Many knots in Chern-Simons field theory

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In this paper, using the $U(1)$ gauge potential decomposition and the ϕ-mapping topological current theory, the many knotlike vortex lines in Chern-Simons field theory are studied. It has been pointed out that the Chern-Simons action is a topological invariant for the family of knots; i.e., it is just the total sum of all the self-linking and all the linking numbers of the knot family. Furthermore, it is also shown that this Chern-Simons knot topological number is preserved in the branch processes (splitting, merging, and intersection) during the evolution of these knotlike vortex lines.

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I. INTRODUCTION

Knotlike configurations as string structures of finite energy appear in a variety of physical, chemical, and biological scenarios, including the structure of elementary particles [1,2], early universe cosmology [3–5], Bose-Einstein condensation [6], polymer folding [7], and DNA replication, transcription, and recombination [8].

In itself, a knot γ is in fact an embedding map in geometry,

$$\gamma : S^1 \rightarrow \mathbf{R}^3, \tag{1}$$

and two or more such knots together are called a link, i.e., a family of knots. It is known that for a knot family there are important characteristic numbers to describe its topology, such as the self-linking and the linking numbers. So in research into knotlike configurations in physics, one should also pay much attention to these knot characteristics. In this paper we will just use the topological viewpoint to study the many knots inherent in Chern-Simons field theory, and reveal the inner relationship between the Chern-Simons action and the topological characteristic numbers of the knot family [9,10].

This paper is arranged as follows. In Sec. II, using the $U(1)$ gauge potential decomposition and the ϕ-mapping topological current theory, it is revealed that there are knotlike vortex lines in the Chern-Simons field. In Sec. III, we point out that the Chern-Simons action is a topological invariant for the family of knots, i.e., it is just the total sum of all the self-linking and all the linking numbers of the knot family. In Sec. IV, it is shown that this Chern-Simons knot topological number is preserved in the branch processes (splitting, merging, and intersection) during the evolution of the knotlike vortex lines.

II. CHERN-SIMONS KNOTS

Let M be the four-dimensional Euclidean space, and $\mathbf{P}(M, U(1), \pi)$ be the principal $U(1)$ bundle on base M. The complex line bundle on M is the associate bundle $\mathbf{P} \times_{U(1)} \mathbf{C}$, and the basic field $\psi(x)$ [the wave function in $U(1)$ quantum mechanics] is a section of this complex line bundle, i.e., a section of the two-dimensional real vector bundle on M:

$$\psi(x) = \phi^1(x) + i\phi^2(x). \tag{2}$$

The covariant derivative of ψ is defined as

$$D_\mu \psi = \partial_\mu \psi - iA_\mu \psi, \tag{3}$$

where $\mu = 0,1,2,3$ denotes the four-dimensional space-time, and A_μ is the $U(1)$ gauge potential. The $U(1)$ gauge field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{4}$$

The Chern-Simons action in three-dimensional space is defined as [11,12]

$$I = \frac{1}{4\pi}\int_M A \wedge F = \frac{1}{8\pi}\int_M \epsilon^{ijk} A_i F_{jk} d^3x, \tag{5}$$

where $i,j,k = 1,2,3$ denote the three-dimensional space.

In this section we will show that in Chern-Simons field theory there exist vortex line structures. Defining a two-dimensional unit vector

$$m^a = \frac{\phi^a}{\|\phi\|} \left(a = 1,2; \|\phi\|^2 = \phi^a \phi^a = \psi^* \psi\right), \tag{6}$$

it can be proved [13] that A_μ can be decomposed in terms of m^a: $A_\mu = \epsilon_{ab} m^a \partial_\mu m^b - \partial_\mu \theta$, where θ is a phase factor. Since the $(\partial_\mu \theta)$ term does not contribute to the field tensor $F_{\mu\nu}$ of Eq. (4), A_μ can be expressed as

$$A_\mu = \epsilon_{ab} m^a \partial_\mu m^b, \tag{7}$$

and $F_{\mu\nu}$ is

$$F_{\mu\nu} = 2\epsilon_{ab} \partial_\mu m^a \partial_\nu m^b. \tag{8}$$

According to Ref. [14], the two-dimensional topological tensor current is defined as

