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Majorana decomposition for two-qubit pure states

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Abstract

The Majorana representation (MR), which represents a quantum state and its evolution with the Majorana stars (MSs) on the Bloch sphere, provides us an intuitive way to study symmetric multiqubit pure states physical system with $SU(2)$ symmetry. In this work, we propose a method to extend the MR for generic two-qubit pure states. By adding an additional reference star (RS), which represents the symmetry of the qubits, on the symmetric two qubit state with two base stars, we give a more general MR for both symmetric, anti-symmetric and generic two-qubit states. Using this method, we study the entanglement, Berry phase and their relation by this Majorana decomposition. Furthermore, the symmetries for different type of two-qubit states are also discussed via the RSs and correlation between the two MSs.

Keywords: Majorana representation, two-qubit state, Berry phase, concurrence

(Some figures may appear in colour only in the online journal)

1. Introduction

Multipartite states and high-dimensional states are important resources for many fields of quantum science, such as quantum information processing and ultracold atomic physics. However, due to the large dimension of projective Hilbert space, it is hard to study these states intuitively. Recently, the Majorana representation (MR) [1] has been reintroduced to visualize high-dimensional spin states. Majorana's insight was that we can describe a spin-J state (which is equivalent to an n-body two-mode boson state or a symmetric n-qubit state with $n = 2J$) by $2J$ points on the two-dimensional Bloch sphere rather than one point on a

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high-dimensional geometric structure. These $2J$ points are called Majorana stars (MSs) of the state. Consequently, this representation becomes an efficient method to study the geometric related properties of multipartite or high-dimensional state such as entanglement [2–11], Berry phase [12–18], as well as their dynamics, geometric structures and response to geometric transformations (e.g. rotations and inversions) [19–29]. By using MR, entanglement, and Berry phase for symmetric multi-qubit states can be related to each other and intuitively represented by the trajectories and distributions of the stars on the Bloch sphere [16, 18, 30].

However, the MR can only be used to study the entanglement for a symmetric n-qubit state with $SU(2)$ symmetry, although we can similarly parameterization a generic n-dimensional state by $n-1$ stars to study its geometric phase [16] without considering carried features of symmetry. The entanglement of this multi-qubit state cannot be well defined due to the arbitrary of base vector. Recently, with the increasing attention of MR, the MR has been extended to the states with symmetries other than $SU(2)$ [31] and mixed states [32, 33]. However, there is still a lack of decomposition methods similar with MR in which the entanglement and geometric phase of a generic multi-qubit state can be studied intuitively. The symmetric base vectors for MR and only provide part of the bases for the whole Hilbert space of the multi-qubit states. There are still anti-symmetric, partly symmetric, asymmetric base vectors which are related to the symmetries and classification of the states. Therefore, it is a natural question to ask how can we establish a decomposition or representation including all this type of states? How can we use it to study the geometric properties (such as entanglement, geometric phase) of the multi-qubit states? To shed light on these questions, we study the two-qubit pure states in this work which can be generalized into more qubits in the future. For two-qubit states, the base vectors are either symmetric or anti-symmetric. Therefore, we can use the form of two MSs in the MR and add an extra reference star (RS) to represent the symmetry of the qubits. In this Majorana decomposition, the Berry phase are related to the trajectories of the two MSs and the RS. While the entanglement is determined by the distance between the two MSs and the position of the RS.

This paper is organized as follows. In section 2, we introduce the MR and how the proprieties of the states (such as Berry phase, Berry curvature, concurrence) are represented by the MSs. In section 3, we show how to establish a Majorana decomposition with two MSs and a RS for a generic two-qubit states. The star equation for these three stars are also derived. Then, the Berry phase, the Berry curvature, and the concurrence for a two-qubit state are represented by these three stars. Their relation with the RSs and the distance and distribution of the MSs are studied. We also discuss some simple cases to show how the RS is related to the symmetry. A brief conclusion and discussion about the passible decomposition for three qubits are given in section 4.

2. Majorana representation for symmetric two-qubit pure state

For a generic symmetric two-qubit pure state, it can be factorized in MR as

$$
\begin{aligned}
|\Psi\rangle_{\text{sym}}^{(2)} &= C_1 |\uparrow\rangle |\uparrow\rangle + C_0 \left(|\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\uparrow\rangle\right) + C_{-1} |\downarrow\rangle |\downarrow\rangle \\
&= \frac{1}{N_2(\boldsymbol{U})} \left(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + |\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle\right), \qquad (1)
\end{aligned}
$$

where

$$
N_2(\boldsymbol{U}) = \sqrt{(3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2)} \qquad (2)
$$

is the normalization constant, $|\uparrow\rangle$ and $|\downarrow\rangle$ are σ_z eigenstates for a single spin-$1/2$, C_m are the probability amplitudes ($m = 1, 0, -1$), and $|\boldsymbol{u}_j\rangle = \cos\frac{\theta_j}{2}|\uparrow\rangle + \sin\frac{\theta_j}{2}e^{i\phi_j}|\downarrow\rangle$ ($j = 1, 2$) are the qubit states of star $\boldsymbol{u}_j = (\theta_j, \phi_j)$ which can be determined by the roots $x_j \equiv \tan\frac{\theta_j}{2}e^{i\phi_j}$ of star equation [1, 16, 18]

$$\sum_{k=0}^{2J} \frac{(-1)^k C_{J-k}}{(2J-k)!k!} x^{2J-k} = 0 \tag{3}$$

with $J = 1$. These stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ yield many useful insights for the characters of the quantum state $|\Psi\rangle^{(2)}_{\text{sym}}$. For example, the Berry connection of $|\Psi\rangle^{(2)}_{\text{sym}}$ is [16, 18, 21]

$$A^{\text{sym}}_{\mu} = -\sum_{j=1}^{2} \frac{(1-\cos\theta_j)\partial_\mu\phi_j}{2} + \frac{1}{2}\frac{\boldsymbol{u}_1 \times \boldsymbol{u}_2 \cdot \partial_\mu(\boldsymbol{u}_2 - \boldsymbol{u}_1)}{3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}. \tag{4}$$

Therefore, the Berry phase accumulated in an adiabatic cyclic evolution of the state $|\Psi\rangle^{(2)}_{\text{sym}}$ takes the form

$$\gamma_{\text{sym}} = -\frac{\Omega_1}{2} - \frac{\Omega_2}{2} + \frac{1}{2}\oint \frac{\boldsymbol{u}_1 \times \boldsymbol{u}_2 \cdot (d\boldsymbol{u}_2 - d\boldsymbol{u}_1)}{3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}, \tag{5}$$

which not only contains the solid angles $\Omega_{\boldsymbol{u}_j} = \oint d\Omega_{\boldsymbol{u}_j} \equiv \oint(1-\cos\theta_j)d\phi_j$ subtended by the closed evolution paths of the MSs on the Bloch sphere like the Berry phase for a spin-1/2 state but is also associated with the correlation between the stars. The corresponding Berry curvature can also be represented by the vector operation of stars as [21]

$$\begin{aligned} F^{\text{sym}}_{\mu\nu} = -\frac{1}{(3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2)^2} \big[& 2\partial_\mu\boldsymbol{u}_1 \times \partial_\nu\boldsymbol{u}_1 \cdot \boldsymbol{u}_1 + 2\partial_\mu\boldsymbol{u}_2 \times \partial_\nu\boldsymbol{u}_2 \cdot \boldsymbol{u}_2 \\ & + (\partial_\mu\boldsymbol{u}_2 \times \partial_\nu\boldsymbol{u}_1 + \partial_\mu\boldsymbol{u}_1 \times \partial_\nu\boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_2) \big]. \end{aligned} \tag{6}$$

The Berry phase can thus be written alternatively as $\gamma_{\text{sym}} = \int_S \sum_{\mu,\nu} \frac{1}{2} F_{\mu\nu} d\mu \wedge d\nu$.

