Measure of bipartite correlation for quantum metrology

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Recently, researchers have found that not only entanglement but also quantum discord and even classical correlation can enhance the precision of parameter estimation. Thus the correlation which contributes quantum metrology should be treated as a new type of resource. In this paper, we directly construct a measure of the resource named as the *relative quantum Fisher information* (rQFI). It quantifies the improvement of QFI caused by correlations naturally. Operationally, rQFI quantifies the improvement of optimal precision achieved by joint measurement on a composite system, instead of the local measurement on the subsystem. Furthermore, rQFI itself is an alternative type of total correlation which captures effects of quantum entanglement, quantum discord, and classical correlation. It reduces to an entanglement measure for pure states. A detailed study for general two-qubit states is presented. Additionally, an alternative generalized discord measure can be extracted out of the rQFI.

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I. INTRODUCTION

High-precision measurement of parameters is significant in almost every branch of physics. In theoretical studies, it relates to the detection of gravitational waves [1], permanent electric dipole moment [2], local Lorentz invariance [3], etc. In applications, it relates to biological magnetic field detection [4,5], global positioning systems, quantum gyroscopes, etc. The theories of quantum parameter estimation [6,7] and quantum metrology [8] show that quantum Fisher information (QFI) sets the upper bound of the precision of quantum parameter estimation, and this bound can be improved from the standard quantum limit to the Heisenberg limit with the help of quantum entanglement. Therefore, entanglement is treated as the main resource in quantum metrology. There are many achievements in reaching larger QFI with highly entangled states [9] and detecting entanglement via QFI [10–14].

Recently, researchers have found that "There is more to quantum interferometry than entanglement" [15–17]: quantum discord and even classical correlation contribute to the enhancement of precision. Furthermore, there is no monotonic relationship between QFI and entanglement measures. Therefore, the correlations which can enhance the precision of quantum parameter estimation are still an interesting topic under study.

Specifically for state ρ_{AB} of bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$, two parametrization schemes are widely used. Figure 1(a) exemplifies the standard quantum metrology scheme [8]: parameters to be estimated are "imprinted" into ρ_{AB} through the product of the same rotations on every single particle. Figure 1(b) exemplifies the other scheme [16–19]: parameters to be estimated are imprinted into quantum state ρ_{AB} with local rotations in a subsystem, e.g., $\mathcal{H}_A$. We use this scheme in this paper, for its convenience to construct a local invariant measure.

In this paper, we define the correlation which can enhance the precision of quantum parameter estimation as a new type of resource, and construct a measure of it named as the *relative quantum Fisher information* (rQFI) for bipartite systems. The rQFI is defined with the difference of QFI between ρ_{AB} and $\rho_A \otimes \rho_B$, with the reduced density matrix $\rho_{A(B)} = \mathrm{tr}_{B(A)}(\rho_{AB})$. rQFI measures the enhancement of precision caused by correlations between $\mathcal{H}_A$ and $\mathcal{H}_B$. Furthermore, rQFI itself is a measure of total correlation; classical correlation, discord, and entanglement all contribute to rQFI, and rQFI reduces to an entanglement measure for pure states. A detailed study will be presented with the codistribution figure of rQFI and entanglement for two-qubit cases. In addition, we will give a discrimination about the total sensitivity, the contribution of coherence, and correlation. Finally, a comparison between quantum discord and rQFI will be presented. As a result, a generalized discord (quantum correlation) measure is extracted out of the rQFI.

II. RELATIVE QUANTUM FISHER INFORMATION

We study state ρ_{AB} of a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where the dimension of system $\mathcal{H}_A$ (ancillary $\mathcal{H}_B$) is $N(M)$. Subjecting the sensor to a local rotation $U(G,\theta) = \exp(-iG\theta)$ generated by $G \in \mathcal{H}_A$, the state evolves to $\rho_{AB}(\theta) = U(G,\theta)\rho_{AB}U^\dagger(G,\theta)$. Thus one can infer the local rotation by measure $\rho_{AB}(\theta)$ and estimate the parameter θ. The precision of θ's estimator $\theta_{\rm est}$ is bounded by QFI via $\delta^2\theta_{\rm est} \geqslant 1/[\nu\mathcal{F}(\rho_{AB},G)]$, where ν is the repetition and $\mathcal{F}(\rho_{AB},G) \equiv \mathrm{tr}[\rho_{AB}L(\rho_{AB})^2]$ is the quantum Fisher information with the symmetric logarithmic derivative (SLD) $L(\rho)$ defined by $\partial_\theta\rho = -i[G,\rho] \equiv [L(\rho)\rho + \rho L(\rho)]/2$ [6,7].

Naturally, the difference $\Delta\mathcal{F}(\rho_{AB},G) \equiv \mathcal{F}(\rho_{AB},G) - \mathcal{F}(\rho_A \otimes \rho_B, G)$ measures the enhancement of optimal

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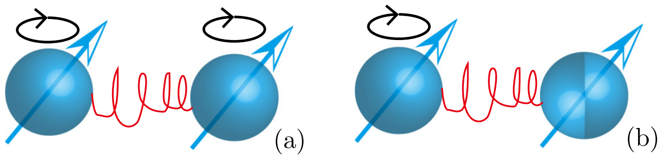

FIG. 1. Two schemes to study the contribution of correlations to quantum metrology. (a) Standard quantum metrology scheme. The parametrization process is the product of the same rotations on all of the particles. (b) Our scheme. The parametrization process is rotations on only one of those particles.

precision of $\theta_{\rm est}$ contributed by correlations between A and B. To quantify the correlation, a measure should only depend on ρ_{AB}. We let G run over the generators $\{T_i|i=1,\dots,N^2-1\}$ of the $SU(N)$ group, which imprints parameter $\vec{\theta}=(\theta_1,\theta_2,\dots,\theta_{N^2-1})$ into ρ_{AB} via the corresponding local rotation $\exp(-iT_i^A\theta_i)$, with $T_i^A=T_i\otimes I_B$. The relative quantum Fisher information for a given state ρ_{AB} is defined as

$$\Delta\mathcal{F}(\rho_{AB})\equiv\sum_i\mathcal{F}\big(\rho_{AB},T_i^A\big)-\sum_i\mathcal{F}\big(\rho_A\otimes\rho_B,T_i^A\big). \quad (1)$$

By summing up over all the generators of local rotations in $\mathcal{H}_A$, rQFI represents a total enhancement of $\vec{\theta}_{\rm est}$'s precision induced by correlations. From the perspective of measurements, $\mathcal{F}(\rho_{AB},T_i^A)$ sets the upper bound of $1/\delta^2\theta_{i,{\rm est}}$ over the set of positive operator-valued measures (POVMs) on $\mathcal{H}_{AB}$ [6,7,20]. But if only local measurement on $\mathcal{H}_A$ is allowed, the precision is bounded by $\mathcal{F}(\rho_A\otimes\rho_B,T_i^A)=\mathcal{F}(\rho_A,T_i)$ from above (see Appendix A). Therefore, $\Delta\mathcal{F}(\rho_{AB})$ quantifies the improvement of the optimal precision of $\vec{\theta}_{\rm est}$ by joint measurement on composite system AB, instead of the local measurement on subsystem A.

rQFI not only measures the effects of correlation, it is a measure of total correlation by itself. Specifically, we prove rQFI has the following properties (the proof is provided in Appendix B).

(a1) $\Delta\mathcal{F}(\rho_{AB})\geqslant 0$, where the minimum is reached and only reached by product states. When $M=N=D$, $\Delta\mathcal{F}(\rho_{AB})\leqslant 2(D-1/D)$, the maximum is reached and only reached by maximally entangled states.

