states as an example to verify our results.

Dicke states are eigenstates of the total angular momentum, $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$, and the angular momentum component in the z direction, $\hat{J}_z$, where $\hat{J}_\alpha = \frac{1}{2}\sum_{i=1}^N \sigma_{i\alpha}$ with $\alpha = x, y, z$. It can be marked as $|J, M\rangle$, where J is the angular momentum quantum number, and M is the magnetic quantum number. Dicke states are completely symmetric states coupled by the eigenstates of σ_z, i.e., $|J, M\rangle = \frac{1}{\mathcal{N}_N}\sum_{P\in S_N}\otimes_{i=1}^N|\psi_{P(i)}\rangle$, where $|\psi_{P(i)}\rangle = |0\rangle$ or $|1\rangle$. Then after combining like terms, it is usually expressed as

$$|J, M\rangle = \frac{1}{\sqrt{C_N^K}}\sum_i P_i\left(\bigotimes_{i=1}^{K}|0\rangle\bigotimes_{i=1}^{N-K}|1\rangle\right), \tag{53}$$

where $N = 2J$, $K = J + M$ is the number of excitations (here we set $|0\rangle$ as the excited state and $|1\rangle$ as the ground state, i.e., $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$), and $\sum_i P_i(\dots)$ means the sum over all N qubits in C_N^K possible distinct ways.

As a Hermitian operator in the two-dimensional complex Hilbert space, the normalized single-particle RDM can be written as $\varrho^{[N]} = \frac{1}{2}(\mathbb{I} + \boldsymbol{u}\cdot\boldsymbol{\sigma})$. Furthermore, it can be expanded as

$$\varrho^{[N]} = \begin{bmatrix} \frac{1}{2}(1+\langle\sigma_z\rangle) & \langle\sigma_-\rangle \\ \langle\sigma_+\rangle & \frac{1}{2}(1-\langle\sigma_z\rangle)\end{bmatrix}, \tag{54}$$

where $\langle\sigma_i\rangle = \mathrm{Tr}(\varrho^{[N]}\sigma_i)$ and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. As the composite system coupled by N identical particles, we can obtain the expectation values for σ_z and $\sigma_\pm$ in arbitrary single-particle subsystems as

$$\langle\sigma_z\rangle = \frac{2\langle\hat{J}_z\rangle}{N} = \frac{2M}{N}, \tag{55}$$

and

$$\langle\sigma_\pm\rangle = \frac{\langle\hat{J}_\pm\rangle}{N} = 0, \tag{56}$$

since $\langle J, M|J, M'\rangle = \delta_{MM'}$, $\hat{J}_z|J, M\rangle = M|J, M\rangle$, and $\hat{J}_\pm|J, M\rangle = \sqrt{J(J+1) - M(M\pm 1)}|J, M\pm 1\rangle$. So in this way, the single-particle RDM for Dicke states is

$$\varrho^{[N]} = \frac{\mathbb{I}}{2} + \frac{M}{N}\sigma_z. \tag{57}$$

Next, we use the result of Eq. (52) to calculate the RDM of Dicke states. Because the quantum state of a two-level system can be expressed as Eq. (1), and the corresponding Bloch vector is $\boldsymbol{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, it is easy to get that when the particle is on the excited state $|\psi_{P(i)}\rangle = |0\rangle$, the corresponding Bloch vector is

$$\boldsymbol{u}_+ = (0, 0, 1), \tag{58}$$

and when the particle is on the ground state $|\psi_{P(i)}\rangle = |1\rangle$, the corresponding Bloch vector is

$$\boldsymbol{u}_- = (0, 0, -1). \tag{59}$$

For Dicke state $|J, M\rangle$ excited by $K = J + M$ particles, there are $J + M$ number of $\boldsymbol{u}_+$ and $J - M$ number of $\boldsymbol{u}_-$ in $N = 2J$ particles. Since the particles are identical, we might as well set $\boldsymbol{u}_i = \begin{cases}\boldsymbol{u}_-, & i = 1, \dots, J-M \\ \boldsymbol{u}_+, & i = J-M+1, \dots, N\end{cases}$.

In order to get the corresponding normalization constant $\mathcal{N}_N$ and RDM $\varrho^{[N]}$, we first calculate the values of $D_{T_N}^{(k)}$ and $\boldsymbol{V}_{T_N}^{(k)}$. For example, when $N = 5$ and $J - M = 1$, $\boldsymbol{u}_1 = \boldsymbol{u}_-$ and $\boldsymbol{u}_i = \boldsymbol{u}_+$, $i = 2, 3, 4, 5$, and to calculate $D_{T_5}^{(2)} = \sum_{i_1 j_1 i_2 j_2}(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})$, we find the term $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})$ equals 1 when there are even $\boldsymbol{u}_-$ in the term, and it equals -1 when there are odd $\boldsymbol{u}_-$. In this example, to get terms equal -1, there are C_{J-M}^1 ways to get $\boldsymbol{u}_-$ and C_{J+M}^{2k-1} ways to get $\boldsymbol{u}_+$, and after the Bloch vectors are selected, there are $\frac{(2k)!}{k!2^k} = (2k-1)!!$ ways to arrange them into $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})$. So there are $3!!C_1^1C_4^3 = 12$ terms in $D_{T_5}^{(2)}$ with the value of -1. In the same way, we can get that there are $(2k-1)!!C_{J-M}^0C_{J+M}^{2k} = 3!!C_1^0C_4^4 = 3$ terms with the value of 1.