Besides, the entanglement between the two qubits, as another unique character of quantum state, can be intuitively measured by the concurrence [34]

$$\mathcal{C}_{\text{sym}} = 2|C_0^2 - C_1 C_{-1}| = \frac{d_{12}}{N_2^2(\boldsymbol{U})}, \tag{7}$$

where $d_{12} = 1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2$ characterize the distance between the two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. If the state $|\Psi\rangle^{(2)}_{\text{sym}}$ is separable, the two stars must overlap at one point ($\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}$) with $|\Psi\rangle^{(2)\text{sym}} = |\boldsymbol{u}\rangle|\boldsymbol{u}\rangle$. While the two qubits will be maximal entangled ($\mathcal{C} = 1$), iff the two stars are symmetrical about the center of the Bloch sphere. The state $|\Psi\rangle^{(2)}_{\text{sym}}$ becomes

$$\frac{1}{\sqrt{2}}(|\boldsymbol{u}\rangle\,|-\boldsymbol{u}\rangle + |-\boldsymbol{u}\rangle\,|\boldsymbol{u}\rangle), \tag{8}$$

where the two-qubit states are orthogonal, i.e. $\langle\boldsymbol{u}| - \boldsymbol{u}\rangle = 0$. Thus, the trajectories and distributions of MSs provide an intuitive way to study the symmetric related properties of a symmetric two-qubit pure state. Next, we discuss how to extend this representation to a generic two qubit pure state.

3. Majorana decomposition for two-qubit pure state

For a generic two-qubit pure state

$$|\psi\rangle = C_1 |\uparrow\rangle |\uparrow\rangle + C_2 |\uparrow\rangle |\downarrow\rangle + C_3 |\downarrow\rangle |\uparrow\rangle + C_4 |\downarrow\rangle |\downarrow\rangle , \tag{9}$$

(suppose $C_1 > 0$ and $|C_1|^2 + |C_2|^2 + |C_3|^2 + |C_4|^2 = 1$), it can be represented by the symmetric basis ($|\uparrow\rangle |\uparrow\rangle$, $1/\sqrt{2}\left(|\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\uparrow\rangle\right)$, $|\downarrow\rangle |\downarrow\rangle$) and the antisymmetric basis

$$|\Psi\rangle^{(2)}_{\text{asym}} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle\right). \tag{10}$$

Since the MR can represent all the symmetric states, a generic two-qubit pure state can be consequently decomposed by

$$|\psi\rangle = \cos\frac{\alpha}{2}|\Psi\rangle^{(2)}_{\text{sym}} + \sin\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\beta}|\Psi\rangle^{(2)}_{\text{asym}} \tag{11}$$

with two MSs $\boldsymbol{u}_1 = (\theta_1, \phi_1)$ and $\boldsymbol{u}_2 = (\theta_2, \phi_2)$ for $|\Psi\rangle^{(2)}_{\text{sym}}$ on the Bloch sphere S_{M} which can be determined by the roots $x_j \equiv \tan\frac{\theta_j}{2}\,\mathrm{e}^{\mathrm{i}\phi_j}$ of star equation (see appendix A for details)

$$C_1 x^2 - (C_2 + C_3)x + C_4 = 0, \tag{12}$$

and an extra star $\bar{\boldsymbol{u}} = (\alpha, \beta)$ on another sphere $S_{\bar{\boldsymbol{u}}}$ which represents the superposition of the symmetric part and the antisymmetric part. Its coordinates can be derived by (see appendix A for details)

$$\bar{x} \equiv \tan\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\beta} = \frac{C_2 - C_3}{\sqrt{2 - |C_2 - C_3|^2}}. \tag{13}$$

As shown in figure 1, we can use two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ to represent the symmetric part $|\Psi\rangle^{(2)}_{\text{sym}}$ on the Bloch sphere S_{M} with north pole $|\uparrow\rangle$ and south pole $|\downarrow\rangle$. Together with this representation, the state $|\psi\rangle$ is represented by point $\bar{\boldsymbol{u}}$ on the sphere $S_{\bar{\boldsymbol{u}}}$ with the north pole $|\Psi\rangle^{(2)}_{\text{sym}}$ and the south pole $|\Psi\rangle^{(2)}_{\text{asym}}$. Thus, a generic two-qubit pure state can be perfectly represented by these three stars on two Bloch spheres.

Besides, the adiabatic evolution $|\psi\rangle$ can also be described by the adiabatic evolution and trajectories of the stars $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\bar{\boldsymbol{u}}$. The Berry phase accumulated in this adiabatic evolution can be written as (see appendix A for details)

$$\gamma = -\frac{\Omega_{\bar{\boldsymbol{u}}}}{2} + \oint \frac{1 + \bar{u}_z}{2}\sum_{\mu} A^{\text{sym}}_{\mu}\,\mathrm{d}\mu \tag{14}$$

with $\bar{u}_z = \cos\alpha$ and a solid angle $\Omega_{\bar{\boldsymbol{u}}}$ subtended by the closed evolution paths of $\bar{\boldsymbol{u}}$. The integration of the symmetric Berry connection is weighted by the symmetric proportion $\frac{1+\bar{u}_z}{2}$. The Berry curvature now takes the form

$$F_{\mu\nu} = -\frac{1}{2}\partial_\mu\bar{\boldsymbol{u}} \times \partial_\nu\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{u}} + \frac{1 + \bar{u}_z}{2}F^{\text{sym}}_{\mu\nu} + \frac{1}{2}(\partial_\mu\bar{u}_z A^{\text{sym}}_\nu - \partial_\nu\bar{u}_z A^{\text{sym}}_\mu), \tag{15}$$

which contains not only the symmetric Berry curvature weighted by the symmetric proportion $\frac{1+\bar{u}_z}{2}$ but also the correlation between the MSs $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and the point $\bar{\boldsymbol{u}}$.

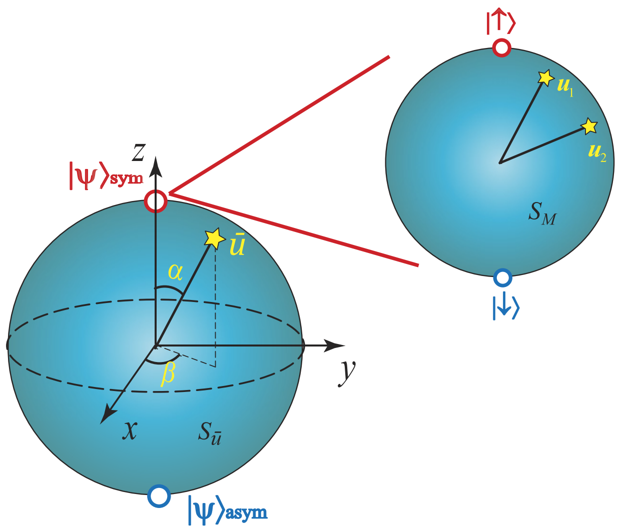

Figure 1. Schematic illustration of the Majorana decomposition of a generic two-qubit state.