(a2) $\Delta\mathcal{F}(\rho_{AB})$ is independent of the local generator's representation. Setting $T_i'^A=UT_i^AU^\dagger, i=1,\dots,N^2-1$, where $U\in SU(N)\otimes I_M$ is the local rotation in $\mathcal{H}_A$, we have $\Delta\mathcal{F}(\rho_{AB})=\Delta\mathcal{F}'(\rho_{AB})$, with $\Delta\mathcal{F}'(\rho_{AB})\equiv\sum_i\Delta\mathcal{F}(\rho_{AB},T_i'^A)$.

(a3) $\Delta\mathcal{F}(\rho_{AB})$ is invariant under local rotations in state space, i.e., $\forall U\in SU(N)\otimes SU(M)$, we have $\Delta\mathcal{F}(\rho_{AB})=\Delta\mathcal{F}(\rho'_{AB})$ with $\rho'_{AB}=U\rho_{AB}U^\dagger$.

(a4) $\Delta\mathcal{F}(\rho_{AB})$ is not increased under the local completely positive and trace preserving (CPTP) map $\mathcal{I}_A\otimes\mathcal{E}_B$ on subsystem B, i.e., $\Delta\mathcal{F}(\rho_{AB})\geqslant\Delta\mathcal{F}[\mathcal{I}_A\otimes\mathcal{E}_B(\rho_{AB})]$.

Property (a1) shows that all the correlations have positive contributions to rQFI, and the maximum of rQFI is obtained by states with maximal entanglement. Property (a2) shows that rQFI is well defined as a property of ρ_{AB} itself. Property (a3) shows that rQFI satisfies one of the necessary conditions that a correlation measure should satisfy. These properties indicate that rQFI itself is an alternative type of total correlation [21]. And by quantifying the enhancement of QFI, rQFI is a natural measure of the bipartite correlation for quantum metrology.

III. PURE STATES

In this section, we will show that rQFI reduces to an entanglement measure for pure states $\rho_{AB}=|\psi\rangle\langle\psi|$. Based on the local unitary invariance (a3), we can study the rQFI of $|\psi\rangle$ in Schmidt decomposition form $|\psi\rangle=\sum_{a=1}^D\sqrt{d_a}|a\rangle_A|a\rangle_B$, with the coefficient $\boldsymbol{d}_{|\psi\rangle}\equiv(d_1,d_2,\dots,d_D)$, $d_a\geqslant 0$, $\sum_a d_a=1$, and $D=\min\{M,N\}$. The rQFI of $|\psi\rangle$ is [22] (for details, see Appendix C)

$$\Delta\mathcal{F}(|\psi\rangle\langle\psi|)=2\sum_{a\neq b}\left(\frac{2d_ad_b}{d_a+d_b}+d_ad_b\right). \quad (2)$$

$\Delta\mathcal{F}(|\psi\rangle\langle\psi|)$ is Schur concave. This means $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)\geqslant\Delta\mathcal{F}(|\phi\rangle\langle\phi|)$ iff the Schmidt coefficient $\boldsymbol{d}_{|\psi\rangle}$ is majorized by $\boldsymbol{d}_{|\phi\rangle}$, i.e., $\boldsymbol{d}_{|\psi\rangle}\prec\boldsymbol{d}_{|\phi\rangle}$[23]. As a direct result of the Schur concavity, the bound of $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)$ is

$$0\leqslant\Delta\mathcal{F}(|\psi\rangle\langle\psi|)\leqslant 2(D-1/D), \quad (3)$$

where the lower bound is reached by product states with $\boldsymbol{d}^{\downarrow}_{|\psi\rangle}=(1,0,\dots,0)$, where $\downarrow$ indicates rearranging elements in descending order; the upper bound is reached by maximally entangled states with $\boldsymbol{d}_{|\psi\rangle}=(1/D,1/D,\dots,1/D)$.

With the above discussions, we can prove that $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)$ is a valid entanglement measure for pure states by showing it fulfills the criteria proposed in [24].

Proof. (i) $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)=0$, iff $|\psi\rangle$ is separable. This is proved with Eq. (3), where the minimum zero is reached and only reached by product states.

(ii) The local unitary invariance of $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)$ has been proved by (a3).

(iii) $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)$ is not increased under local operations and classical communication (LOCC). Nielsen's theorem [25] says that $|\psi\rangle$ transforms to $|\phi\rangle$ using LOCC iff $\boldsymbol{d}_{|\psi\rangle}\prec\boldsymbol{d}_{|\phi\rangle}$. And the Schur concavity of Eq. (2) means $\boldsymbol{d}_{|\psi\rangle}\prec\boldsymbol{d}_{|\phi\rangle}$ iff $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)\geqslant\Delta\mathcal{F}(|\phi\rangle\langle\phi|)$. Therefore, $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)$ is not increased under LOCC. ■

As an entanglement measure, $\Delta\mathcal{F}(|\psi\rangle\langle\psi|)$ is a monotone increasing function of other entanglement measures for pure states $|\psi_2\rangle$ of the two-qubit system. For example,

$$\Delta\mathcal{F}(|\psi_2\rangle\langle\psi_2|)=3C^2(|\psi_2\rangle\langle\psi_2|), \quad (4)$$

where $\Delta\mathcal{F}(|\psi_2\rangle\langle\psi_2|)=12d_1d_2$ is derived from Eq. (2), and $C(|\psi_2\rangle\langle\psi_2|)=2\sqrt{d_1d_2}$ is the concurrence [26–28].

IV. TWO-QUBIT SYSTEMS

A general two-qubit state ρ_{AB} can be expanded as

$$\rho_{AB}=\frac{\boldsymbol{I}}{4}+\vec{n}\cdot\vec{\sigma}^A+\vec{m}\cdot\vec{\sigma}^B+\sum_{ij}\beta^{ij}\sigma_i\otimes\sigma_j, \quad (5)$$

with $\sigma_i^A=\sigma_i\otimes I_2$, $\sigma_i^B=I_2\otimes\sigma_i$, $i,j=1,2,3$. The generator of rotation in $\mathcal{H}_A$ is $\{\sigma_i^A/2\}$. The analytic calculation of rQFI for general two-qubit mixed states ρ_{AB} is too complex. We study it numerically instead. By randomly generating 200 000 states in the form of Eq. (5), we get the codistribution of rQFI and concurrence as shown in Fig. 2(a). To explore more details

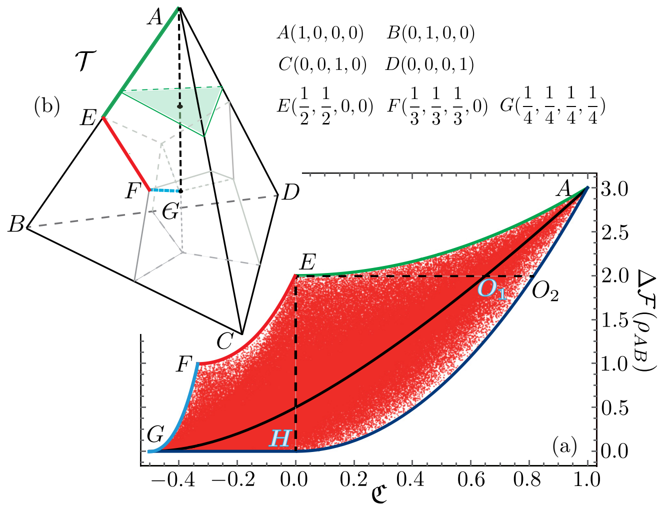

FIG. 2. rQFI ($\Delta\mathcal{F}$) vs concurrence ($\mathfrak{C}$). (a) Red dots are data of random generated two-qubit states in the general form of Eq. (5). The line AO_2H is entangled pure states. GH is direct product states. $AEFGO_1A$ is the bound of MS, where GO_1A is the well-known Werner states. (b) The tetrahedron $\mathcal{T}$ with four maximally entangled states A, B, C, and D as its vertices shows the geometry of MS in the barycentric coordinate system, and the coordinates of those points are given. For simplification, we only denote the points satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ in (b). Points in (a) and (b) with the same label represent the same states.

for separable states, we remove the truncation in the definition of concurrence to define

$$\mathfrak{C}(\rho) \equiv \tilde{\lambda}_1 - \tilde{\lambda}_2 - \tilde{\lambda}_3 - \tilde{\lambda}_4, \tag{6}$$

where $\tilde{\lambda}_i$s are eigenvalues of $R(\rho) = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ in descending order, and $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$ is the spin-flipped state [26,27]. Surprisingly, the numerical data points are bounded by some special class of states. Next we will study those bounds in detail.