Similarly, for arbitrary $D_{T_N}^{(k)}$, the number of the terms that equal -1 is

$$n_{-1} = (2k-1)!!\sum_{j=1,3,\dots}C_{J-M}^{j}C_{J+M}^{2k-j}, \tag{60}$$

and the number of the terms that equal 1 is

$$n_{+1} = (2k-1)!!\sum_{j=0,2,\dots}C_{J-M}^{j}C_{J+M}^{2k-j}. \tag{61}$$

In this way, we have

$$\begin{aligned} D_{T_N}^{(k)} &= n_{+1} + (-1)n_{-1} \\ &= (2k-1)!!\sum_{j=0}^{J-M}(-1)^j C_{J-M}^{j} C_{J+M}^{2k-j}. \end{aligned} \tag{62}$$

Further, we can get

$$\begin{aligned} &\sum_{k=0}^{[\frac{N}{2}]}\frac{1}{(2k+1)!!}D_{T_N}^{(k)} \\ &= \sum_{k=0}^{[J]}\frac{1}{2k+1}\sum_{j=0}^{J-M}(-1)^j C_{J-M}^{j} C_{J+M}^{2k-j} \qquad (63) \\ &= 2^{2J}\frac{(J-M)!(J+M)!}{(2J+1)!}. \qquad (64) \end{aligned}$$

The last step of the calculation is shown in Appendix B.

Similarly, in $\boldsymbol{V}_{T_N}^{(k)}$ for Dicke states, the value of the terms is $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})\boldsymbol{u}_{j_{k+1}} = (0,0,\pm1)$, and the numbers of terms are

$$\begin{aligned} n_{(0,0,-1)} &= \sum_{j=1,3,\ldots} C_{J-M}^{j} C_{J+M}^{2k+1-j}\frac{(2k+1)!}{k!2^k} \\ &= (2k+1)!!\sum_{j=1,3,\ldots} C_{J-M}^{j} C_{J+M}^{2k+1-j}, \end{aligned} \tag{65}$$

and

$$\begin{aligned} n_{(0,0,1)} &= \sum_{j=0,2,\cdots} C_{J-M}^{j} C_{J+M}^{2k+1-j}\frac{(2k+1)!}{k!2^k} \\ &= (2k+1)!!\sum_{j=0,2,\ldots} C_{J-M}^{j} C_{J+M}^{2k+1-j}, \end{aligned} \tag{66}$$

respectively. Then we have

$$\begin{aligned} \boldsymbol{V}_{T_N}^{(k)} &= n_{(0,0,1)}(0,0,1) + n_{(0,0,-1)}(0,0,-1) \\ &= (2k+1)!!\sum_{j=0}^{J-M}(-1)^j C_{J-M}^{j} C_{J+M}^{2k+1-j}(0,0,1), \end{aligned} \tag{67}$$

and get

$$\begin{aligned} &\sum_{k=0}^{[\frac{N-1}{2}]}\frac{1}{(2k+3)!!}\boldsymbol{V}_{T_N}^{(k)}\cdot\boldsymbol{\sigma} \\ &= \sum_{k=0}^{[\frac{2J-1}{2}]}\frac{1}{2k+3}\sum_{j=0}^{J-M}(-1)^j C_{J-M}^{j} C_{J+M}^{2k+1-j}\sigma_z \qquad (68) \\ &= 2^{2J}\frac{(J+M)!(J-M)!2M}{(2J+2)!}\sigma_z. \qquad (69) \end{aligned}$$

The last step of the calculation is also shown in Appendix B.

Then by substituting the results of Eqs. (64) and (69) into Eq. (51), we can get the unnormalized RDM as

$$\tilde{\varrho}^{[N]} = \frac{(N-1)!N!}{C_N^{J+M}}\left(\frac{N}{2}\mathbb{I} + M\sigma_z\right). \tag{70}$$

Then with Eq. (8), the normalization constant is obtained as

$$\begin{aligned} \mathcal{N}_N &= \frac{N!}{\sqrt{C_N^{J+M}}} \\ &= \frac{N!}{\sqrt{C_N^{K}}}, \end{aligned} \tag{71}$$

and the normalized RDM is obtained as

$$\varrho^{[N]} = \frac{\mathbb{I}}{2} + \frac{M}{N}\sigma_z, \tag{72}$$

which is the same as Eq. (57). So far, we have verified the correctness of Eq. (52) in Dicke states.

IV. APPLICATION OF RDM IN MSR

By calculating the RDM of a single particle in MSR, we can further study some other properties of the system. In this section, to illustrate the application of the RDM in MSR, we use the result of Eq. (52) to solve the spin-$\frac{N}{2}$ state in a uniform magnetic field and some symmetric structures on the Bloch sphere.