Furthermore, the distribution of the MSs $\boldsymbol{u}_1, \boldsymbol{u}_2$ and star $\bar{\boldsymbol{u}}$ can also represent the concurrence between the two stars

$$\mathcal{C} = \left|\cos^2\frac{\alpha}{2}\mathcal{C}_{\text{sym}}\mathrm{e}^{\mathrm{i}\delta} - \sin^2\frac{\alpha}{2}\mathrm{e}^{2\mathrm{i}\beta}\right| \tag{16}$$

with $\delta = \pi + \phi_1 + \phi_2 + 2\arctan(\sin\frac{\theta_1+\theta_2}{2}\tan\frac{\phi_1-\phi_2}{2}/\sin\frac{\theta_1-\theta_2}{2})$. Note that, the expressions of Berry phase and concurrence in decomposition (11) seem rather complex and cannot be mapping intuitively to the evolution and distribution of stars. Therefore, we separate the decomposition (11) into two type of states to find a more intuitive form of Majorana decomposition.

3.1. States with two overlapped Majorana stars $u_1 = u_2 = u$

The simplest case of the MSs is the two stars are coincident on the sphere S_{M}. The corresponding state becomes

$$|\psi\rangle_{\mathrm{O}} = \cos\frac{\alpha}{2}|\boldsymbol{u}\rangle|\boldsymbol{u}\rangle + \sin\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}\beta}\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right) \tag{17}$$

with $\alpha \in [0, \pi)$ (to remove the repetition of the two different types, we suppose that $\bar{\boldsymbol{u}}$ cannot reach the south poles).

As shown in figure 2, the north pole $|\boldsymbol{u}\rangle|\boldsymbol{u}\rangle$ of the sphere $S_{\bar{\boldsymbol{u}}}$ can now be represented by one star on the sphere S_{M}. Therefore, the dynamic evolution of the state $|\psi\rangle_{\mathrm{O}}$ can be represented by the trajectories of $\bar{\boldsymbol{u}}$ about a rotating axis $\boldsymbol{u}$. When the MS $\boldsymbol{u}$ is fixed, the dynamics of $|\psi\rangle_{\mathrm{O}}$ is just the same with that of a spin-1/2 state represented by star $\boldsymbol{u}$. As the state adiabatic evolves, the Berry phase

$$\gamma_{\mathrm{O}} = -\frac{1}{2}\Omega_{\bar{\boldsymbol{u}}} - \frac{1}{2}\oint(1+\bar{u}_z)\mathrm{d}\Omega_{\boldsymbol{u}}, \tag{18}$$

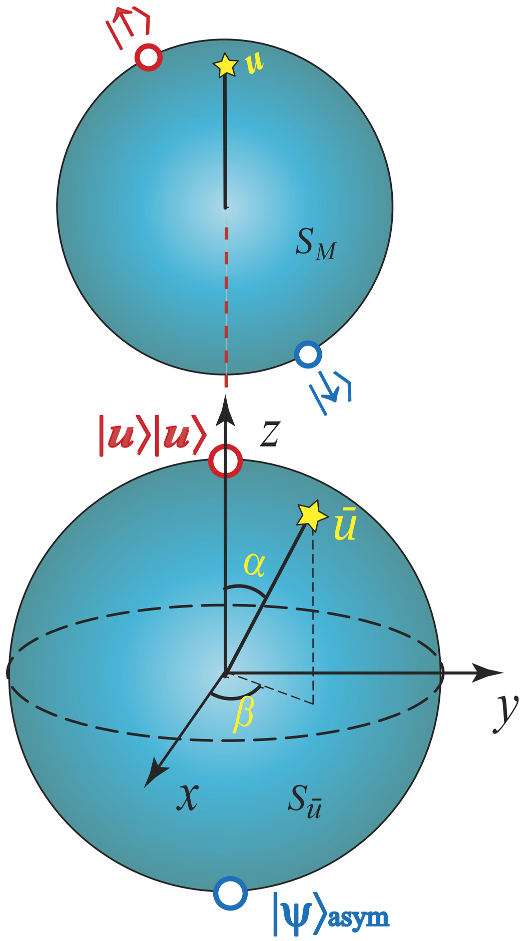

Figure 2. Schematic illustration of the Majorana decomposition for a two-qubit state with $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}$.

simply reduces to the weighted sums of the solid angle $\mathrm{d}\Omega_{\boldsymbol{u}} \equiv (1 - \cos\theta)\mathrm{d}\phi$ of MSs $\boldsymbol{u}$ and $\Omega_{\bar{\boldsymbol{u}}} \equiv \oint(1 - \cos\alpha)\mathrm{d}\beta$ of point $\bar{\boldsymbol{u}}$. If one of the two stars $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$ is fixed, the Berry phase is only determined (weighted) by the solid angle subtended by the trajectory of the other star. Furthermore, the entanglement between the two qubits can be directly decided by the latitude of the star $\bar{\boldsymbol{u}}$ as

$$\mathcal{C}_{\mathrm{O}} = \frac{1}{2}(1 - \cos\alpha), \tag{19}$$

the lower $\bar{\boldsymbol{u}}$'s latitude $(\pi/2 - \alpha)$ is, the more the two qubit entangles. Note that the two-qubit state in this case can reach the separable state but not the maximal entangled state with the limitation of $\alpha \in [0, \pi)$.

3.2. States with two separated Majorana stars $u_1 \neq u_2$

If the two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ do not coincide, the Berry phase and the entanglement of the state $|\psi\rangle_2$ is rather complex under the decomposition (11) as we discussed. Notice that, the antisymmetric basis can also be represented by (see appendix B for details)

$$|\Psi\rangle^{(2)}_{\mathrm{asym}} \sim \frac{1}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle - |\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle), \tag{20}$$

with the two MSs for the symmetric states $|\Psi\rangle^{(2)}_{\mathrm{sym}}$ (this representation is clearly not valid for $\boldsymbol{u}_1 = \boldsymbol{u}_2$). We can alternatively define the Majorana decomposition as (see appendix B for details)

$$|\psi\rangle_{\mathrm{S}} = \frac{1}{N_{\mathrm{S}}}\left(\cos\frac{\theta_3}{2}|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + \sin\frac{\theta_3}{2}\,\mathrm{e}^{\mathrm{i}\phi_3}|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle\right). \tag{21}$$

The normalization coefficient becomes $N_{\mathrm{S}} = [1 + (1+\boldsymbol{u}_1\cdot\boldsymbol{u}_2)u_{3x}/2]^{1/2}$ which is related to the distance between the same MSs $\boldsymbol{u}_1 = (\theta_1, \phi_1)$ and $\boldsymbol{u}_2 = (\theta_2, \phi_2)$ on the sphere S_{M} determined by the roots $x_j \equiv \tan\frac{\theta_j}{2}\,\mathrm{e}^{\mathrm{i}\phi_j}$ of equation (12) and the x coordinate $u_{3x} = \sin\theta_3\cos\phi_3$ of a RS $\boldsymbol{u}_3 = (\theta_3, \phi_3)$ on sphere S_{R}. The coordinates of $\boldsymbol{u}_3$ can be derived by (see appendix B for details)

$$x_3 = \tan\frac{\theta_3}{2}\,\mathrm{e}^{\mathrm{i}\phi_3} = -\frac{C_3 - C_2 - \sqrt{(C_2+C_3)^2 - 4C_1C_4}}{C_3 - C_2 + \sqrt{(C_2+C_3)^2 - 4C_1C_4}}. \tag{22}$$