A. Upper bound and maximally mixed marginal states

The line $AEFG$ in Fig. 2(a) is upper bound of the maximally mixed marginal states (MS) $\rho'^{\rm MS}_{AB} = \boldsymbol{I}/4 + \sum_{ij} \beta^{ij}\sigma_i \otimes \sigma_j$, which is local unitary equivalent to

$$\rho^{\rm MS}_{AB} = \frac{\boldsymbol{I}}{4} + \sum_i \beta_i \sigma_i \otimes \sigma_i \tag{7}$$

via singular-value decomposition. $\rho^{\rm MS}_{AB}$ is diagonal in the Bell basis, hence called Bell diagonal states [29–31] too. It is widely used in the study of correlations such as entanglement [29,30] and discord [31–33]. Its rQFI is (for details, see Appendix D1)

$$\Delta\mathcal{F}\big(\rho^{\rm MS}_{AB}\big) = 3 - 2\sum_{j\neq i} \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j}, \tag{8}$$

where $\lambda_j = 1/4 + \beta - 2\beta_j,\ j = 1, 2, 3,$ and $\lambda_4 = 1/4 - \beta$ are the eigenvalues of $\rho^{\rm MS}_{AB}$ with $\beta = \sum_{j=1}^{3} \beta_j$. $\Delta\mathcal{F}(\rho^{\rm MS}_{AB}) \equiv f(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is Schur convex, which means that $f(\boldsymbol{\lambda}) \leqslant f(\boldsymbol{\lambda}')$ iff $\boldsymbol{\lambda} \prec \boldsymbol{\lambda}'$.

Geometrically, the set of $\rho^{\rm MS}_{AB}$ can be represented by a 3-simplex (tetrahedron) with the barycentric coordinate system as shown by Fig. 2(b) [30,31,34] (for details, see Appendix D2). The point with coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ represents the MS with $\boldsymbol{\lambda}$ as its eigenvalues. We denote points corresponding to A, E, F, and G in Fig. 2(a) with the same letters. For MS, $\mathfrak{C}(\rho^{\rm MS}_{AB}) = 2\lambda_{\max} - 1$, where $\lambda_{\max}$ is the maximum element of $\boldsymbol{\lambda}$. Finding the bound of $\Delta\mathcal{F}(\rho^{\rm MS}_{AB})$ for certain $\mathfrak{C}$ is an extremum problem in the intersection of plane $\lambda_{\max} = (\mathfrak{C}+1)/2$ with tetrahedron $\mathcal{T}$. For the Schur convexity of $\Delta\mathcal{F}(\rho^{\rm MS}_{AB})$, the upper (lower) bound is reached by the vertices (center) of the intersection. For simplification, we set $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$. When λ_1 decreases from 1 to 1/4 continuously, the vertices move along AE, EF, and FG in Fig. 2(b) successively, where E, F, and G are the center of the edge, surface, and body, respectively. According to Eq. (8), the corresponding upper bounds in Fig. 2(a) are

$$\Delta\mathcal{F}^{\rm MS}_{\max} = \begin{cases} 2 + \mathfrak{C}^2 & \mathfrak{C} \in [0, 1] \\ 16/(1-\mathfrak{C}) - 9\mathfrak{C} - 14 & \mathfrak{C} \in \left[-\frac{1}{3}, 0\right) \\ -3(1+2\mathfrak{C})^2/\mathfrak{C} & \mathfrak{C} \in \left[-\frac{1}{2}, -\frac{1}{3}\right) \end{cases}. \tag{9}$$

The centers of those intersections lying in AG are Werner states with $\boldsymbol{\lambda} = (\lambda_1, (1-\lambda_1)/3, (1-\lambda_1)/3, (1-\lambda_1)/3)$, $1/4 \leqslant \lambda_1 \leqslant 1$. Therefore, MS's lower bound (AO_1G) is $\Delta\mathcal{F}^{\rm MS}_{\min} = (1+2\mathfrak{C})^2/(2+\mathfrak{C})$, $\mathfrak{C} \in [-1/2, 1]$.

$\Delta\mathcal{F}(\rho^{\rm MS}_{AB})$ is invariant under rearranging elements of $\boldsymbol{\lambda}$, and $\mathcal{T}$ has tetrahedral symmetry $\boldsymbol{T}$ geometrically. This indicates a single point in Fig. 2(a) representing several points of $\mathcal{T}$. For example, E in the codistribution figure corresponds to all the middle points of six edges of $\mathcal{T}$.

B. Lower bound and separable states

Pure state $|\psi_2\rangle$ sets the lower bound AO_2H of entangled states ($\mathfrak{C} > 0$) with Eq. (4). And the lower bound GH for separable states ($\mathfrak{C} \leqslant 0$) is given by direct product states. To show this more vividly, we study a class of separable states

$$\rho^{\rm SS}_{AB} = \frac{\boldsymbol{I}}{4} + n\sigma^A_z + m\sigma^B_z + \beta_3 \sigma_z \otimes \sigma_z. \tag{10}$$

It is diagonal in product state basis $|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle$, and $|\downarrow\downarrow\rangle$. $\rho^{\rm SS}_{AB}$ only contains classical correlations. Its rQFI is (for details, see Appendix E)

$$\Delta\mathcal{F}\big(\rho^{\rm SS}_{AB}\big) = \frac{8(\lambda_1\lambda_4 - \lambda_2\lambda_3)^2}{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)}, \tag{11}$$

where $\lambda_k = 1/4 + in + jm + ij\beta_3$ with $k = (5-i)/2 - j$, $i, j = 1, -1$ are eigenvalues of $\rho^{\rm SS}_{AB}$. $\Delta\mathcal{F}(\rho^{\rm SS}_{AB})$ is non-negative and vanishes iff $\rho^{\rm SS}_{AB}$ is product states (with $\lambda_1\lambda_4 = \lambda_2\lambda_3$). Specifically, the states $\rho^{\rm SS}_{AB}$ locate on G (H) with $\boldsymbol{\lambda}^{\downarrow} = (1/4, 1/4, 1/4, 1/4)$ [(1,0,0,0)]. We mention that $\rho^{\rm SS}_{AB}$ is a mixture of four product states with zero rQFI. Therefore, *$\Delta\mathcal{F}(\rho_{AB})$ is not convex in general cases.* It is consistent with property (a1), indicating classical correlation has a positive contribution to rQFI. We take $\rho^{\rm SS}_{AB}$ with $\lambda_2 = \lambda_3 = 0$ as an example, which is the mixture of $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. By measuring $|\uparrow\rangle_b$ and $|\downarrow\rangle_b$ in $\mathcal{H}_B$, one can separate $U(\vec{\theta})|\uparrow\rangle_a$ and $U(\vec{\theta})|\downarrow\rangle_a$ in $\mathcal{H}_A$ as conditional states for the existence of classical correlation. This provides the QFI $\lambda_1 \sum_i \mathcal{F}(|\uparrow\rangle_{aa}\langle\uparrow|, T_i) + \lambda_4 \sum_i (|\downarrow\rangle_{aa}\langle\downarrow|, T_i)$, which is larger than $\sum_i \mathcal{F}(\rho_A, T_i)$ with $\rho_A = \lambda_1 |\uparrow\rangle_{aa}\langle\uparrow| + \lambda_4 |\downarrow\rangle_{aa}\langle\downarrow|$, hence a nonzero rQFI is acquired.