A. RDM for the spin-$\frac{N}{2}$ state in a uniform magnetic field

The first typical case is a spin-$\frac{N}{2}$ system in a uniform magnetic field $\boldsymbol{B} = B(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, which has the eigenstates $|\Psi_{(N,M)}\rangle = e^{-\hat{J}_y\theta}e^{-\hat{J}_z\phi}|J,M\rangle$. Then in MSR, there are $\frac{N}{2}+M$ coincident stars on $\boldsymbol{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ [or $\boldsymbol{u} = (\theta,\phi)$ in spherical coordinate system with $r=1$], and the other $\frac{N}{2}-M$ coincident stars are on $-\boldsymbol{u}$ for the eigenstates [45]. By writing the corresponding states as $|\boldsymbol{u}\rangle$ and $|-\boldsymbol{u}\rangle$, the eigenstates are

$$|\Psi_{(N,M)}\rangle = \sum_{P\in S_N}\underbrace{|-\boldsymbol{u}\rangle|-\boldsymbol{u}\rangle\cdots|-\boldsymbol{u}\rangle}_{\frac{N}{2}-M}\underbrace{|\boldsymbol{u}\rangle|\boldsymbol{u}\rangle\cdots|\boldsymbol{u}\rangle}_{\frac{N}{2}+M}. \tag{73}$$

It is like the Dicke states, and in the same way, we can get the same result as Eq. (64) and

$$\sum_{k=0}^{[\frac{N-1}{2}]}\frac{\boldsymbol{V}_{T_N}^{(k)}}{(2k+3)!!} = 2^N\frac{\left(\frac{N}{2}+M\right)!\left(\frac{N}{2}-M\right)!2M}{(N+2)!}\boldsymbol{u}. \tag{74}$$

Then, by substituting the results into Eq. (51), we can get the unnormalized RDM as

$$\tilde{\varrho}^{[N]} = \frac{(N-1)!N!}{C_N^{\frac{N}{2}+M}}\left(\frac{N}{2}\mathbb{I} + M\boldsymbol{u}\cdot\boldsymbol{\sigma}\right). \tag{75}$$

With Eq. (8), the normalization constant is obtained as

$$\mathcal{N}_N = \frac{N!}{\sqrt{C_N^{\frac{N}{2}+M}}}, \tag{76}$$

and the normalized RDM is obtained as

$$\varrho^{[N]} = \frac{\mathbb{I}}{2} + \frac{M}{N}\boldsymbol{u}\cdot\boldsymbol{\sigma}. \tag{77}$$

It can be seen that the RDM of the state in a uniform magnetic field in MSR is similar to the Dicke state since they are both W-type entangled states whose Majorana stars are

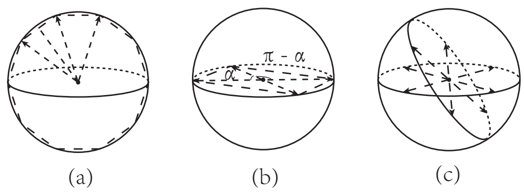

FIG. 1. (a) The Bloch vectors located at the regular N-gon falling on the great circle of the Bloch sphere. (b) The four Bloch vectors located at the inscribed rectangle of the great circle of the Bloch sphere. (c) The general cases with $N = 2m$ pairwise symmetric Majorana stars.

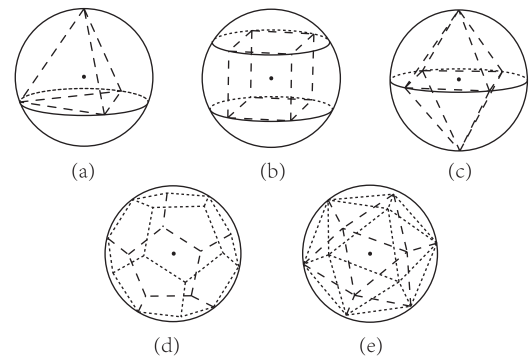

FIG. 2. Five kinds of inscribed regular polyhedrons of the Bloch sphere.

distributed on a pair of antipodal points on the Bloch sphere, and the Dicke state is the case of $\boldsymbol{B} = 0$ [36].

B. RDM of symmetric structures on Bloch sphere

Next, we will study some symmetric structures on the Bloch sphere with the result of Eq. (52), including the regular N-gon and the regular polyhedrons on the Bloch sphere, and so on.

Generally, when the N stars form symmetric structures, they satisfy the relation as $\sum_{i=1}^{N}\boldsymbol{u}_i = 0$ and there is $\boldsymbol{V}_{T_N}^{(0)} = \sum_{i=1}^{N}\boldsymbol{u}_i = 0$. Furthermore, because of

$$\begin{aligned}\left(\sum_{i=1}^{N}\boldsymbol{u}_i\right)\cdot\left(\sum_{i=1}^{N}\boldsymbol{u}_i\right) &= \sum_{i_1,i_2=1}^{N}\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{i_2}\\ &= \sum_{i_1=i_2}^{N}\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{i_2} + \sum_{i_1\neq i_2}^{N}\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{i_2}\\ &= N + 2D_{T_N}^{(1)}\\ &= 0, \end{aligned} \tag{78}$$