Similar with $\bar{\boldsymbol{u}}$, the RS $\boldsymbol{u}_3$ measures the symmetry of the state $|\psi\rangle_{\mathrm{S}}$ (see figure 3(a)). For $\boldsymbol{u}_3 = (\pi/2, 0)$ and $(\pi/2, \pi)$, the two-qubit state become the symmetric state $|\Psi\rangle_{\mathrm{sym}}$ and the antisymmetric state $|\Psi\rangle_{\mathrm{asym}}$ respectively (as shown in figures 3(b) and (c)). While the separable states $|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle$ and $|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle$ can be represented by $\boldsymbol{u}_3 = (0, 0)$, and $(0, \pi)$ respectively (as shown in figures 3(d) and (e)). The Berry phase

$$\gamma_{\mathrm{S}} = -\sum_{i=1}^{3}\frac{\Omega_{\boldsymbol{u}_i}}{2} + \frac{1}{2}\sum_{i,j=1}^{2}\oint\frac{\boldsymbol{u}_i\times\boldsymbol{u}_j\cdot\mathrm{d}\boldsymbol{u}_j}{2N_{\mathrm{S}}^2}u_{3x} + \oint\frac{1+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}{4N_{\mathrm{S}}^2}(\boldsymbol{u}_3\times\mathrm{d}\boldsymbol{u}_3)_x \tag{23}$$

can also be interpreted as the solid angles of the two MSs and their correlation (modulated by the x coordinate of the RS) like that for the symmetric state [16, 18]. Besides, it now contains the contribution of the solid angles of the RS and its correlation with the pair of the MSs. Similarly, the Berry curvature

$$\begin{aligned}
F^{\mathrm{S}}_{\alpha\beta} = &-\frac{1+u_{3x}}{2[2+(1+\boldsymbol{u}_1\cdot\boldsymbol{u}_2)u_{3x}]^2}\{2\partial_\alpha\boldsymbol{u}_1\times\partial_\beta\boldsymbol{u}_1\cdot\boldsymbol{u}_1 + 2\partial_\alpha\boldsymbol{u}_2\times\partial_\beta\boldsymbol{u}_2\cdot\boldsymbol{u}_2 \\
&+ u_{3x}(\partial_\alpha\boldsymbol{u}_2\times\partial_\beta\boldsymbol{u}_1 + \partial_\alpha\boldsymbol{u}_1\times\partial_\beta\boldsymbol{u}_2)\cdot(\boldsymbol{u}_1+\boldsymbol{u}_2)\} \\
&+ \frac{1}{2[2+(1+\boldsymbol{u}_1\cdot\boldsymbol{u}_2)u_{3x}]^2} \\
&\times\{2\partial_\alpha(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)(\boldsymbol{u}_3\times\partial_\beta\boldsymbol{u}_3)_x + 2\partial_\beta(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)(\boldsymbol{u}_3\times\partial_\alpha\boldsymbol{u}_3)_x \\
&- (1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2)(3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2)\partial_\alpha\boldsymbol{u}_3\times\partial_\beta\boldsymbol{u}_3\cdot\boldsymbol{u}_3 \\
&+ 2\boldsymbol{u}_2\times\boldsymbol{u}_1\cdot\left[\partial_\beta(\boldsymbol{u}_1-\boldsymbol{u}_2)\partial_\alpha u_{3x} - \partial_\alpha(\boldsymbol{u}_1-\boldsymbol{u}_2)\partial_\beta u_{3x}\right]\}
\end{aligned} \tag{24}$$

can be decomposed into two parts. One part is the correlation between the two MSs modulated by the RS, the other part is caused by the adiabatic evolution of the RS. To illustrate the modulation of RS onto the two MSs, we discuss some specific situations for $|\psi\rangle_2$ with different symmetry.

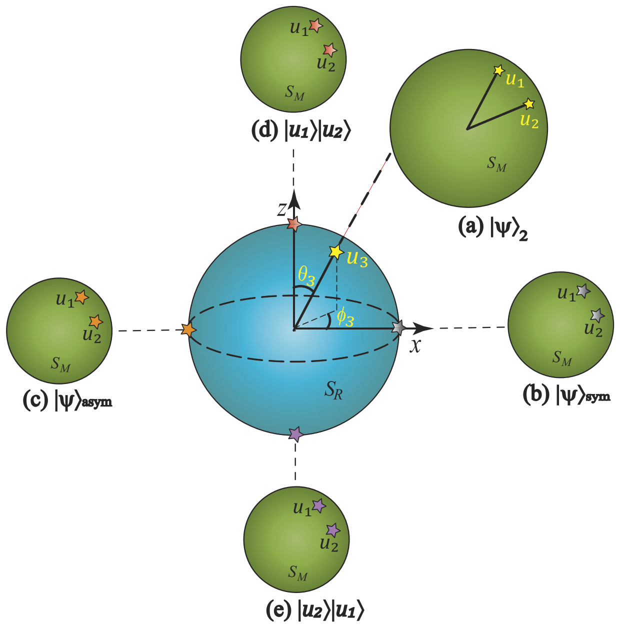

Figure 3. Schematic illustration of the Majorana decomposition with $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$ for (a) a two-qubit state $|\psi\rangle_2$, (b) the symmetric state $|\Psi\rangle^{(2)}_{\mathrm{sym}}$, (c) the antisymmetric state $|\Psi\rangle^{(2)}_{\mathrm{asym}}$, the separable state (d) $|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle$ and (e) $|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle$.

(a) The RS $\boldsymbol{u}_3 = (\theta_3, \phi_3)$ is fixed on the Bloch sphere. As $\boldsymbol{u}_3$ is time independent, the Berry phase and curvature become

$$\gamma_{\mathrm{S}} = -\sum_{i=1}^{2} \frac{\Omega_{\boldsymbol{u}_i}}{2} + \frac{1}{2}\sum_{i,j=1}^{2} \oint \frac{\boldsymbol{u}_i \times \boldsymbol{u}_j \cdot \mathrm{d}\boldsymbol{u}_j}{2N_2^2} u_{3x}, \tag{25}$$

and

$$F^{\mathrm{S}}_{\alpha\beta} = -\frac{1 + u_{3x}}{2[2 + (1 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2)u_{3x}]^2} \left[2\partial_\alpha \boldsymbol{u}_1 \times \partial_\beta \boldsymbol{u}_1 \cdot \boldsymbol{u}_1 + 2\partial_\alpha \boldsymbol{u}_2 \times \partial_\beta \boldsymbol{u}_2 \cdot \boldsymbol{u}_2 + u_{3x}(\partial_\alpha \boldsymbol{u}_2 \times \partial_\beta \boldsymbol{u}_1 + \partial_\alpha \boldsymbol{u}_1 \times \partial_\beta \boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_2)\right]. \tag{26}$$

For $\boldsymbol{u}_3 = (\pi/2, 0)$, the results become the one for symmetric states (5) and (6). On the contrary, the antisymmetric state (10) (with $\boldsymbol{u}_3 = (\pi/2, \pi)$) has zero curvature

$$F^{\mathrm{S}}_{\alpha\beta} = 0, \tag{27}$$

since the states $|\psi\rangle_{\mathrm{S}}$ now differs only a total phase $\mathrm{e}^{-\mathrm{i}\,\arg(x_2 - x_1)}$ from the antisymmetric state (10) with all the three stars fixed on the Bloch sphere (see appendix B for details).