Figure 2(a) shows that nonentanglement correlation (NEC), which contains classical correlation and quantum discord, contributes to rQFI. It has two signatures: separable states on the left side of EH have nonzero rQFI, and pure entangled states have the minimal rQFI when concurrence ($\mathfrak{C}$) is fixed. The contribution of classical correlation is studied above. A more general and formal study about quantum discord will be presented with an additional section below. Compared with NEC, entanglement is necessary to acquire rQFI higher than 2, which is the upper bound of rQFI provided by separable states. And the maximum of rQFI is reached by states with maximal entanglement. We mention that MS embodies the optimal combination of entanglement and NEC to reach high rQFI: on one hand, MS of rank 2 has the maximal rQFI when entanglement is fixed; on the other hand, it costs the least entanglement to reach a certain rQFI ($\geqslant 2$).

V. SENSITIVITY, rQFI, AND COHERENCE

As the sum of QFI over the generator $\{T_i^A\}$, $\mathcal{F}(\rho_{AB}) \equiv \sum_i \mathcal{F}(\rho_{AB}, T_i^A)$ is a natural measure of ρ_{AB}'s "sensitivity" to the local rotation (SLR) on the state space of $\mathcal{H}_A$, the SLR can be decomposed as

$$\mathcal{F}(\rho_{AB}) = \Delta\mathcal{F}(\rho_{AB}) + \mathcal{F}(\rho_A), \tag{12}$$

where the rQFI $\Delta\mathcal{F}(\rho_{AB})$ measures the contribution of correlation, and $\mathcal{F}(\rho_A) \equiv \sum_i \mathcal{F}(\rho_A, T_i)$ measures the contribution of ρ_A's coherence, with $\mathcal{F}(\rho_A, T_i)$ as a measure of ρ_A's coherence between T_i's eigenspaces [35,36]. For state $\rho_A = \sum_a d_a |a\rangle\langle a|$ in diagonal form, we have $\mathcal{F}(\rho_A) = 2N - \sum_{ab} 4d_a d_b/(d_a + d_b)$; it is convex, its maximum $2(N-1)$ is reached by pure states with $\boldsymbol{d}^{\downarrow} = (1, 0, \dots, 0)$, and its minimum zero is reached by maximally mixed states with $\boldsymbol{d} = (1/N, 1/N, \dots, 1/N)$.

For the convexity of QFI, the SLR $\mathcal{F}(\rho_{AB})$ is convex. Together with Eq. (C8), the maximum of SLR is $2(N - 1/D)$, reached by the maximally entangled states; its minimum zero is reached by $\boldsymbol{I}_A/N \otimes \rho_B$; and the states have no correlation and no coherence in reduced density matrix ρ_A for any orthogonal complete basis of $\mathcal{H}_A$.

Generally speaking, the local coherence is easier to manipulate than the quantum correlation. But with the help of quantum correlation, one can attain a higher SLR. Exemplified with two-qubit states ρ_{AB}, the maximum of $\mathcal{F}(\rho_A)$ is 2, and the maximal SLR contributed by the maximally entangled states is 3. If there only exists classical correlation, the maximum of $\Delta\mathcal{F}(\rho_{AB})$ is 2, attained by the state, e.g., $(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)/2$, which is equivalent to a pure state of qubit A in SLR.

VI. rQFI AND QUANTUM DISCORD

Since classical correlation contributes to the SLR, the free states of discord—the well-known classical states $\chi_{aB} = \sum_a p_a |a\rangle\langle a| \otimes \rho_{B|a}$—may have nonzero rQFI, where $\{|a\rangle\}$ is an orthogonal complete basis of $\mathcal{H}_A$ and $p_a \geqslant 0$. It depicts χ_{aB}'s "sensitivity" to the rotation generated by the nondiagonal generator $\{T_{ab}^{(i)}\}$ in the basis $\{|a\rangle\}$ with $i = 1, 2$ [22]. Apart from this difference, rQFI satisfies all other established criteria to define a generalized discord (quantum correlation) measure (GDM) listed in [37,38].

If one only focuses on the generator $\{T_{cc}^{(3)}\}$ diagonal in an unknown orthogonal complete basis $\{|e\rangle\}$ of $\mathcal{H}_A$ [22], the minimal contribution of correlation to SLR is characterized by a GDM, which is defined as

$$\delta\mathcal{F}(\rho_{AB}) = \min_{\{|e\rangle\}} \sum_c \left[\mathcal{F}\left(\rho_{AB}, T_{cc}^{(3)A}\right) - \mathcal{F}\left(\rho_A, T_{cc}^{(3)}\right)\right], \tag{13}$$

where $T_{cc}^{(3)}$ is generator diagonal in a basis $\{|e\rangle\}$, the summation is done over the set $\{T_{cc}^{(3)}\}$, and the minimization is done over all complete orthogonal bases of $\mathcal{H}_A$. $\delta\mathcal{F}(\rho_{AB})$ satisfies all of the necessary properties to define the GDM given in [37,38]. Specifically, we have the following (the proof is provided in Appendix F).

(b1) $\delta\mathcal{F}(\rho_{AB})$ is invariant under local unitary operations in state space, i.e., $\delta\mathcal{F}(\rho_{AB}) = \delta\mathcal{F}(\rho'_{AB})$, with $\rho'_{AB} = (U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger)$.

(b2) $\delta\mathcal{F}(\rho_{AB})$ is non-negative.

(b3) $\delta\mathcal{F}(\rho_{AB})$ vanishes if ρ_{AB} is classical state χ_{aB}.

(b4) $\delta\mathcal{F}(\rho_{AB})$ is not increased under the local completely positive and trace preserving map $\mathcal{I}_A \otimes \mathcal{E}_B$ on subsystem B, i.e., $\Delta\mathcal{F}(\rho_{AB}) \geqslant \Delta\mathcal{F}[\mathcal{I}_A \otimes \mathcal{E}_B(\rho_{AB})]$.

Properties (b1)–(b3) are the necessary properties to define a GDM given in [37,38]; property (b4) is also satisfied by the "interferometric power" proposed by Girolami *et al.* [16]. Hence, we claim that $\delta\mathcal{F}(\rho_{AB})$ is a qualified GDM.

VII. SUMMARY AND DISCUSSION

To summarize, we have proposed a measure named as relative quantum Fisher information to quantify the contribution of correlation for quantum metrology. We have shown that rQFI is a measure of total correlation. It can capture effects of entanglement, quantum discord, and classical correlation. rQFI reduces to an entanglement measure for pure states. A detailed study with the codistribution of rQFI and concurrence has been presented. Furthermore, a discussion about rQFI and discord has been given. As a result, an alternative generalized discord measure has been extracted out of the rQFI.

rQFI quantifies the contribution of bipartite correlation to metrology. The comparison between rQFI and entanglement, discord, etc., will help us to clarify the contributions of those correlations in metrology. Experimentally, rQFI quantifies the enhancement of SLR attained by joint measurements on a composite system over the local measurements.