we can get

$$D_{T_N}^{(1)} = -\frac{N}{2}. \tag{79}$$

Then we focus on some completely symmetric Majorana star structures, that is, the regular N-gon falling on the great circle of the Bloch sphere [as shown in Fig. 1(a)] and the inscribed regular polyhedrons of the Bloch sphere, including the regular tetrahedron ($N = 4$), the regular hexahedron ($N = 8$), the regular octahedron ($N = 6$), the regular dodecahedron ($N = 20$), and the regular icosahedron ($N = 12$) (as shown in Fig. 2). For these completely symmetric Majorana star structures, we can select any Bloch vector as the reference point for numbering. As long as the relative positions are not changed during numbering, the included angle between the Bloch vectors will not change. In this way, we have $(\boldsymbol{u}_{i'_1}\cdot\boldsymbol{u}_{j'_1})(\boldsymbol{u}_{i'_2}\cdot\boldsymbol{u}_{j'_2})\cdots(\boldsymbol{u}_{i'_k}\cdot\boldsymbol{u}_{j'_k}) = (\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})$, where $\{i_1,\ldots,i_k,j_1,\ldots,j_k\}$ represents the original index set and $\{i'_1,\ldots,i'_k,j'_1,\ldots,j'_k\}$ represents the index set after changing the reference point and renumbering the stars. Therefore, in the process of calculating $D_{T_N}^{(k)}$, we can merge many of the terms. Similarly, when calculating $\boldsymbol{V}_{T_N}^{(k)}$, for each $\boldsymbol{u}_{i_{k+1}}$, we can find a set of $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})$ with the same value and get

$$\begin{aligned}\boldsymbol{V}_{T_N}^{(k)} &= \sum_{T_N''}(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})\boldsymbol{u}_{i_{k+1}}\\ &= \sum_{T_N'''}(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})\sum_{i_{k+1}=1}^{N}\boldsymbol{u}_{i_{k+1}}\\ &= 0. \end{aligned} \tag{80}$$

Substituting Eq. (80) into Eq. (52), we can get

$$\varrho^{[N]} = \frac{\mathbb{I}}{2} \tag{81}$$

as the RDM in MSR when the Majorana stars are completely symmetric on the Bloch sphere.

Next, let us consider an even number of Majorana stars which are symmetric in pairs as $\boldsymbol{u}_i = -\boldsymbol{u}_j$.

As the simplest case, when there are two stars as $\boldsymbol{u}_2 = -\boldsymbol{u}_1$, we have $D_{T_2}^{(1)} = -1$, and the corresponding results are $\mathcal{N}_2 = \sqrt{\frac{2!3!}{2^2}(1 + \frac{1}{3!!}D_{T_2}^{(1)})} = \sqrt{2}$ and $\varrho^{[2]} = \frac{\mathbb{I}}{2}$.

Then when $N = 4$ and $\boldsymbol{u}_1 = -\boldsymbol{u}_3$, $\boldsymbol{u}_2 = -\boldsymbol{u}_4$, the stars fall on the inscribed rectangle of a great circle. When the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is $\alpha \in (0, \pi)$, as shown in Fig. 1(b), we have $D_{T_4}^{(1)} = -2$ and

$$\begin{aligned} D_{T_4}^{(2)} &= (\boldsymbol{u}_1\cdot\boldsymbol{u}_2)(\boldsymbol{u}_3\cdot\boldsymbol{u}_4) + (\boldsymbol{u}_1\cdot\boldsymbol{u}_3)(\boldsymbol{u}_2\cdot\boldsymbol{u}_4)\\ &\quad + (\boldsymbol{u}_1\cdot\boldsymbol{u}_4)(\boldsymbol{u}_2\cdot\boldsymbol{u}_3)\\ &= (|\boldsymbol{u}_1||\boldsymbol{u}_2|\cos\alpha)(|\boldsymbol{u}_3||\boldsymbol{u}_4|\cos\alpha)\\ &\quad + (|\boldsymbol{u}_1||\boldsymbol{u}_3|\cos\pi)(|\boldsymbol{u}_2||\boldsymbol{u}_4|\cos\pi)\\ &\quad + [|\boldsymbol{u}_1||\boldsymbol{u}_4|\cos(\pi-\alpha)][|\boldsymbol{u}_2||\boldsymbol{u}_3|\cos(\pi-\alpha)]\\ &= 2\cos^2\alpha + 1, \end{aligned} \tag{82}$$

and

$$\begin{aligned}\mathcal{N}_4 &= \sqrt{\frac{4!5!}{2^4}\left(1 + \frac{1}{3!!}D_{T_4}^{(1)} + \frac{1}{5!!}D_{T_4}^{(2)}\right)}\\ &= 2\sqrt{6\cos^2\alpha + 18} \end{aligned} \tag{83}$$

can be obtained from the above results. When $\alpha = \frac{\pi}{2}$, we have $\mathcal{N}_4 = 6\sqrt{2}$. At the same time, because $\boldsymbol{V}_{T_4}^{(0)} = \sum_{i=1}^{4}\boldsymbol{u}_i = 0$