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$$K^{\mu\nu}=\frac{1}{8\pi}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}. \tag{9}$$

Then using $\partial_\mu\phi^a/\|\phi\|=\partial_\mu\phi^a/\|\phi\|+\phi^a\partial_\mu 1/\|\phi\|$ and the Green's function relation in ϕ space, $\partial_a\partial_a\ln\|\phi\|=2\pi\delta^2(\vec{\phi})$ $\times(\partial_a=\partial/\partial\phi^a)$, one can prove that [15]

$$K^{\mu\nu}=\delta^2(\vec{\phi})D^{\mu\nu}\left(\frac{\phi}{x}\right), \tag{10}$$

where $D^{\mu\nu}(\phi/x)=\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ab}\partial_\lambda\phi^a\partial_\rho\phi^b$. Defining the spatial components of $K^{\mu\nu}$ as

$$j^i=K^{0i}=\frac{1}{8\pi}\epsilon^{ijk}F_{jk}\quad (i,j,k=1,2,3), \tag{11}$$

we have

$$j^i=\delta^2(\vec{\phi})D^i\left(\frac{\phi}{x}\right), \tag{12}$$

where $D^i(\phi/x)=\frac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j\phi^a\partial_k\phi^b$ is the Jacobian vector.

The expression (12) provides an important conclusion:

$$j^i\begin{cases}=0 & \text{if and only if } \vec{\phi}\neq 0,\\ \neq 0 & \text{if and only if } \vec{\phi}=0,\end{cases} \tag{13}$$

so it is necessary to study the zero points of $\vec{\phi}$ to determine the nonzero solutions of j^i. The implicit function theory shows [16] that under the regular condition

$$D^{\mu\nu}(\phi/x)\neq 0 \tag{14}$$

the general solutions of

$$\phi^1(t,x^1,x^2,x^3)=0,\ \phi^2(t,x^1,x^2,x^3)=0 \tag{15}$$

can be expressed as

$$x^1=x_k^1(s,t),\ x^2=x_k^2(s,t),\ x^3=x_k^3(s,t), \tag{16}$$

which represent the world surfaces of N moving isolated singular strings L_k with string parameter s $(k=1,2,\dots,N)$. These singular string solutions are just the Chern-Simons vortex lines.

In δ-function theory [17], one can prove that in three-dimensional space

$$\delta^2(\vec{\phi})=\sum_{k=1}^{N}\beta_k\int_{L_k}\frac{\delta^3(\vec{x}-\vec{x}_k(s))}{\left|D\left(\frac{\phi}{u}\right)\right|_{\Sigma_k}}ds, \tag{17}$$

where $D(\phi/u)_{\Sigma_k}=[\frac{1}{2}\epsilon^{jk}\epsilon_{mn}(\partial\phi^m/\partial u^j)(\partial\phi^n/\partial u^k)]$, and Σ_k is the kth planar element transverse to L_k with local coordinates (u^1,u^2). The positive integer β_k is the Hopf index of ϕ mapping, which means that when $\vec{x}$ covers the neighborhood of the zero point $\vec{x}_k(s)$ once, the vector field $\vec{\phi}$ covers the corresponding region in ϕ space β_j times. Meanwhile the direction vector of L_k is given by [15]

$$\left.\frac{dx^i}{ds}\right|_{x_k}=\left.\frac{D^i(\phi/x)}{D(\phi/u)}\right|_{x_k}. \tag{18}$$