The composite system of sensor and ancillary is widely used in the studies of black box estimation [16], non-Markovianity of quantum dynamics [39], quantum error correction [40–42], etc. As an alternative figure of merit of this setup, rQFI has an intrinsic connection with previous studies. The comparison between rQFI and interferometric power [16] is given above, and brief comments about rQFI and quantum error correction are given in Appendix G. Apart from those fields, rQFI has potential applications in the study of quantum coherence [36,43], quantum speed limit [44,45], asymmetry [35], etc., as QFI plays important roles in those fields, and rQFI measures the correlations' contribution to QFI quantitatively.

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APPENDIX A: PROOF OF $\mathcal{F}(\rho_A \otimes \rho_B, T_i^A) = \mathcal{F}(\rho_A, T_i)$

For

$$\begin{aligned} -i\big[T_i^A, \rho_A \otimes \rho_B\big] &= -i[T_i, \rho_A] \otimes \rho_B \\ &\equiv \tfrac{1}{2}\{L_i(\rho_A), \rho_A\} \otimes \rho_B \\ &= \tfrac{1}{2}\{L_i(\rho_A) \otimes I_M, \rho_A \otimes \rho_B\}, \end{aligned} \tag{A1}$$

where $[,]$($\{,\}$) denotes the commutation (anticommutation), one can define $L_i(\rho_A \otimes \rho_B) \equiv L_i(\rho_A) \otimes I_M$. It is a valid symmetric logarithmic derivative corresponding to generator T_i^A. Thus,

$$\begin{aligned} &\mathcal{F}\big(\rho_A \otimes \rho_B, T_i^A\big) \equiv \mathrm{tr}_{AB}\big[\rho_A \otimes \rho_B L_i^2(\rho_A \otimes \rho_B)\big] \\ &= \mathrm{tr}_A\big[\rho_A L_i^2(\rho_A)\big] \equiv \mathcal{F}(\rho_A, T_i). \end{aligned} \tag{A2}$$

APPENDIX B: PROOF OF THE PROPERTIES OF $\Delta\mathcal{F}(\rho_{AB})$

1. Proof of (a1)

a. $\Delta\mathcal{F}(\rho_{AB}, G), \Delta\mathcal{F}(\rho_{AB}) \geqslant 0$

Setting the parametrized state as $\rho_{AB}(\theta) = \exp(-iG\theta)\rho_{AB}\exp(iG\theta)$, where $G \in \mathcal{H}_A$ is the generator of local rotation, and θ is the parameter to be estimated, we have the inequality [20]

$$\mathcal{F}(\rho_{AB}, G) \geqslant \mathcal{F}^c, \tag{B1}$$

where $\mathcal{F}^c = \sum_n (\partial_\theta p_n)^2/p_n$ is the classical Fisher information of probability distribution $p_n = \mathrm{tr}_{AB}[E_n \rho_{AB}(\theta)]$, and $\{E_n\}$ is a set of POVM measurement operators in $\mathcal{H}_{AB}$. The equality in Eq. (B1) holds over $\mathcal{M}_{AB}$, the set of all of these POVM measurements. Similarly, $\mathcal{F}(\rho_A, G)$ sets the upper bound of the classical Fisher information of $q_n = \mathrm{tr}_A[\rho_A(\theta)E'_n] = \mathrm{tr}[\rho_{AB}(\theta)E'_n \otimes I]$. It is the bound of $\mathcal{F}^c$ over $\mathcal{M}_A$, the set of all local POVM measurements in the form $\{E'_n \otimes I\}$. For $\mathcal{M}_A \subset \mathcal{M}_{AB}$, we have

$$\mathcal{F}(\rho_{AB}, G) = \max\nolimits_{\mathcal{M}_{AB}} \mathcal{F}^c \geqslant \max\nolimits_{\mathcal{M}_A} \mathcal{F}^c = \mathcal{F}(\rho_A, G), \tag{B2}$$

i.e.,

$$\Delta\mathcal{F}(\rho_{AB}, G) = \mathcal{F}(\rho_{AB}, G) - \mathcal{F}(\rho_A, G) \geqslant 0, \tag{B3}$$

together with Eq.(A2). Replacing G with local rotation generator T_j^A, then summing $\Delta\mathcal{F}(\rho_{AB}, T_j^A)$ over all the generators, we have $\Delta\mathcal{F}(\rho_{AB}) \geqslant 0$.

b. The upper bound of $\Delta\mathcal{F}(\rho_{AB})$ is reached and only reached by maximally entangled states when $N = M$

For a mixed state $\rho_{AB} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ in diagonal form with $p_i > 0$ and $\sum_i p_i = 1$, we have

$$\mathcal{F}(\rho_{AB}) = \sum_i p_i \mathcal{F}(|\psi_i\rangle\langle\psi_i|) - \sum_{k;i\neq j} \frac{8p_i p_j}{p_i + p_j}|\langle\psi_i|T_k^A|\psi_j\rangle|^2. \tag{B4}$$

Based on the Schur concaveness of Eq. (C8), we have the inequality

$$\mathcal{F}(|\psi_i\rangle\langle\psi_i|) \leqslant 2(D - 1/D), \tag{B5}$$

with $D = M = N$, where the equality holds and only holds by maximally entangled states. Hence, straightforwardly, the maximum of Eq. (B4) is reached by maximally entangled states. The remaining question is whether the mixture of maximally entangled states can reach the upper bound of $\mathcal{F}(\rho_{AB})$. Next, we will show it is false.

Two maximally entangled states $|\psi_i\rangle$ and $|\psi_j\rangle$ with $\langle\psi_i|\psi_j\rangle = 0$ can be formalized as

$$|\psi_i\rangle = D^{-1/2}\sum_a |a\rangle_A|a\rangle_B \tag{B6}$$

and

$$|\psi_j\rangle = U_A \otimes U_B|\psi_i\rangle = U'_A \otimes I_B|\psi_i\rangle \tag{B7}$$

with $(U'_A)_{ba} = \sum_c (U_A)_{bc}(U_B)_{ac}$. One can verify that

$$\sum_k |\langle\psi_i|T_k^A|\psi_j\rangle|^2 > 0, \tag{B8}$$

for U'_A is neither $\mathbf{0}$ nor the identity matrix. It decreases the $\mathcal{F}(\rho_{AB})$ by contributing the last term of Eq. (B4). Hence the upper bound of $\mathcal{F}(\rho_{AB})$ is reached and only reached by maximally entangled states.

Furthermore, the reduced density matrix ρ_A of a maximally entangled state is a maximally mixed state with $\mathcal{F}(\rho_A) = 0$. Hence, the rQFI

$$\begin{aligned} \Delta\mathcal{F}(\rho_{AB}) &\leqslant \max[\mathcal{F}(\rho_{AB})] - \min[\mathcal{F}(\rho_A)] \\ &= 2(D - 1/D) - 0, \end{aligned} \tag{B9}$$

where the equality is held and only held by maximally entangled states. This completes the proof.

c. $\Delta\mathcal{F}(\rho_{AB}) = 0$ iff ρ_{AB} is a product state

The sufficiency [$\Delta\mathcal{F}(\rho_{AB}) = 0 \leftarrow \rho_{AB}$ is a product state] is easy to verify with the definition of $\Delta\mathcal{F}(\rho_{AB})$.