and

$$\begin{aligned}\boldsymbol{V}_{T_4}^{(1)} &= (\boldsymbol{u}_1\cdot\boldsymbol{u}_2)\boldsymbol{u}_3 + (\boldsymbol{u}_1\cdot\boldsymbol{u}_2)\boldsymbol{u}_4 + (\boldsymbol{u}_1\cdot\boldsymbol{u}_3)\boldsymbol{u}_2 + (\boldsymbol{u}_1\cdot\boldsymbol{u}_3)\boldsymbol{u}_4\\ &\quad + (\boldsymbol{u}_1\cdot\boldsymbol{u}_4)\boldsymbol{u}_2 + (\boldsymbol{u}_1\cdot\boldsymbol{u}_4)\boldsymbol{u}_3 + (\boldsymbol{u}_2\cdot\boldsymbol{u}_3)\boldsymbol{u}_1\\ &\quad + (\boldsymbol{u}_2\cdot\boldsymbol{u}_3)\boldsymbol{u}_4 + (\boldsymbol{u}_2\cdot\boldsymbol{u}_4)\boldsymbol{u}_1 + (\boldsymbol{u}_2\cdot\boldsymbol{u}_4)\boldsymbol{u}_3\\ &\quad + (\boldsymbol{u}_3\cdot\boldsymbol{u}_4)\boldsymbol{u}_1 + (\boldsymbol{u}_3\cdot\boldsymbol{u}_4)\boldsymbol{u}_2\\ &= [|\boldsymbol{u}_1||\boldsymbol{u}_2|(\boldsymbol{u}_3+\boldsymbol{u}_4) + |\boldsymbol{u}_3||\boldsymbol{u}_4|(\boldsymbol{u}_1+\boldsymbol{u}_2)]\cos\alpha\\ &\quad + [|\boldsymbol{u}_1||\boldsymbol{u}_3|(\boldsymbol{u}_2+\boldsymbol{u}_4) + |\boldsymbol{u}_2||\boldsymbol{u}_4|(\boldsymbol{u}_1+\boldsymbol{u}_3)]\cos\pi\\ &\quad + [|\boldsymbol{u}_1||\boldsymbol{u}_4|(\boldsymbol{u}_2+\boldsymbol{u}_3) + |\boldsymbol{u}_2||\boldsymbol{u}_3|(\boldsymbol{u}_1+\boldsymbol{u}_4)]\cos(\pi-\alpha)\\ &= 0, \end{aligned} \tag{84}$$

we still have $\varrho^{[4]} = \frac{\mathbb{I}}{2}$.

Further, considering $N = 2m, m \in \mathbb{Z}_+$ pairwise symmetric Majorana stars, as shown in Fig. 1(c), we mark the pair of stars as $\boldsymbol{u}_i$ and $\boldsymbol{u}_{-i}$, respectively, and have $\boldsymbol{u}_i + \boldsymbol{u}_{-i} = 0$. In the process of solving $D_{T_N}^{(k)}$, it is easy to get each $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k}) = (\boldsymbol{u}_{-i_1}\cdot\boldsymbol{u}_{-j_1})(\boldsymbol{u}_{-i_2}\cdot\boldsymbol{u}_{-j_2})\cdots(\boldsymbol{u}_{-i_k}\cdot\boldsymbol{u}_{-j_k})$. Thus, in the process of solving $\boldsymbol{V}_{T_N}^{(k)}$, we have $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})\boldsymbol{u}_{i_{k+1}} = -(\boldsymbol{u}_{-i_1}\cdot\boldsymbol{u}_{-j_1})(\boldsymbol{u}_{-i_2}\cdot\boldsymbol{u}_{-j_2})\cdots(\boldsymbol{u}_{-i_k}\cdot\boldsymbol{u}_{-j_k})\boldsymbol{u}_{-i_{k+1}}$, and then we can get all $\boldsymbol{V}_{T_N}^{(k)} = 0$ in this case, that is, the RDM of a single particle in MSR is $\varrho^{[N]} = \frac{1}{2}\mathbb{I}$ in these symmetric cases.

The inscribed regular N-gon of the great circle of the Bloch sphere when N is even, and the inscribed regular hexahedron ($N = 8$), regular octahedron ($N = 6$), regular dodecahedron ($N = 20$), and regular icosahedron ($N = 12$) of the Bloch sphere, are also the above cases. The results tell that if the stars remain some kind of symmetric structure or evolve between different symmetric structures, the RDM of a single particle remains constant and the expectation value of the observable $\boldsymbol{F}$ in the subsystem is

$$\begin{aligned}\langle \boldsymbol{F}\rangle &= \tfrac{1}{2}\mathrm{Tr}\boldsymbol{F}\\ &= F_0,\end{aligned} \tag{85}$$

where $\boldsymbol{F}$ is decomposed into $\boldsymbol{F} = F_0\mathbb{I} + F_x\sigma_x + F_y\sigma_y + F_z\sigma_z$.