For separable state $|\psi\rangle = |\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle$ with $\boldsymbol{u}_3 = (0, 0)$ or $(0, \pi)$, it corresponds to two stars on the Bloch sphere with no correlation, the Berry phase

$$\gamma_{\mathrm{SM}} = -\sum_{i=1}^{2} \frac{\Omega_{\boldsymbol{u}_i}}{2} \tag{28}$$

becomes the sum of the two solid angles of the two stars, and the Berry curvature becomes

$$F^{\mathrm{S}}_{\alpha\beta} = -\frac{1}{2}\partial_\alpha \boldsymbol{u}_1 \times \partial_\beta \boldsymbol{u}_1 \cdot \boldsymbol{u}_1 - \frac{1}{2}\partial_\alpha \boldsymbol{u}_2 \times \partial_\beta \boldsymbol{u}_2 \cdot \boldsymbol{u}_2. \tag{29}$$

(b) The two stars are symmetrical about the center of S_{M} ($\boldsymbol{u}_1 = -\boldsymbol{u}_2 = \boldsymbol{u}$). The state $|\psi\rangle_{\mathrm{S}}$ becomes

$$|\psi\rangle_{\mathrm{S}} = \cos\frac{\theta_3}{2}|\boldsymbol{u}\rangle\,|-\boldsymbol{u}\rangle + \sin\frac{\theta_3}{2}\,\mathrm{e}^{\mathrm{i}\phi_3}\,|-\boldsymbol{u}\rangle\,|\boldsymbol{u}\rangle. \tag{30}$$

In this case, the phases accumulated by the adiabatic evolutions of the two MSs cancel each other. The Berry phase only contains the solid angle of the RS, and the Berry curvature takes the form

$$F^{\mathrm{S}\boldsymbol{u}}_{\alpha\beta} = -\frac{1}{2}\partial_\alpha \boldsymbol{u}_3 \times \partial_\beta \boldsymbol{u}_3 \cdot \boldsymbol{u}_3. \tag{31}$$

Therefore, this case is just the opposite of case (a) in which the Berry phase is determined by the adiabatic evolution of the two MSs only. These two different cases reveal different symmetries as interpreted by the Berry curvatures (6), (27), (29) and (31).

Not only the symmetry of the adiabatic evolution of $|\psi\rangle_{\mathrm{S}}$ can be measured by the RS $\boldsymbol{u}_3$, the entanglement of the two qubits in $|\psi\rangle_{\mathrm{S}}$ can also be classified by it. The concurrence for $|\psi\rangle_{\mathrm{S}}$ can be written as

$$C = \frac{(1 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2)\sin\,\theta_3}{2 + (1 + \boldsymbol{u}_1\cdot\boldsymbol{u}_2)\sin\,\theta_3\,\cos\,\phi_3}, \tag{32}$$

where the entanglement of the two qubits is determined by the distance between the two MSs and the RSs $\boldsymbol{u}_3$ together. For fixed RS, the entanglement in $|\psi\rangle_{\mathrm{S}}$ is determined by the distance $d_{12} = 1 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2$ or the angle $\theta = \arccos(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)$ between the two MSs like the symmetric state $|\Psi\rangle_{\mathrm{sym}}$. On the contrary, the entanglement is decided by the location of $\boldsymbol{u}_3$ on the Bloch sphere when the distance between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is fixed.

As shown in figure 4, the distribution of concurrence is always symmetric about the x-axis. No matter how far the distance between the two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ ($\boldsymbol{u}_1 \neq \boldsymbol{u}_2$), the maximal entangled states $|\Psi\rangle_{\mathrm{asym}}$, separable states $|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle$, and $|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle$ are always correspond to $\boldsymbol{u}_3 = (\pi/2, \pi)$ locating at the negative x-axis, $\boldsymbol{u}_3 = (0, 0)$ locating at the north pole, and $\boldsymbol{u}_3 = (\pi, 0)$ locating at the south pole, respectively. For the other area, the closer $\boldsymbol{u}_3$ to the equator on the Bloch sphere, the larger the two qubits entangled. The entanglement between the two qubits becomes larger as the distance between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ increases. Especially, when distance between the two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is maximal (i.e. $\boldsymbol{u}_1 = -\boldsymbol{u}_2 = \boldsymbol{u}$), the concurrence becomes $C = \sin\theta_3$. In this case, the entanglement between the two qubits is only determined by the latitude of $\boldsymbol{u}_3$. All the maximal entangled states are located on the equator (including $|\Psi\rangle_{\mathrm{asym}}$) and can be represented by

$$|\psi\rangle_{\mathrm{max}} = \frac{1}{\sqrt{2}}(|\boldsymbol{u}\rangle\,|-\boldsymbol{u}\rangle + \mathrm{e}^{-\mathrm{i}\phi_3}\,|-\boldsymbol{u}\rangle\,|\boldsymbol{u}\rangle) \tag{33}$$

as the MSs locates symmetrically about the center of S_{M} (see appendix C for details). Unlike the maximal entangled state $|\psi\rangle_{\mathrm{Sch}} = \frac{1}{\sqrt{2}}(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + \mathrm{e}^{-\mathrm{i}\phi_3}\,|-\boldsymbol{u}_1\rangle\,|-\boldsymbol{u}_2\rangle)$ in the representation of Schmidt decomposition, $|\psi\rangle_{\mathrm{max}}$ in the Majorana decomposition only need three parameters $\{\boldsymbol{u} = (\theta_1, \phi_1); \phi_3\}$ to represent all the maximal entangled states (the discussion of minimal real-coefficient number for a generic maximal entangled two qubit pure states can be found in

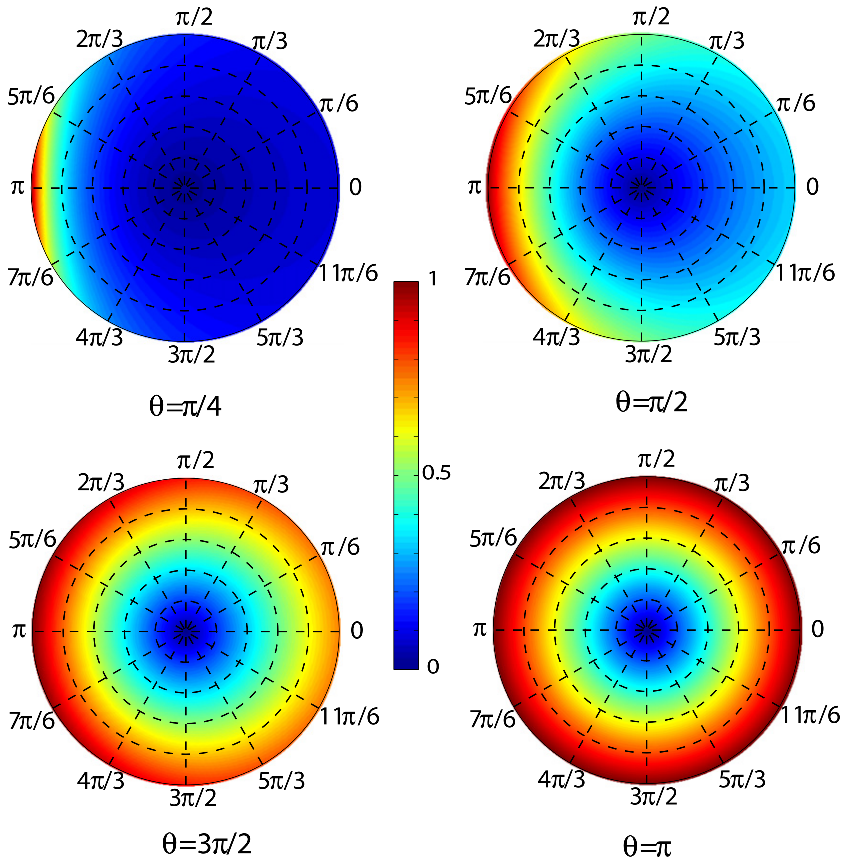

Figure 4. Top-viewed spherical distribution of the concurrence for $|\psi\rangle_S$ on the reference sphere S_R with different angles between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