To prove the necessity [$\Delta\mathcal{F}(\rho_{AB}) = 0 \rightarrow \rho_{AB}$ is a product state], we study state ρ_{AB} with the reduced density matrix $\rho_A = \sum_a p_a|a\rangle\langle a|$.

Set the generator as

$$\begin{aligned} T_{ab}^{(1)} &= \tfrac{1}{2}(|a\rangle\langle b| + |b\rangle\langle a|), 1 \leqslant a < b \leqslant N, \\ T_{ab}^{(2)} &= \tfrac{1}{2i}(|a\rangle\langle b| - |b\rangle\langle a|), 1 \leqslant a < b \leqslant N, \\ T_{aa}^{(3)} &= \frac{1}{\sqrt{2a(a-1)}}(|1\rangle\langle 1| + |2\rangle\langle 2| + \cdots + |a-1\rangle\langle a-1| \\ &\quad + (1-a)|a\rangle\langle a|), 2 \leqslant a \leqslant N. \end{aligned} \tag{B10}$$

For $\Delta\mathcal{F}(\rho_{AB}, T_i^A) \geqslant 0$, then $\Delta\mathcal{F}(\rho_{AB}) > 0$, if there exists at least one generator T_i such that $\Delta\mathcal{F}(\rho_{AB}, T_i^A) > 0$. We will prove this is true when ρ_{AB} is not a product state.

(a) If ρ_{AB} is coherent in the orthogonal complete basis $\{|a\rangle\}$ of $\mathcal{H}_A$, where ρ_A is diagonal, there exists at least one pair of states $|a\rangle$, $|b\rangle \in \{|a\rangle\}$, such that $\langle a|\rho_{AB}|b\rangle \neq \mathbf{0}$. Through the linear composition over the set $\{T_{aa}^{(3)}\}$, one can always construct an operator $T_{ab}^{\prime(3)} = (|a\rangle\langle a| - |b\rangle\langle b|)/2$, such that $[\rho_{AB}, T_{ab}^{\prime(3)} \otimes I_B] \neq 0$ for ρ_{AB} with a nonzero element $\langle a|\rho_{AB}|b\rangle$. This indicates the QFI $\mathcal{F}(\rho_{AB}, T_{ab}^{\prime(3)} \otimes I_B) \geqslant -2\mathrm{tr}([\sqrt{\rho_{AB}}, T_{ab}^{\prime(3)} \otimes I_B]^2) > 0$ according to [46]. Furthermore, one can always find an alternative representation of $\{T_i\}$ with $T_{ab}^{\prime(3)}$ as an element in it. And Eq. (B23) reveals that rQFI $\Delta\mathcal{F}(\rho_{AB})$ is independent of the representation of the generator. Hence if ρ_{AB} is coherent in the basis $\{|a\rangle\}$, where ρ_A is diagonal, the rQFI $\Delta\mathcal{F}(\rho_{AB})$ is nonzero.

(b) If ρ_{AB} is incoherent in the orthogonal complete basis $\{|a\rangle\}$ of $\mathcal{H}_A$, ρ_{AB} can be represented as [43,47–49]

$$\rho_{AB} = \sum_a p_a |a\rangle\langle a| \otimes \rho_{B|a}. \tag{B11}$$

Performing a POVM measurement $\{E_i^B\}$ on system B, one can acquire state $\rho_i = \mathrm{tr}_B(E_i^B \rho_{AB})/q_i$ with probability $q_i = \mathrm{tr}_{AB}(E_i^B \rho_{AB})$. ρ_i is diagonal in basis $\{|a\rangle\}$, and the reduced density matrix $\rho_A = \sum_i q_i \rho_i$. Then perform the optimal measurement $\{E_{i,j}^A\}$ on ρ_i according to the result of the first measurement, and the probability of acquiring corresponding results is $q_{i,j} = \mathrm{tr}_A(\rho_i E_{i,j}^A)$. Summing up the two measurements, the whole POVM measurement is $\{E_{i,j}^A E_i^B\}$, the corresponding probability is $p_{i,j} = \mathrm{tr}_{AB}(\rho_{AB} E_{i,j}^A E_i^B) = q_i q_{i,j}$, and the classical Fisher information about θ—the parameter imprinted by local rotation $e^{-iG\theta}$—is

$$\begin{aligned}\mathcal{F}^c &= \sum_{i,j} \frac{(\partial_\theta p_{i,j})^2}{p_{i,j}} = \sum_i q_i \frac{(\partial_\theta q_{i,j})^2}{q_{i,j}} \\ &= \sum_i q_i \mathcal{F}(\rho_i, G) \geqslant \mathcal{F}(\rho_A, G).\end{aligned} \tag{B12}$$

Taking G as a first class generator $T_{ab}^{(1)}$, the corresponding QFI is

$$\mathcal{F}\big(\rho_A, T_{ab}^{(1)}\big) = \frac{(p_a - p_b)^2}{p_a + p_b} \equiv f(p_a, p_b). \tag{B13}$$

$f(p_a, p_b)$ is strictly convex, i.e.,

$$\sum_i q_i \mathcal{F}\big(\rho_i, T_{ab}^{(1)}\big) \geqslant \mathcal{F}\left(\sum_i q_i \rho_i, T_{ab}^{(1)}\right), \tag{B14}$$

where the equality holds and only holds by $\Pi_{ab}\rho_i\Pi_{ab} = \lambda\Pi_{ab}\rho_j\Pi_{ab}$ for all i, j, and $\lambda > 0$, $\Pi_{ab} = |a\rangle\langle a| + |b\rangle\langle b|$. But for ρ_{AB}, which is a nonproduct state, there exists at least one pair of states $\rho_B^{(a)} \neq \rho_B^{(b)}$ with $p_a p_b \neq 0$; therefore, one can always find a POVM $\{E_i^B\}$ satisfying $\Pi_{ab}\rho_i\Pi_{ab} \neq \lambda\Pi_{ab}\rho_j\Pi_{ab}$, hence

$$\sum_i q_i \mathcal{F}\big(\rho_i, T_{ab}^{(1)}\big) > \mathcal{F}\left(\sum_i q_i \rho_i, T_{ab}^{(1)}\right) = \mathcal{F}\big(\rho_A, T_{ab}^{(1)}\big). \tag{B15}$$

Together with the quantum Cramer-Rao inequality $\mathcal{F}(\rho_{AB}, T_{ab}^{(1)}) \geqslant \mathcal{F}^c(p_{ij}, T_{AB}^{(1)})$ and Eq. (B12), the rQFI for a nonproduct state ρ_{AB} is

$$\Delta\mathcal{F}\big(\rho_{AB}, T_{ab}^{(1)}\big) = \mathcal{F}\big(\rho_{AB}, T_{ab}^{(1)}\big) - \mathcal{F}\big(\rho_A, T_{ab}^{(1)}\big) > 0. \tag{B16}$$

We have thus proved both sufficiency and necessity.