V. CONCLUSION AND DISCUSSION

In summary, the RDM and MSR provide valuable tools for studying high-dimensional or many-body systems. In this paper, we use them to analyze the completely symmetric states and get the general formula of the RDM in MSR as Eq. (52) for the completely symmetric states coupled by N qubits. In the process, we also reproduce the normalization constant of these states as Eq. (42) with the relation of Eq. (8). The results present intuitive connections between the subsystems and the composite system. Our result of RDM in MSR transforms those high-order matrices of high-dimensional Hilbert space into ordinary algebraic operations, which can be simplified using relations in combinatorics, thus providing a new idea for the numerical solution of multiqubit systems. Furthermore, due to the symmetry of the results, they can be simplified in lots of cases. We apply the results to calculate the RDM of the Dicke states and the spin-$\frac{N}{2}$ state in a uniform magnetic field and find the coupling coefficient as $\frac{2M}{N}$ in the RDM. We also study the states when the stars distribute as symmetric structures on the Bloch sphere, and the RDM are obtained as $\varrho^{[N]} = \frac{1}{2}\mathbb{I}$ in these cases, whose corresponding mixed states are completely mixed states.

For further discussion, these symmetric structures on the Bloch sphere can be regarded as topologically equivalent with topological invariant $\sum_{i=1}^{N}\boldsymbol{u}_i = 0$ and $\varrho^{[N]} = \frac{1}{2}\mathbb{I}$. The expectation values of the observables are also invariant if the stars evolve in the same symmetric structure or between different symmetric structures. In addition, Majorana stars can indicate other physical quantities, such as geometric phases. The RDM in MSR given in this paper can also provide a basis for investigating the relations between these physical quantities. We will also do the corresponding research in the subsequent work.

ACKNOWLEDGMENT

This work was supported by the NSFC through Grant No. 11935012.

APPENDIX A: THE CALCULATION OF $\sum_{I_0} S_{I_0}$

With Eq. (26), we get

$$\begin{aligned}&\left\{\rho_{i_1^{(0)}}, \rho_{i_2^{(0)}}, \ldots, \rho_{i_k^{(0)}}\right\}\\ &\quad = \frac{k!}{2^k}\left[\sum_{i=0}^{[\frac{k}{2}]}\frac{D_{I_0}^{(i)}}{(2i-1)!!} + \sum_{i=0}^{[\frac{k-1}{2}]}\frac{\boldsymbol{V}_{I_0}^{(i)}}{(2i+1)!!}\cdot\boldsymbol{\sigma}\right].\end{aligned} \tag{A1}$$

Here we defined $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})$ as the $(2k)$-order term composed of the Bloch vectors and $(\boldsymbol{u}_{i_1}\cdot\boldsymbol{u}_{j_1})(\boldsymbol{u}_{i_2}\cdot\boldsymbol{u}_{j_2})\cdots(\boldsymbol{u}_{i_k}\cdot\boldsymbol{u}_{j_k})\boldsymbol{u}_{j_{k+1}}$ as the $(2k+1)$-order term. Then we introduce $n(\ldots)$ as the counting function to count the number of the terms. It can be seen that the number of $(2k)$-order terms and $(2k+1)$-order terms in $D_{T_N}^{(k)}$ and $\boldsymbol{V}_{T_N}^{(k)}$ are, respectively,

$$\begin{aligned}n\big(D_{T_N}^{(k)}\big) &= \frac{1}{k!}C_N^2C_{N-2}^2\ldots C_{N-2k+2}^2\\ &= \frac{N!}{(2k)!!(N-2k)!} \qquad \text{(A2)}\\ &= (2k-1)!!C_N^{2k}, \qquad \text{(A3)}\end{aligned}$$

and

$$\begin{aligned}n\big(\boldsymbol{V}_{T_N}^{(k)}\big) &= \frac{1}{k!}C_N^2C_{N-2}^2\ldots C_{N-2k+2}^2C_{N-2k}^1\\ &= \frac{N!}{(2k)!!(N-2k-1)!} \qquad \text{(A4)}\\ &= (2k+1)!!C_N^{2k+1}. \qquad \text{(A5)}\end{aligned}$$

From the definition of the counting function, we can see that the counting function has the following properties: for all $\lambda \in \mathbb{R}$, we have

$$n\big(\lambda D_T^{(k)}\big) = \lambda n\big(D_T^{(k)}\big), \tag{A6}$$

$$n\big(\lambda \boldsymbol{V}_T^{(k)}\big) = \lambda n\big(\boldsymbol{V}_T^{(k)}\big). \tag{A7}$$

In this way, we get the sum of coefficients of $(2i)$-order terms in Eq. (48) as

$$n\left(\frac{(N-1)!(N-k+1)!}{2^{N-k}}\frac{1}{(2i+1)!!}D^{(i)}_{T_N-I_0}\right) = \frac{(N-1)!(N-k)!}{2^{N-k}}C^{2i+1}_{N-k+1}. \tag{A8}$$

In Eq. (A1), we see that the sum of coefficients of $(2i)$-order terms and $(2i+1)$-order terms are

$$n\left(\frac{k!}{2^k}\frac{D^{(i)}_{I_0}}{(2i-1)!!}\right) = \frac{k!}{2^k}C^{2i}_k \tag{A9}$$

and

$$n\left(\frac{k!}{2^k}\frac{\boldsymbol{V}^{(i)}_{I_0}}{(2i+1)!!}\right) = \frac{k!}{2^k}C^{2i+1}_k \tag{A10}$$

respectively, that is, in Eq. (A1), the sum of coefficients of (j)-order terms is $\frac{k!}{2^k}C^j_k$, whether j equals even $2i$ or odd $2i+1$. So the sum of coefficients of (J)-order terms of S_{I_0} is

$$\sum_{2i+j=J}\frac{(N-1)!(N-k)!}{2^{N-k}}C^{2i+1}_{N-k+1}\frac{k!}{2^k}C^j_k = \sum_{2i+j=J}\frac{(N-1)!k!(N-k)!}{2^N}C^{2i+1}_{N-k+1}C^j_k. \tag{A11}$$