Then from Eqs. (17) and (18) we obtain the inner structure of j^i:

$$j^i=\delta^2(\vec{\phi})D^i\left(\frac{\phi}{x}\right)=\sum_{k=1}^{N}W_k\int_{L_k}\frac{dx^i}{ds}\delta^3(\vec{x}-\vec{x}_k(s))ds, \tag{19}$$

where $W_k=\beta_k\eta_k$ is the winding number of $\vec{\phi}$ around L_k, with $\eta_k=\operatorname{sgn}D(\phi/u)_{\vec{x}_j}=\pm 1$ being the Brouwer degree of ϕ mapping. Hence the topological charge of the vortex line L_k is

$$Q_k=\int_{\Sigma_k}j^i d\sigma_i=W_k. \tag{20}$$

Using Eqs. (11) and (19), the Chern-Simons action (5) is expressed as

$$I=\int_M A_i j^i d^3x=\sum_{k=1}^{N}W_k\int_{L_k}A_i dx^i. \tag{21}$$

It can be seen that when these N Chern-Simons vortex lines are N closed curves, i.e., a family of N knots $\gamma_k(k=1,\dots,N)$, Eq. (21) leads to

$$I=\sum_{k=1}^{N}W_k\oint_{\gamma_k}A_i dx^i. \tag{22}$$

This is a very important expression. Consider the $U(1)$ gauge transformation of A_i [18]:

$$A_i'=A_i+\partial_i\alpha, \tag{23}$$

where $\alpha\in\mathbf{R}$ is a phase factor denoting the $U(1)$ transformation. It is seen that the $(\partial_i\alpha)$ term in Eq. (23) contributes nothing to the integral I; hence the expression (22) is invariant under the gauge transformation. Meanwhile we know that I is independent of the metric $g_{\mu\nu}$ (see Sec. V). Therefore one can conclude that I is a topological invariant for the knotlike vortex lines in Chern-Simons field theory.

At the end of this section, it should be addressed that in the above the regular condition (14) has been used; when this condition fails, branch processes in the evolution of vortex lines will occur. This will be detailed in Sec. IV.

III. THE CHERN-SIMONS KNOT TOPOLOGICAL NUMBER

In this section, we will research the relationship between the Chern-Simons action (22) and the self-linking and linking numbers of the knot family.

For this purpose we should first express A_i in terms of the vector field which carries the geometric information of the knot family, namely, we need to decompose A_i in terms of another two-dimensional unit vector $\vec{e}$ which is different from the two-dimensional vector $\vec{m}$ in Sec. II. Define the Gauss mapping $\vec{n}$

$$\vec{n}:S^1\times S^1\rightarrow S^2, \tag{24}$$

$\vec{n}$ is a unit vector

$$\vec{n}(\vec{x},\vec{y})=\frac{\vec{y}-\vec{x}}{\|\vec{y}-\vec{x}\|}, \tag{25}$$

where $\vec{x}$ and $\vec{y}$ are two points, respectively, on the knots γ_k and γ_l (in particular, when $\vec{x}$ and $\vec{y}$ are the same point on the same knot γ, $\vec{n}$ is just the unit tangent vector $\vec{T}$ of γ at $\vec{x}$). Therefore, when $\vec{x}$ and $\vec{y}$, respectively, cover the closed curves γ_k and γ_l once, $\vec{n}$ becomes the section of sphere bundle S^2. So, on this S^2 we can define the two-dimensional unit vector $\vec{e}=\vec{e}(\vec{x},\vec{y})$ as

$$e^a e^a=1 \quad (a=1,2;\vec{e}\perp\vec{n}). \tag{26}$$

Then, according to Ref. [13] A_i can be decomposed in terms of this two-dimensional vector e^a: $A_i=\epsilon_{ab}e^a\partial_i e^b-\partial_i\varphi$, where φ is a phase factor. Since one can see from Eq. (22) that the $(\partial_i\varphi)$ term does not contribute to the integral I, A_i can in fact be expressed as

$$A_i=\epsilon_{ab}e^a\partial_i e^b. \tag{27}$$

Substituting Eq. (27) into Eq. (22), we have

$$I=\sum_{k=1}^{N} W_k \oint_{\gamma_k}\epsilon_{ab}e^a(\vec{x},\vec{y})\partial_i e^b(\vec{x},\vec{y})dx^i. \tag{28}$$