appendix C). Furthermore, in Majorana decomposition, it is easy to find that the topology of the maximal entangled states $|\psi\rangle_{\max}$ and the separable state $|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle$ are quit different with other entangled state represented by equations (17) and (21). The former corresponds to an $SU(3)$ rotation $D^{1/2}(\boldsymbol{u}, \phi)$ on S_M, while the latter is two independent vectors on S_M $[SU(2) \times SU(2)]$.

4. Conclusion and discussion

The MR provides us an intuitive way to study the symmetric multi-qubit states with $SU(2)$ symmetry and their evolution. In this work, we show that we can use the form of two MSs in the MR and add an extra RS which represents the symmetry of the qubits to generalize MR into the generic two-qubit pure states. In this Majorana decomposition, the Berry phase, Berry curvature, and entanglement are found to be related to the trajectories and distances of the two MSs and the position of the RS. Furthermore, we show that the RS is related to the symmetry by discussing different type of states. For the states with more qubits, there are more types of symmetries which relate to the classification of entanglement and geometric phase [35]. For example, according to types of entanglement, a generic three-qubit state contains three types of states: GHZ type, W type, biseparable type, separable type [36]. There is no antisymmetric state for three qubits, while the three-qubit symmetric state takes the form

$$|\Psi\rangle_{\text{sym}}^{(3)} = \frac{1}{\sqrt{6}} \big(|\boldsymbol{u}_1\rangle\, |\boldsymbol{u}_2\rangle\, |\boldsymbol{u}_3\rangle + |\boldsymbol{u}_1\rangle\, |\boldsymbol{u}_3\rangle\, |\boldsymbol{u}_2\rangle + |\boldsymbol{u}_2\rangle\, |\boldsymbol{u}_1\rangle\, |\boldsymbol{u}_3\rangle + |\boldsymbol{u}_2\rangle\, |\boldsymbol{u}_3\rangle\, |\boldsymbol{u}_1\rangle + |\boldsymbol{u}_3\rangle\, |\boldsymbol{u}_1\rangle\, |\boldsymbol{u}_2\rangle + |\boldsymbol{u}_3\rangle\, |\boldsymbol{u}_2\rangle\, |\boldsymbol{u}_1\rangle \big) \tag{34}$$

with three stars $\boldsymbol{u}_1, \boldsymbol{u}_2$, and $\boldsymbol{u}_3$ which only represents part of GHZ type, W type, and separable type [16, 18]. However, we need $(2^3 - 1 =)7$ stars to describe all the three-qubit pure states. One possible way to construct the three-qubit pure states from the decomposition for two-qubit pure states is that

$$\begin{aligned} &\text{1 qubit: } |\boldsymbol{u}_i\rangle\,, \\ &\text{2 qubit: } |\varphi_{12}^{\alpha}\rangle = \cos\frac{\theta_\alpha}{2}\, |\boldsymbol{u}_1\rangle\, |\boldsymbol{u}_2\rangle + \sin\frac{\theta_\alpha}{2}\, \mathrm{e}^{\mathrm{i}\phi_\alpha}\, |\boldsymbol{u}_2\rangle\, |\boldsymbol{u}_1\rangle\,, \\ &\text{3 qubit: } |\psi\rangle = \cos\frac{\theta_\gamma}{2}\, \cos\frac{\theta_\delta}{2} \left|\phi_{12,3}^{\alpha,\beta}\right\rangle + \sin\frac{\theta_\gamma}{2}\, \sin\frac{\theta_\delta}{2}\, \mathrm{e}^{\mathrm{i}(\phi_\gamma+\phi_\delta)} \left|\phi_{23,1}^{\alpha,\beta}\right\rangle \\ &\qquad + \left(\cos\frac{\theta_\gamma}{2}\, \sin\frac{\theta_\delta}{2}\, \mathrm{e}^{\mathrm{i}\phi_\delta} + \cos\frac{\theta_\delta}{2}\, \sin\frac{\theta_\gamma}{2}\, \mathrm{e}^{\mathrm{i}\theta_\gamma}\right) \left|\phi_{13,2}^{\alpha,\beta}\right\rangle, \end{aligned} \tag{35}$$

with

$$\left|\phi_{ij,k}^{\alpha,\beta}\right\rangle = \cos\frac{\theta_\beta}{2}\, \left|\varphi_{ij}^{\alpha}\right\rangle |\boldsymbol{u}_k\rangle + \sin\frac{\theta_\beta}{2}\, \mathrm{e}^{\mathrm{i}\phi_\beta}\, |\boldsymbol{u}_k\rangle \left|\varphi_{ij}^{\alpha}\right\rangle, \tag{36}$$

and four extra stars $x_\alpha = \tan\frac{\theta_\alpha}{2}\,\mathrm{e}^{\mathrm{i}\phi_\alpha}$, $x_\beta = \tan\frac{\theta_\beta}{2}\,\mathrm{e}^{\mathrm{i}\phi_\beta}$, $x_\gamma = \tan\frac{\theta_\gamma}{2}\,\mathrm{e}^{\mathrm{i}\phi_\gamma}$, and $x_\delta = \tan\frac{\theta_\delta}{2}\,\mathrm{e}^{\mathrm{i}\phi_\delta}$. We will discuss this method and its generalizations to the pure states with more qubits in future works.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Derivation of the star equations and Berry phase for a two-qubit pure states

The Majorana decomposition (11) can be described by the complex numbers $x_j = \tan\frac{\theta_j}{2}\,\mathrm{e}^{\mathrm{i}\phi_j}$ for the stars as

$$\begin{aligned} |\psi\rangle^{(2)} &= \cos\frac{\alpha}{2} |\Psi\rangle_{\text{sym}}^{(2)} + \sin\frac{\alpha}{2}\, \mathrm{e}^{\mathrm{i}\beta} |\Psi\rangle_{\text{asym}}^{(2)} \\ &= \cos\frac{\alpha}{2}\, \cos\frac{\theta_1}{2}\, \cos\frac{\theta_2}{2} \left\{ \frac{1}{\sqrt{3 + \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}} \big[(|\uparrow\rangle + x_1\, |\downarrow\rangle)(|\uparrow\rangle + x_2\, |\downarrow\rangle) \right. \end{aligned}$$

$$+ (|\uparrow\rangle + x_2 |\downarrow\rangle)(|\uparrow\rangle + x_1 |\downarrow\rangle)\big] + \frac{\bar{x}}{\sqrt{2}}(|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle)\Big\}$$

$$= \frac{2\cos\frac{\alpha}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}{\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}\Bigg[|\uparrow\rangle|\uparrow\rangle + x_1x_2|\downarrow\rangle|\downarrow\rangle + \left(\frac{x_1+x_2}{2} + \frac{\bar{x}\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}{2\sqrt{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}\right)|\uparrow\rangle|\downarrow\rangle + \left(\frac{x_1+x_2}{2} - \frac{\bar{x}\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}{2\sqrt{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}\right)|\downarrow\rangle|\uparrow\rangle\Bigg]. \tag{A.1}$$