Now consider all pairs of P and $\tilde{P}$ whose character sequence is of size k. The number of them is just C^k_N times as the number of pairs of P and $\tilde{P}$ whose character sequence is a permutation of I_0. So the sum of coefficients of (J)-order terms in their contributions to $\sum_{I_0}S_{I_0}$ is C^k_N times as Eq. (A11), that is,

$$\sum_{2i+j=J}\frac{(N-1)!k!(N-k)!}{2^N}C^k_N C^{2i+1}_{N-k+1}C^j_k = \sum_{2i+j=J}\frac{(N-1)!N!}{2^N}C^{2i+1}_{N-k+1}C^j_k. \tag{A12}$$

Equation (A12) is precisely the Jth power series coefficient of

$$\frac{(N-1)!N!}{2^N}\left(\sum_{i=0}^{\infty}C^{2i+1}_{N-k+1}x^{2i}\right)\left(\sum_{j=0}^{\infty}C^j_k x^j\right) = \frac{(N-1)!N!}{2^N}\frac{(1+x)^{N-k+1}-(1-x)^{N-k+1}}{2x}(1+x)^k. \tag{A13}$$

Finally, by summing Eq. (A13) from $k=1$ to N, we get

$$\begin{aligned}\sum_{k=1}^{N}&\left[\frac{(N-1)!N!}{2^N}\frac{(1+x)^{N-k+1}-(1-x)^{N-k+1}}{2x}(1+x)^k\right]\\ &= \frac{(N-1)!N!}{2^N}\frac{1}{4x^2}\big[2nx(1+x)^{N+1}+(1+x)(1-x)^{N+1}\\ &\quad-(1-x)(1+x)^{N+1}\big]\\ &= \frac{(N-1)!N!}{2^N}\left[\sum_{J=0}^{\infty}\frac{N}{2}C^{2J+1}_{N+1}x^{2J}\right.\\ &\quad\left.+\sum_{J=0}^{\infty}(J+1)C^{2J+3}_{N+2}x^{2J+1}\right],\end{aligned} \tag{A14}$$

which shows that the sum of coefficients of $(2J)$-order terms in $\sum_{I_0}S_{I_0}$ is

$$\frac{(N-1)!N!}{2^N}\frac{N}{2}C^{2J+1}_{N+1} = \frac{N!^2}{2^{N+1}}C^{2J+1}_{N+1}, \tag{A15}$$

and the sum of coefficients of $(2J+1)$-order terms in $\sum_{I_0}S_{I_0}$ is

$$\frac{(N-1)!N!}{2^N}(J+1)C^{2J+3}_{N+2} = \frac{(N-1)!N!(2J+2)}{2^{N+1}}C^{2J+3}_{N+2}. \tag{A16}$$

Again, using the completely symmetric property, the sum of all $(2J)$-order terms in $\sum_{I_0}S_{I_0}$ must be $D^{(J)}_{T_N}$ multiplied by a coefficient. Since it can be seen from Eq. (A2) that there are

$$n\big(D^{(J)}_{T_N}\big) = \frac{N!}{(2J)!!(N-2J)!} \tag{A17}$$

terms in $D^{(J)}_{T_N}$, the coefficient of each term is

$$\frac{N!^2}{2^{N+1}}C^{2J+1}_{N+1}\frac{(2J)!!(N-2J)!}{N!} = \frac{N!(N+1)!}{2^{N+1}}\frac{1}{(2J+1)!!} \tag{A18}$$

from Eqs. (A15) and (A17). The sum of all $(2J+1)$-order terms in $\sum_{I_0}S_{I_0}$ must be $\boldsymbol{V}^{(J)}_{T_N}\cdot\boldsymbol{\sigma}$ multiplied by a coefficient. Since it can be seen from Eq. (A4) that there are

$$n\big(\boldsymbol{V}^{(J)}_{T_N}\big) = \frac{N!}{(2J)!!(N-2J-1)!} \tag{A19}$$

terms in $\boldsymbol{V}^{(J)}_{T_N}\cdot\boldsymbol{\sigma}$, the coefficient of each term is

$$\frac{(N-1)!N!(2J+2)}{2^{N+1}}C^{2J+3}_{N+2}\frac{(2J)!!(N-2J-1)!}{N!} = \frac{(N-1)!(N+2)!}{2^{N+1}}\frac{1}{(2J+3)!!} \tag{A20}$$

from Eqs. (A16) and (A19). In conclusion, we get

$$\sum_{I_0}S_{I_0} = \frac{(N-1)!(N+1)!}{2^{N+1}}\left[N\sum_{J=0}^{[\frac{N}{2}]}\frac{1}{(2J+1)!!}D^{(J)}_{T_N} + (N+2)\sum_{J=0}^{[\frac{N-1}{2}]}\frac{1}{(2J+3)!!}\boldsymbol{V}^{(J)}_{T_N}\cdot\boldsymbol{\sigma}\right]. \tag{A21}$$

This is Eq. (51).