Noticing the symmetry between the points $\vec{x}$ and $\vec{y}$ in Eq. (25), Eq. (28) should be reexpressed as

$$I=\sum_{k,l=1}^{N} W_k W_l \oint_{\gamma_k}\oint_{\gamma_l}\epsilon_{ab}\partial_i e^a(\vec{x},\vec{y})\partial_j e^b(\vec{x},\vec{y})dx^i\wedge dy^j. \tag{29}$$

One should notice that in Eq. (29) there are three cases: (1) γ_k and γ_l are two different knots $(k\neq l)$ and $\vec{x}$ and $\vec{y}$ are therefore two different points $(\vec{x}\neq\vec{y})$; (2) γ_k and γ_l are the same knot $(k=l)$ but $\vec{x}$ and $\vec{y}$ are two different points $(\vec{x}\neq\vec{y})$; (3) γ_k and γ_l are the same knot $(k=l)$ and $\vec{x}$ and $\vec{y}$ are the same point $(\vec{x}=\vec{y})$. So Eq. (29) can be written as three terms:

$$\begin{aligned}I=&\sum_{k=1(k=l,\vec{x}\neq\vec{y})}^{N} W_k^2 \oint_{\gamma_k}\oint_{\gamma_k}\epsilon_{ab}\partial_i e^a\partial_j e^b dx^i\wedge dy^j\\&+\sum_{k=1}^{N} W_k^2 \oint_{\gamma_k}\epsilon_{ab}e^a\partial_i e^b dx^i\\&+\sum_{k,l=1(k\neq l)}^{N} W_k W_l \oint_{\gamma_k}\oint_{\gamma_l}\epsilon_{ab}\partial_i e^a\partial_j e^b dx^i\wedge dy^j.\end{aligned} \tag{30}$$

Using the relation $\epsilon_{ab}\partial_i e^a\partial_j e^b=\frac{1}{2}\vec{n}\cdot(\partial_i\vec{n}\times\partial_j\vec{n})$, Eq. (30) is just

$$\begin{aligned}I=&\sum_{k=1(\vec{x}\neq\vec{y})}^{N}\frac{1}{2}W_k^2\oint_{\gamma_k}\oint_{\gamma_k}\vec{n}^*(dS)\\&+\sum_{k=1}^{N} W_k^2\oint_{\gamma_k}\epsilon_{ab}e^a\partial_i e^b dx^i\\&+\sum_{k,l=1(k\neq l)}^{N}\frac{1}{2}W_k W_l\oint_{\gamma_k}\oint_{\gamma_l}\vec{n}^*(dS),\end{aligned} \tag{31}$$

where $\vec{n}^*(dS)=\vec{n}\cdot(\partial_i\vec{n}\times\partial_j\vec{n})dx^i\wedge dy^j(\vec{x}\neq\vec{y})$ denotes the pullback of the S^2 surface element.

Let us discuss these three terms in detail. First, the first term of Eq. (31) is just related to the writhing number $\mathcal{W}(\gamma_k)$ of γ_k [19,20]:

$$\mathcal{W}(\gamma_k)=\frac{1}{4\pi}\oint_{\gamma_k}\oint_{\gamma_k}\vec{n}^*(dS). \tag{32}$$

For the second term of Eq. (31), since this is the $\vec{x}=\vec{y}$ term, one can prove that it is related to the twisting number $\mathcal{T}(\gamma_k)$ of γ_k:

$$\frac{1}{2\pi}\oint_{\gamma_k}\epsilon_{ab}e^a\partial_i e^b dx^i=\frac{1}{2\pi}\oint_{\gamma_k}(\vec{T}\times\vec{V})\cdot d\vec{V}=\mathcal{T}(\gamma_k), \tag{33}$$

where $\vec{T}$ is the unit tangent vector of knot γ_k at $\vec{x}$ ($\vec{n}=\vec{T}$ when $\vec{x}=\vec{y}$), and $\vec{V}$ is defined as $e^a=\epsilon^{ab}V^b(a,b=1,2;\vec{V}\perp\vec{T},\vec{e}=\vec{T}\times\vec{V})$. From the White formula [19,20]

$$\mathcal{S}(\gamma_k)=\mathcal{W}(\gamma_k)+\mathcal{T}(\gamma_k) \tag{34}$$

[where $\mathcal{S}(\gamma_k)$ is the self-linking number of γ_k], we see that the first and second terms of Eq. (31) just compose the self-linking numbers of the N knots.