Compare with equation (9), we find that

$$x_1 + x_2 = \frac{C_2+C_3}{C_1}, \quad x_1x_2 = \frac{C_4}{C_1}, \quad \bar{x} = \frac{(C_2-C_3)\sqrt{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}{C_1\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}. \tag{A.2}$$

Therefore, we can derive the two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ by the star equation

$$C_1x^2 - (C_2+C_3)x + C_4 = 0, \tag{A.3}$$

i.e.

$$x_{1,2} = \frac{C_2+C_3 \pm \sqrt{(C_2+C_3)^2 - 4C_1C_4}}{2C_1}. \tag{A.4}$$

Furthermore, the two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ corresponds to the symmetric state

$$|\psi\rangle^{(2)}_{\text{sym}} = \frac{1}{A}\left(C_1|\uparrow\uparrow\rangle + \frac{C_2+C_3}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + C_4|\downarrow\downarrow\rangle\right), \tag{A.5}$$

with $A = \sqrt{|C_1|^2 + |C_4|^2 + \frac{|C_2+C_3|^2}{2}} = \sqrt{1 - \frac{|C_2-C_3|^2}{2}}$. By equation (7), the concurrence of this symmetric state can be written as

$$\mathcal{C}_{\text{sym}} = 2\left|\left(\frac{C_2+C_3}{2}\right)^2 - C_1C_4\right|/A^2 = \frac{|C_1|^2|x_1-x_2|^2}{2A^2} = \frac{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}. \tag{A.6}$$

Together with

$$\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = \sqrt{\frac{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}{2|x_1-x_2|^2}}, \tag{A.7}$$

and equation (A.2), we have

$$\bar{x} = \frac{(C_2-C_3)\sqrt{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}{C_1\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}} = \frac{C_2-C_3}{C_1}\sqrt{\frac{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}{(3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2)|x_1-x_2|^2}} = \frac{C_2-C_3}{C_1}\frac{C_1}{\sqrt{2}A} = \frac{C_2-C_3}{\sqrt{2-|C_2-C_3|^2}}. \tag{A.8}$$

With this Majorana decomposition (11), and equation (4), the Berry phase accumulated by an adiabatic evolution process of $|\psi\rangle$ can be derived by

$$\begin{aligned}\gamma &= \oint \mathrm{i}\,\langle \psi|\psi\rangle \\ &= \oint \cos\frac{\alpha}{2}\mathrm{d}\left(\cos\frac{\alpha}{2}\right) + \sin\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\beta}\mathrm{d}\left(\sin\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\beta}\right) + \oint \cos^2\frac{\alpha}{2}\mathrm{i}\left\langle \Psi^{(2)}_{\mathrm{sym}}|\mathrm{d}\Psi^{(2)}_{\mathrm{sym}}\right\rangle \\ &= -\frac{\Omega_{\bar{\boldsymbol{u}}}}{2} + \oint \frac{1+\bar{u}_z}{2}\sum_{\mu} A^{\mathrm{sym}}_{\mu}\mathrm{d}\mu \end{aligned} \tag{A.9}$$

with $\Omega_{\bar{\boldsymbol{u}}} = \oint (1-\cos\,\alpha)\mathrm{d}\beta$, $\bar{u}_z = \cos\,\alpha$ and $\sum_{\mu} A^{\mathrm{sym}}_{\mu}\,\mathrm{d}\mu \equiv \mathrm{i}\left\langle \Psi^{(2)}_{\mathrm{sym}}|\mathrm{d}\Psi^{(2)}_{\mathrm{sym}}\right\rangle$.

Appendix B. Derivation of the star equations for states with two separated Majorana stars $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$

In equation (11), we know that a generic two-qubit state can be decomposed by a symmetric part and an antisymmetric part. Similar with the MR for the symmetric state, we can also define an antisymmetric state by the stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ as

$$\begin{aligned}&\frac{1}{M}(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle - |\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle) \\ &\quad= \frac{\cos\,\frac{\theta_1}{2}\,\cos\,\frac{\theta_2}{2}}{M}[(|{\uparrow}\rangle + x_1\,|{\downarrow}\rangle)(|{\uparrow}\rangle + x_2\,|{\downarrow}\rangle) - (|{\uparrow}\rangle + x_2\,|{\downarrow}\rangle)(|{\uparrow}\rangle + x_1\,|{\downarrow}\rangle)] \\ &\quad= \frac{\cos\,\frac{\theta_1}{2}\,\cos\,\frac{\theta_2}{2}}{M}(x_2 - x_1)(|{\uparrow}\rangle\,|{\downarrow}\rangle - |{\downarrow}\rangle\,|{\uparrow}\rangle) \\ &\quad= \frac{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}\,\mathrm{e}^{\mathrm{i}\phi_{21}}}{\sqrt{2}M}(|{\uparrow}\rangle\,|{\downarrow}\rangle - |{\downarrow}\rangle\,|{\uparrow}\rangle),\end{aligned} \tag{B.1}$$

where the normalization coefficient $M \equiv \sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}$ and $\phi_{21} \equiv \arg(x_2 - x_1)$ are derived by equation (A.7). Therefore, the antisymmetric state $|\Psi\rangle^{(2)}_{\mathrm{asym}}$ can be equivalently represented by

$$|\Psi\rangle^{(2)}_{\mathrm{asym}} = \frac{\mathrm{e}^{-\mathrm{i}\phi_{21}}}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle - |\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle). \tag{B.2}$$

It is worth to notice that this representation for antisymmetric state is only valid for the case that the two stars $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are separated. If $\boldsymbol{u}_1 = \boldsymbol{u}_2$, this representation cannot derive the state $\frac{1}{\sqrt{2}}(|{\uparrow}\rangle\,|{\downarrow}\rangle - |{\downarrow}\rangle\,|{\uparrow}\rangle)$. By equations (A.6),(A.8) and (B.2), the Majorana decomposition (11) can now take the form

$$\begin{aligned}|\psi\rangle &= \cos\,\frac{\alpha}{2}|\Psi\rangle^{(2)}_{\mathrm{sym}} + \sin\,\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\beta}|\Psi\rangle^{(2)}_{\mathrm{asym}} \\ &= \frac{\cos\,\frac{\alpha}{2}}{\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + |\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle) + \frac{\sin\,\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}(\beta-\phi_{21})}}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle - |\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle) \\ &= \left[\frac{\cos\,\frac{\alpha}{2}}{\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}} + \frac{\sin\,\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}(\beta-\phi_{21})}}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}\right]|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + \left[\frac{\cos\,\frac{\alpha}{2}}{\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}} - \frac{\sin\,\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}(\beta-\phi_{21})}}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}\right]|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle\end{aligned}$$

$$= \left[\frac{\cos\frac{\alpha}{2}}{\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}} + \frac{\sin\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}(\beta-\phi_{21})}}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}\right]\left(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + \frac{1-\bar{x}\,\mathrm{e}^{-\mathrm{i}\phi_{21}}\sqrt{\frac{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}}{1+\bar{x}\,\mathrm{e}^{-\mathrm{i}\phi_{21}}\sqrt{\frac{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}}|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle\right)$$