2. Proof of (a2): rQFI is independent of representation

Two equivalent unitary representations of local rotation generators $\{T_i^A\}$ and $\{T_i^{\prime A}\}$ can be connected through a unitary transformation U as

$$T_i^{\prime A} = U T_i^A U^\dagger = \sum_j R_{ji} T_j^A, \tag{B17}$$

where $U \in SU(N) \otimes I_M$ is a local rotation in $\mathcal{H}_A$, and R is the corresponding unitary transformation in the tangent space of $\mathcal{H}_A$. The definition of SLD creates a linear map between $L_i(\rho_{AB})$ and T_i^A, and we have

$$-i\big[U T_i^A U^\dagger, \rho_{AB}\big] = \frac{1}{2}\left\{\sum_j R_{ji} L_j(\rho_{AB}), \rho_{AB}\right\}, \tag{B18}$$

where {, } denotes the anticommutation. For $T_i^{\prime A}$, a generator in the alternative representation, one can define a new SLD $L_i'(\rho_{AB})$ with

$$L_i'(\rho_{AB}) = \sum_j R_{ji} L_j(\rho_{AB}), \tag{B19}$$

which satisfies

$$-i\big[U T_i^A U^\dagger, \rho_{AB}\big] \equiv \tfrac{1}{2}\big\{L_i'(\rho_{AB}), \rho_{AB}\big\}. \tag{B20}$$

Furthermore, $\sum_j L_j(\rho_{AB})^2$ is invariant in that

$$\begin{aligned}\sum_i L_i'(\rho_{AB}) L_i'(\rho_{AB}) &= \sum_{ijk} R_{ji} R_{ik}^{-1} L_j(\rho_{AB}) L_k(\rho_{AB}) \\ &= \sum_j L_j(\rho_{AB}) L_j(\rho_{AB}).\end{aligned} \tag{B21}$$

Hence $\mathcal{F}(\rho_{AB})$ is invariant in that

$$\begin{aligned}\mathcal{F}(\rho_{AB}) &\equiv \sum_i \mathcal{F}\big(\rho_{AB}, T_i^A\big) \\ &= \sum_i \mathcal{F}\big(\rho_{AB}, T_i^{\prime A}\big) \equiv \mathcal{F}'(\rho_{AB}).\end{aligned} \tag{B22}$$

Using the same argument, one can prove that $\mathcal{F}(\rho_A) = \mathcal{F}'(\rho_A)$. Hence,

$$\Delta\mathcal{F}(\rho_{AB}) = \Delta\mathcal{F}'(\rho_{AB}). \tag{B23}$$

3. Proof of (a3): rQFI is invariant under local rotations

The definition of SLD $-i[T_i^A, \rho_{AB}] = \frac{1}{2}\{L_i(\rho_{AB}), \rho_{AB}\}$ creates a linear map between T_i^A and $L_i(\rho_{AB})$, and

$$-i\big[U T_i^A U^\dagger, \rho_{AB}'\big] = \tfrac{1}{2}\{U L_i(\rho_{AB}) U^\dagger, \rho_{AB}'\}, \tag{B24}$$

where $\rho'_{AB} = U\rho_{AB}U^\dagger$, and $U = U_A \otimes U_B$ is a local rotation in $\mathcal{H}_A \otimes \mathcal{H}_B$. According to Eq. (B17), we have

$$UL_i(\rho_{AB})U^\dagger = \sum_j R_{ji}L_j(\rho'_{AB}), \tag{B25}$$

where R is unitary transformation corresponding to U_A, and $L_j(\rho'_{AB})$ is a SLD defined by

$$-i[T_j^A, \rho'_{AB}] = \tfrac{1}{2}\{L_j(\rho'_{AB}), \rho'_{AB}\}. \tag{B26}$$

Corresponding to Eq. (B21), Eq. (B25) implies

$$\begin{aligned}\sum_i UL_i^2(\rho_{AB})U^\dagger &= \sum_{ijk} R_{ji}R_{ki}L_j(\rho'_{AB})L_k(\rho'_{AB})\\ &= \sum_{ijk} R_{ji}R_{ik}^{-1}L_j(\rho'_{AB})L_k(\rho'_{AB})\\ &= \sum_j L_j(\rho'_{AB})L_j(\rho'_{AB}).\end{aligned} \tag{B27}$$

And we have an invariance under the rotation:

$$\begin{aligned}\mathrm{tr}\left[\sum_i L_i^2(\rho_{AB})\rho_{AB}\right] &= \mathrm{tr}\left[\sum_i UL_i^2(\rho_{AB})U^\dagger\rho'_{AB}\right]\\ &= \mathrm{tr}\left[\sum_i L_i^2(\rho'_{AB})\rho'_{AB}\right],\end{aligned} \tag{B28}$$

i.e., $\sum_i \mathcal{F}(\rho_{AB}, T_i^A) = \sum_i \mathcal{F}(\rho'_{AB}, T_i^A)$, with $\rho'_{AB} = U\rho_{AB}U^\dagger$. Using the same argument, one can prove $\sum_i \mathcal{F}(\rho_A, T_i) = \sum_i \mathcal{F}(\rho'_A, T_i)$, hence $\Delta\mathcal{F}(\rho_{AB}) = \Delta\mathcal{F}(\rho'_{AB})$. We have thus proved the local invariance of rQFI.

4. Proof of (a4): rQFI is not increased under the completely positive and trace preserving map on subsystem *B*

The CPTP map $\mathcal{E}_B$ on subsystem B can be expressed as

$$\mathcal{I}_A \otimes \mathcal{E}_B(\rho_{AB}) = \mathrm{tr}_C(I_A \otimes U_{BC}\rho_{AB} \otimes \rho_C I_A \otimes U_{BC}^\dagger), \tag{B29}$$

where ρ_C is a density matrix of an ancillary system C, and U_{BC} is a unitary operation in state space of composite system BC. Then for any local rotation generator G_A and state ρ_{AB}, we have

$$\begin{aligned}&\mathcal{F}(\mathcal{I}_A \otimes \mathcal{E}_B(\rho_{AB}), G_A \otimes I_B)\\ &\quad= \mathcal{F}(\mathrm{tr}_C(I_A \otimes U_{BC}\rho_{AB} \otimes \rho_C I_A \otimes U_{BC}^\dagger), G_A \otimes I_B)\\ &\quad\leqslant \mathcal{F}(I_A \otimes U_{BC}\rho_{AB} \otimes \rho_C I_A \otimes U_{BC}^\dagger, G_A \otimes I_{BC})\end{aligned} \tag{B30}$$

$$\begin{aligned}&= \mathcal{F}(\rho_{AB} \otimes \rho_C, I_A \otimes U_{BC}^\dagger(G_A \otimes I_{BC})I_A \otimes U_{BC})\\ &= \mathcal{F}(\rho_{AB} \otimes \rho_C, G_A \otimes I_{BC})\\ &= \mathcal{F}(\rho_{AB}, G_A \otimes I_B),\end{aligned} \tag{B31}$$

with the reduced density matrix $\rho_A = \mathrm{tr}_B(\rho_{AB})$, where the line (B30) is proved with Eq. (B3). Subtracting $\mathcal{F}(\rho_A, G_A)$ on both sides of Eq. (B31), we have

$$\Delta\mathcal{F}(\mathcal{I}_A \otimes \mathcal{E}_B(\rho_{AB}), G_A \otimes I_B) \leqslant \Delta\mathcal{F}(\rho_{AB}, G_A \otimes I_B). \tag{B32}$$

Taking G_A as the generator T_i^A, then summing the inequality Eq. (B32) over all the generators, we have thus proved that the rQFI is not increased under the CPTP map on subsystem B.