Second, for the third term, one can prove that

$$\begin{aligned}\frac{1}{4\pi}\oint_{\gamma_k}\oint_{\gamma_l}\vec{m}^*(dS)=&\frac{1}{4\pi}\epsilon^{ijk}\oint_{\gamma_k}dx^i\oint_{\gamma_l}dy^j\frac{(x^k-y^k)}{\|\vec{x}-\vec{y}\|^3}\\&=\mathcal{L}(\gamma_k,\gamma_l)(k\neq l)\end{aligned} \tag{35}$$

where $\mathcal{L}(\gamma_k,\gamma_l)$ is the Gauss linking number between γ_k and γ_l [9,21]. So the third term is just related to the linking numbers between the N knots.

Therefore, third, from Eqs. (32), (33), (34), and (35), we arrive at the important result

$$I=2\pi\left[\sum_{k=1}^{N} W_k^2\mathcal{S}(\gamma_k)+\sum_{k,l=1(k\neq l)}^{N} W_kW_l\mathcal{L}(\gamma_k,\gamma_l)\right]. \tag{36}$$

This precise expression just reveals the relationship between the Chern-Simons action and the self-linking and linking numbers of the N-knot family [9,10]. Since the self-linking and linking numbers are both invariant characteristic numbers of the knotlike closed curves in topology, I is an important invariant required to describe the knotlike vortex lines in Chern-Simons field theory.

IV. THE CONSERVATION OF THE CHERN-SIMONS KNOT TOPOLOGICAL NUMBER

In Sec. II, we have used the regular condition $D^{\mu\nu}(\vec{\phi})\neq 0$; when this condition fails, branch processes will occur. In this section, we discuss the conservation of the Chern-Simons knot topological number in branch processes (splitting, merging, and intersection) during the evolution of Chern-Simons knots.

Generally speaking, the evolution of a Chern-Simons vortex line L can be discussed from Eq. (10). Here we fix the $x^3=z$ coordinate for simplicity and take the XOY plane as the cross section, so the intersection line between the L's evolution surface and the cross section is just the motion curve of L [12,22]. In this case the two-dimensional topological current is defined as

$$j^3=K^{03}=\delta^2(\vec{\phi})D^0(\phi/x) \tag{37}$$

and

$$K^i=K^{i3}=\delta^2(\vec{\phi})D^i(\phi/x) \quad (i=1,2), \tag{38}$$

which satisfy the continuity equation [22]

$$\partial_t j^3+\partial_i K^i=0. \tag{39}$$

The velocity of the intersection point between L and the cross section is given by

$$v^i=\frac{dx^i}{dt}=\left.\frac{D^i(\phi/x)}{D^0(\phi/x)}\right|_{\vec{x}} \quad (i=1,2), \tag{40}$$

where $D^0(\phi/x)=\epsilon_{ab}\partial_1\phi^a\partial_2\phi^b$, $D^1(\phi/x)=\epsilon_{ab}\partial_2\phi^a\partial_0\phi^b$, and $D^2(\phi/x)=\epsilon_{ab}\partial_0\phi^a\partial_1\phi^b$.