$$= \left[\frac{\cos\frac{\alpha}{2}}{\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}} + \frac{\sin\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}(\beta-\phi_{21})}}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}\right]\left(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + \frac{1-\frac{C_2-C_3}{C_1(x_2-x_1)}}{1+\frac{C_2-C_3}{C_1(x_2-x_1)}}|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle\right)$$

$$= \left[\frac{\cos\frac{\alpha}{2}}{\sqrt{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}} + \frac{\sin\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}(\beta-\phi_{21})}}{\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}\right]\left(|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + x_3|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle\right), \tag{B.3}$$

where

$$x_3 \equiv \frac{C_1(x_2-x_1)-(C_2-C_3)}{C_1(x_2-x_1)+(C_2-C_3)} = -\frac{C_3-C_2-\sqrt{(C_2+C_3)^2-4C_1C_4}}{C_3-C_2+\sqrt{(C_2+C_3)^2-4C_1C_4}}$$

$$= \frac{\sqrt{2}A\bar{x}}{C_1(x_2-x_1)}. \tag{B.4}$$

Therefore, we can alternatively define the Majorana decomposition as

$$|\psi\rangle_{\mathrm{S}} = \frac{1}{N_2}\left(\cos\frac{\theta_3}{2}|\boldsymbol{u}_1\rangle|\boldsymbol{u}_2\rangle + \sin\frac{\theta_3}{2}\,\mathrm{e}^{\mathrm{i}\phi_3}|\boldsymbol{u}_2\rangle|\boldsymbol{u}_1\rangle\right). \tag{B.5}$$

with $N_2 = [1+(1+\boldsymbol{u}_1\cdot\boldsymbol{u}_2)u_{3x}/2]^{1/2}$ and $x_3 \equiv \tan\frac{\theta_3}{2}\,\mathrm{e}^{\mathrm{i}\phi_3}$ when $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$. It is worth noticing that, even we set $\boldsymbol{u}_1 = \boldsymbol{u}_2$, we still cannot describe the type of states $|\psi\rangle_{\mathrm{O}} = \cos\frac{\alpha}{2}|\boldsymbol{u}\rangle|\boldsymbol{u}\rangle + \sin\frac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\beta}\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$ except for $|\boldsymbol{u}\rangle|\boldsymbol{u}\rangle$.

Appendix C. The minimal number of real coefficients to represent a generic maximal entangled two qubit pure states

For a generic two qubit pure state

$$|\psi\rangle = C_1|\uparrow\uparrow\rangle + C_2|\uparrow\downarrow\rangle + C_3|\downarrow\uparrow\rangle + C_4|\downarrow\downarrow\rangle, \tag{C.1}$$

the maximal entangled state satisfies the relation $\mathcal{C} = 2|C_1C_4 - C_2C_3| = 1$, i.e. $|C_1C_4 - C_2C_3|^2 = 1/4$. Note that

$$|C_1C_4 - C_2C_3|^2 \leqslant (|C_1C_4| + |C_2C_3|)^2, \quad |C_1C_4| \leqslant \frac{1}{2}(|C_1|^2 + |C_4|^2)$$

$$|C_2C_3| \leqslant \frac{1}{2}(|C_2|^2 + |C_3|^2). \tag{C.2}$$

The equal sign holds for $\arg(C_1C_4) = \pi + \arg(C_2C_3)$, $|C_1| = |C_4|$, $|C_2| = |C_3|$, respectively. When all of the three equal sign hold, we have the maximal entangled relation $2|C_1C_4 - C_2C_3| = 1$. Therefore, combine with the normalization condition and the total phase, a generic maximal entangled two-qubit pure state can be presented by three real coefficients.

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References

[1] Majorana E 1932 *Il Nuovo Cimento* **9** 43–50
[2] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
[3] Bastin T, Krins S, Mathonet P, Godefroid M, Lamata L and Solano E 2009 *Phys. Rev. Lett.* **103** 070503
[4] Mathonet P, Krins S, Godefroid M, Lamata L, Solano E and Bastin T 2010 *Phys. Rev.* A **81** 052315
[5] Aulbach M, Markham D and Murao M 2010 *New J. Phys.* **12** 073025
[6] Ribeiro P and Mosseri R 2011 *Phys. Rev. Lett.* **106** 180502
[7] Markham D J H 2010 *Phys. Rev.* A **83** 042332
[8] Wang Z and Markham D 2012 *Phys. Rev. Lett.* **108** 210407
[9] Ganczarek W, Kuś M and Życzkowski K 2012 *Phys. Rev.* A **85** 032314
[10] Wang Z and Markham D 2013 *Phys. Rev.* A **87** 012104
[11] Kam C F and Liu R B 2020 *Phys. Rev.* A **101** 032318
[12] Berry M V 1984 *Proc. R. Soc.* A **392** 45–57
[13] Hannay J H 1998 *J. Phys. A: Math. Gen.* **31** L53–9
[14] Tamate S, Ogawa K and Kitano M 2011 *Phys. Rev.* A **84** 052114
[15] Bruno P 2012 *Phys. Rev. Lett.* **108** 240402
[16] Liu H D and Fu L B 2014 *Phys. Rev. Lett.* **113** 240403
[17] Ogawa K, Tamate S, Kobayashi H, Nakanishi T and Kitano M 2015 *Phys. Rev.* A **91** 062118
[18] Liu H D and Fu L B 2016 *Phys. Rev.* A **94** 022123
[19] Ribeiro P, Vidal J and Mosseri R 2007 *Phys. Rev. Lett.* **99** 050402
[20] Ribeiro P, Vidal J and Mosseri R 2008 *Phys. Rev.* E **78** 021106
[21] Barnett R, Podolsky D and Refael G 2009 *Phys. Rev.* B **80** 024420
[22] Lamacraft A 2010 *Phys. Rev.* B **81** 184526
[23] Lian B, Ho T L and Zhai H 2012 *Phys. Rev.* A **85** 051606
[24] Cui X, Lian B, Ho T-L, Lev B L and Zhai H 2013 *Phys. Rev.* A **88** 11601
[25] Kawaguchi Y and Ueda M 2012 *Phys. Rep.* **520** 253–381
[26] Stamper-Kurn D M and Ueda M 2013 *Rev. Mod. Phys.* **85** 1191–244
[27] Yao F, Li D, Liu H, Fu L and Wang X 2017 *Sci. Rep.* **7** 15558
[28] Bruno P 2019 arXiv:1903.08166
[29] Dogra S, Vepsäläinen A and Paraoanu G S 2020 *Phys. Rev. Res.* **2** 043079
[30] Loredo J C, Broome M A, Smith D H and White A G 2014 *Phys. Rev. Lett.* **112** 143603
[31] Liu H-D, Fu L-B and Wang X-G 2017 *Commun. Theor. Phys.* **67** 611
[32] Giraud O, Braun D, Baguette D, Bastin T and Martin J 2015 *Phys. Rev. Lett.* **114** 080401
[33] Serrano-Ensástiga E and Braun D 2020 *Phys. Rev.* A **101** 022332
[34] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245–8
[35] Bastin T, Thiel C, Zanthier J V, Lamata L, Solano E and Agarwal G S 2009 *Phys. Rev. Lett.* **102** 053601
[36] Acín A, Bruß D, Lewenstein M and Sanpera A 2001 *Phys. Rev. Lett.* **87** 040401