APPENDIX B: THE CALCULATION OF EQS. (64) AND (69)

Given $n_{+1}+n_{-1}=n(D_{T_N}^{(k)})$, it corresponds exactly to Vandermonde's identity. For Eq. (63), we need to find the corresponding generating function. First, construct

$$f(x)=\sum_{k=0}^{[J]}\frac{1}{2k+1}\sum_{j=0}^{J-M}(-1)^jC_{J-M}^{j}C_{J+M}^{2k-j}x^{2k+1}, \tag{B1}$$

that is, multiply the right side of the equal sign of Eq. (63) by x^{2k+1}, and then derive it to get

$$\begin{aligned}f'(x)&=\sum_{k=0}^{[J]}\sum_{j=0}^{J-M}(-1)^jC_{J-M}^{j}C_{J+M}^{2k-j}x^{2k}=\sum_{j=0,2,\dots}C_{J-M}^{j}x^j\left(\sum_{k=0}^{[J]}C_{J+M}^{2k-j}x^{2k-j}\right)-\sum_{j=1,3,\dots}C_{J-M}^{j}x^j\left(\sum_{k=0}^{[J]}C_{J+M}^{2k-j}x^{2k-j}\right)\\&=\frac{(1+x)^{J-M}+(1-x)^{J-M}}{2}\frac{(1+x)^{J+M}+(1-x)^{J+M}}{2}-\frac{(1+x)^{J-M}-(1-x)^{J-M}}{2}\frac{(1+x)^{J+M}-(1-x)^{J+M}}{2}\\&=\frac{1}{2}[(1-x)^{J-M}(1+x)^{J+M}+(1+x)^{J-M}(1-x)^{J+M}],\end{aligned} \tag{B2}$$

and then get

$$f(x)=\int f'(x)dx, \tag{B3}$$

then Eq. (63) can be simplified to

$$\begin{aligned}\int_0^1 f'(x)dx&=\frac{1}{4}\int_{-1}^{1}[(1-x)^{J-M}(1+x)^{J+M}+(1+x)^{J-M}(1-x)^{J+M}]dx\\&=\frac{1}{4}[2^{N+1}\mathrm{B}(J+M+1,J-M+1)+2^{N+1}\mathrm{B}(J-M+1,J+M+1)]\\&=2^N\frac{(J-M)!(J+M)!}{(N+1)!},\end{aligned} \tag{B4}$$

where $\mathrm{B}(\dots)$ is the Beta function. This is Eq. (64).

Construct

$$g(x)=\sum_{k=0}^{[J-\frac{1}{2}]}\frac{1}{2k+3}\sum_{j=0}^{J-M}(-1)^jC_{J-M}^{j}C_{J+M}^{2k+1-j}x^{2k+3}, \tag{B5}$$

that is, multiply x^{2k+3} to the right side of the equal sign of Eq. (68) and divide by σ_z, and then derive it to get

$$\begin{aligned}g'(x)&=\sum_{k=0}^{[J-\frac{1}{2}]}\sum_{j=0}^{J-M}(-1)^jC_{J-M}^{j}C_{J+M}^{2k+1-j}x^{2k+2}=x\sum_{j=0,2,\dots}C_{J-M}^{j}x^j\left(\sum_{k=0}^{[J-\frac{1}{2}]}C_{J+M}^{2k+1-j}x^{2k+1-j}\right)\\&\quad-x\sum_{j=1,3,\dots}C_{J-M}^{j}x^j\left(\sum_{k=0}^{[J-\frac{1}{2}]}C_{J+M}^{2k+1-j}x^{2k+1-j}\right)\\&=x\left[\frac{(1+x)^{J-M}+(1-x)^{J-M}}{2}\frac{(1+x)^{J+M}-(1-x)^{J+M}}{2}-\frac{(1+x)^{J-M}-(1-x)^{J-M}}{2}\frac{(1+x)^{J+M}+(1-x)^{J+M}}{2}\right]\\&=\frac{x}{2}[(1-x)^{J-M}(1+x)^{J+M}-(1+x)^{J-M}(1-x)^{J+M}],\end{aligned} \tag{B6}$$

and then get

$$g(x)=\int g'(x)dx. \tag{B7}$$

In this way, Eq. (68) can be simplified to

$$\begin{aligned}\sigma_z \int_0^1 g'(x)dx &= \frac{\sigma_z}{4}\int_{-1}^1 [(1-x)^{J-M}(1+x)^{J+M+1} - (1-x)^{J-M+1}(1+x)^{J+M}]dx \\ &= \frac{\sigma_z}{4}[2^{2J+2}\mathrm{B}(J+M+2, J-M+1) - 2^{2J+2}\mathrm{B}(J+M+1, J-M+2)] \\ &= 2^{2J}\frac{(J+M)!(J-M)!2M}{(2J+2)!}\sigma_z. \end{aligned} \tag{B8}$$

This is Eq. (69).

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