From Eq. (40) it is obvious that when

$$D^0(\phi/x)=0 \tag{41}$$

at the very point $(t^*,\vec{x}^*)$, the velocity

$$\frac{dx^1}{dt}=\left.\frac{D^1(\phi/x)}{D^0(\phi/x)}\right|_{(t^*,\vec{x}^*)}, \quad \frac{dx^2}{dt}=\left.\frac{D^2(\phi/x)}{D^0(\phi/x)}\right|_{(t^*,\vec{x}^*)} \tag{42}$$

is not unique in the neighborhood of $(t^*,\vec{x}^*)$. This very point $(t^*,\vec{x}^*)$ is called the bifurcation point. Without loss of generality, we discuss only the branch of the velocity component (dx^1/dt) at $(t^*,\vec{x}^*)$. It is known [12,15] that the Taylor expansion of the solution of Eq. (15) in the neighborhood of $(t^*,\vec{x}^*)$ can generally be expressed as $A(x^1-x^{1*})^2+2B(x^1-x^{1*})(t-t^*)+C(t-t^*)^2+\cdots=0$, where A, B, and C are three constants. (In this paper we do not consider the intersection of three knots, so we assume $A\neq 0$ and the one-split-into-three or the three-merge-into-one branch cases are not considered [12].) Then the above Taylor expansion leads to

$$A\left(\frac{dx^1}{dt}\right)^2+2B\frac{dx^1}{dt}+C=0 \quad (A\neq 0). \tag{43}$$

The solutions of Eq. (43) give different motion directions of the zero point on the cross section at the bifurcation point. There are two possible cases (the illustrations can be found in [12]).

Case 1. For $\Delta=4(B^2-AC)=0$, from Eq. (43) we get only one motion direction of the zero point on the cross section at the bifurcation point: $(dx^1/dt)|_{1,2}=-B/A$, which includes three sub-cases: (a) one vortex line split into two vortex lines; (b) two vortex lines merge into one vortex line; (c) two vortex lines tangentially intersect at the bifurcation point.

Case 2. For $\Delta=4(B^2-AC)>0$, from Eq. (43) we get two different motion directions of the zero point on the cross section: $(dx^1/dt)|_{1,2}=(-B\pm\sqrt{B^2-AC})/A$. This is the intersection of two vortex lines, which means that the two vortex lines meet and then depart at the bifurcation point.

In both cases 1 and 2, from the continuity equation (39) we know that the sum of the topological charges of final vortex line(s) must be equal to that of the initial vortex line(s) at the bifurcation point for fixed k:

$$\sum_f W_{kf}=\sum_i W_{ki}; \tag{44}$$

namely, (a) for the case that one line L split into two lines L_1 and L_2, we have $W_L=W_{L_1}+W_{L_2}$, (b) for the case that two lines L_1 and L_2 merge into one line L, $W_{L_1}+W_{L_2}=W_L$, and (c) for the case that two lines L_1 and L_2 meet and then depart as two other lines L_3 and L_4, $W_{L_1}+W_{L_2}=W_{L_3}+W_{L_4}$.

In the following we will discuss the conservation of the Chern-Simons topological number (36) in the branch processes of knots.

(i) The splitting case. We consider one knot γ split into two knots γ_1 and γ_2 which are of the same self-linking number as $\gamma[\mathcal{S}(\gamma)=\mathcal{S}(\gamma_1)=\mathcal{S}(\gamma_2))$, and then we will compare the two numbers I_γ and $I_{\gamma_1+\gamma_2}$ (where I_γ is the contribution of γ to I before splitting, and $I_{\gamma_1+\gamma_2}$ is the total contribution

of γ_1 and γ_2 to I after splitting). First, from the above text we have $W_\gamma = W_{\gamma_1} + W_{\gamma_2}$ in the splitting process. Second, on the one hand, we note that in the neighborhood of the bifurcation point $(\vec{x}^*, t^*)$, γ_1 and γ_2 are infinitesimally displaced from each other; on the other hand, for a knot γ its self-linking number $\mathcal{S}(\gamma)$ is defined as

$$\mathcal{S}(\gamma) = \mathcal{L}(\gamma, \gamma_V) \tag{45}$$

where γ_V is another knot obtained by infinitesimally displacing γ in the normal direction $\vec{V}$ [9]. Therefore

$$\mathcal{S}(\gamma) = \mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = \mathcal{L}(\gamma_1, \gamma_2) = \mathcal{L}(\gamma_2, \gamma_1), \tag{46}$$

and

$$\mathcal{L}(\gamma, \gamma_k') = \mathcal{L}(\gamma_1, \gamma_k') = \mathcal{L}(\gamma_2, \gamma_k') \tag{47}$$

[where γ_k' denotes another arbitrary knot in the family ($\gamma_k' \neq \gamma, \gamma_k' \neq \gamma_{1,2}$)]. Then, third, we can compare I_γ and $I_{\gamma_1+\gamma_2}$ as follows: before splitting, we have

$$I_\gamma = 2\pi\left[W_\gamma^2 \mathcal{S}(\gamma) + \sum_{k=1(\gamma_k' \neq \gamma)}^{N} 2 W_\gamma W_{\gamma_k'} \mathcal{L}(\gamma, \gamma_k') \right], \tag{48}$$

where $\mathcal{L}(\gamma, \gamma_k') = \mathcal{L}(\gamma_k', \gamma)$; after splitting,

$$\begin{aligned} I_{\gamma_1+\gamma_2} = 2\pi\Bigg[& W_{\gamma_1}^2 \mathcal{S}(\gamma_1) + W_{\gamma_2}^2 \mathcal{S}(\gamma_2) + 2 W_{\gamma_1} W_{\gamma_2} \mathcal{L}(\gamma_1, \gamma_2) \\ & + \sum_{k=1(\gamma_k' \neq \gamma_{1,2})}^{N} 2 W_{\gamma_1} W_{\gamma_k'} \mathcal{L}(\gamma_1, \gamma_k') \\ & + \sum_{k=1(\gamma_k' \neq \gamma_{1,2})}^{N} 2 W_{\gamma_2} W_{\gamma_k'} \mathcal{L}(\gamma_2, \gamma_k') \Bigg]. \end{aligned} \tag{49}$$

Comparing Eqs. (48) and (49), we just have

$$I_\gamma = I_{\gamma_1+\gamma_2}. \tag{50}$$

This means that in the splitting process of a knot the Chern-Simons knot topological number is conserved.

(ii) The merging case. We consider two knots γ_1 and γ_2, which are of the same self-linking number, merging into one knot γ which is of the same self-linking number as γ_1 and γ_2. This is obviously the inverse process of the above splitting case; therefore we have

$$I_{\gamma_1+\gamma_2} = I_\gamma. \tag{51}$$

(iii) The intersection case. This case is related to the collision of two knots [10,4]. We consider two knots γ_1 and γ_2, which are of the same self-linking number, meet, and then depart as other two knots γ_3 and γ_4, which are of the same self-linking number as γ_1 and γ_2. This process can be identified as two subprocesses: γ_1 and γ_2 merge into one knot γ, and then γ splits into γ_3 and γ_4. Thus from the above two cases (ii) and (i) we have

$$I_{\gamma_1+\gamma_2} = I_{\gamma_3+\gamma_4}. \tag{52}$$

Therefore we obtain the result that, in branch processes during the evolution of knotlike vortex lines (splitting, merging, and intersection), the Chern-Simons knot topological number I is preserved.

V. CONCLUSION

In this paper, using the $U(1)$ gauge potential decomposition and the ϕ-mapping topological current theory, the topology of a family of N knots in Chern-Simons field theory is studied. In Sec. II, it is revealed that there are knotlike vortex lines existing in Chern-Simons field theory. In Sec. III, we point out in Eq. (36) that the Chern-Simons action I is just a topological invariant for the knot family, i.e., it is the total sum of all the self-linking and all the linking numbers of the knot family. Furthermore, in Sec. IV it is shown that this Chern-Simons knot topological number I is preserved in the branch processes (splitting, mergence and intersection) during the evolution of these knotlike vortex lines.

Finally, it should be pointed out that in the present paper the Chern-Simons action is given on the flat Euclidean space M; however, in fact on a curved base manifold the Chern-Simons action is also defined, and is independent of the choice of metric $g_{\mu\nu}$ of the manifold [9]. The topology of knotlike strings in curved space-time, such as the Riemann-Cartan space-time, will be discussed in further papers.

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