5. rQFI under the local CPTP map on subsystem *A*

Brodutch and Modi have proposed a set of necessary criteria that a total correlation measure should satisfy [37]. According to those criteria, in order to prove $\Delta\mathcal{F}(\rho_{AB})$ is a measure of total correlation, we need to prove the following property in addition to properties (a1)–(a4): $\Delta\mathcal{F}(\rho_{AB})$ is not increased under local CPTP operation as

$$\Delta\mathcal{F}(\rho_{AB}) \geqslant \Delta\mathcal{F}[\mathcal{E}_A \otimes \mathcal{E}_B(\rho_{AB})], \tag{B33}$$

where $\mathcal{E}_i$ is a CPTP map on state space of $\mathcal{H}_i$ with $i = A, B$. Together with $\mathcal{E}_A \otimes \mathcal{E}_B = (\mathcal{E}_A \otimes \mathcal{I}_B)(\mathcal{I}_A \otimes \mathcal{E}_B)$, we need to prove

$$\Delta\mathcal{F}(\rho_{AB}) \geqslant \Delta\mathcal{F}[\mathcal{E}_A \otimes \mathcal{I}_B(\rho_{AB})], \tag{B34}$$

in addition to property (a4). Whether it is valid for general cases is still in question. But it is true at least when reduced density matrix ρ_A is a maximally mixed state $\boldsymbol{I}/N$ and $\mathcal{E}_A$ is a random unitary operation

$$\mathcal{E}_A^{\mathrm{RU}} \otimes \mathcal{I}_B(\rho_{AB}) = \sum_i p_i U_i^A \rho_{AB} U_i^{A\dagger}, \tag{B35}$$

with the probability $p_i \geqslant 0$, and $\sum_i p_i = 1$, where $U_i^A = U_i \otimes I_M$ is a unitary operation in the state space of $\mathcal{H}_A$.

Since $\mathcal{E}_A^{\mathrm{RU}}(I_A) = I_A$, we have

$$\mathcal{F}\big[\mathcal{E}_A^{\mathrm{RU}}(\rho_A)\big] = \mathcal{F}(\rho_A) \tag{B36}$$

with $\rho_A = I_A/N$. Based on the invariance Eq. (B28) and the convexity of QFI, we have

$$\begin{aligned}\mathcal{F}(\rho_{AB}) &= \sum_i p_i \mathcal{F}\big(U_i^A \rho_{AB} U_i^{A\dagger}\big)\\ &= \sum_j \sum_i p_i \mathcal{F}\big(U_i^A \rho_{AB} U_i^{A\dagger}, T_j^A\big)\\ &\geqslant \sum_j \mathcal{F}\big(\mathcal{E}_A^{\mathrm{RU}} \otimes \mathcal{I}_B(\rho_{AB}), T_j^A\big)\\ &\equiv \mathcal{F}\big[\mathcal{E}_A^{\mathrm{RU}} \otimes \mathcal{I}_B(\rho_{AB})\big].\end{aligned} \tag{B37}$$

It is straightforward to prove Eq. (B34) is valid when ρ_A is a maximally mixed state and $\mathcal{E}_A$ is a random unitary operation in the state space of $\mathcal{H}_A$. We mention that this method has been used to prove the property of Wigner-Yanase skew information in [19].

APPENDIX C: rQFI OF PURE STATES

We will study rQFI for pure states in Schmidt decomposition form as

$$|\psi\rangle = \sum_{a=1}^{D} \sqrt{d_a}|a\rangle_A|a\rangle_B, \tag{C1}$$

with the reduced density matrix $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|) = \sum_{a=1}^{D} d_a|a\rangle\langle a|$. We take the generators $\{T_i\}$ as Eq. (B10). According to the definition of rQFI, we need to calculate $\sum_i \mathcal{F}(|\psi\rangle\langle\psi|, T_i^A)$ and $\sum_i \mathcal{F}(\rho_A, T_i)$, respectively.

1. Calculation of $\sum_i \mathcal{F}(|\psi\rangle\langle\psi|, T_i^A)$

The QFI of pure state $|\psi\rangle$ is [20]

$$\begin{aligned}\mathcal{F}(|\psi\rangle\langle\psi|, T_i^A) &= 4\langle\psi|(T_i^A)^2|\psi\rangle - 4\langle\psi|T_i^A|\psi\rangle^2 \\ &= 4\mathrm{tr}(\rho_A T_i^2) - 4\mathrm{tr}(\rho_A T_i)^2. \end{aligned} \tag{C2}$$

Summing the first term over generators $\{T_i\}$, we have

$$\sum_i \mathrm{tr}[\rho_A T_i^2] = \frac{N-1}{2} + \frac{N-1}{2N} = \frac{(N^2-1)}{2N}, \tag{C3}$$

with

$$\sum_{a<b}[T_{ab}^{(1)}]^2 = \sum_{a<b}[T_{ab}^{(2)}]^2 = \frac{1}{4}\sum_{a<b}(|a\rangle\langle a| + |b\rangle\langle b|) = \frac{N-1}{4}I_N, \tag{C4}$$

and

$$\begin{aligned}\sum_{a=2}^N [T_{aa}^{(3)}]^2 &= \sum_{a=2}^N \frac{1}{2a(a-1)}[|1\rangle\langle 1| + |2\rangle\langle 2| + \dots \\ &\quad + |a-1\rangle\langle a-1| + (1-a)^2|a\rangle\langle a|] \\ &= \sum_{b=1}^N \left[\frac{(1-b)^2}{2b(b-1)} + \sum_{a=b+1}^N \frac{1}{2a(a-1)}\right]|b\rangle\langle b| \\ &= \frac{N-1}{2N}I_N. \end{aligned} \tag{C5}$$

For ρ_A diagonal in basis $\{|a\rangle\}$, we have

$$\mathrm{tr}[T_{ab}^{(1)}\rho_A]^2 = \mathrm{tr}[T_{ab}^{(2)}\rho_A]^2 = 0, \tag{C6}$$

and

$$\begin{aligned}\sum_i \mathrm{tr}[T_i\rho_A]^2 &= \sum_{a=2}^N \mathrm{tr}[T_{aa}^{(3)}\rho_A]^2 \\ &= \sum_{a=2}^N \frac{1}{2a(a-1)}\left[\sum_{b=1}^{a-1} d_b + d_a(1-a)\right]^2 \\ &= \frac{1}{2}\sum_{a=1}^N d_a^2 - \frac{1}{2N}. \end{aligned} \tag{C7}$$

According to Eqs. (C2), (C3), and (C7),

$$\begin{aligned}\mathcal{F}(|\psi\rangle\langle\psi|) &= 2\left(\frac{N^2-1}{N} - \sum_{a=1}^N d_a^2 + \frac{1}{N}\right) \\ &= 2\left(N - \sum_{a=1}^N d_a^2\right). \end{aligned} \tag{C8}$$

2. Calculation of $\sum_i \mathcal{F}(\rho_A, T_i)$

In this subsection, we will calculate QFI for $\rho_A = \sum_a d_a|a\rangle\langle a|$ according to [50]. Specifically,

$$\begin{aligned}\mathcal{F}(\rho_A, T_{ab}^{(1)}) &= \mathcal{F}(\rho_A, T_{ab}^{(2)}) \\ &= \sum_{e,f}\frac{2(d_f-d_e)^2}{(d_f+d_e)}|\langle f|T_{ab}^{(1)}|e\rangle|^2 = \frac{(d_a-d_b)^2}{d_a+d_b}, \\ \mathcal{F}(\rho_A, T_{aa}^{(3)}) &= \sum_{e,f}\frac{2(d_f-d_e)^2}{d_f+d_e}|\langle f|T_{aa}^{(3)}|e\rangle|^2 = 0. \end{aligned} \tag{C9}$$

Thus the summation over all the generators is

$$\begin{aligned}\mathcal{F}(\rho_A) &= \sum_i \mathcal{F}(\rho_A, T_i) \\ &= 2\sum_{a<b}\left[(d_a+d_b) - \frac{4d_a d_b}{d_a+d_b}\right] \\ &= 2N - \sum_{a,b}\frac{4d_a d_b}{d_a+d_b}. \end{aligned} \tag{C10}$$

Together with Eq. (C8), the rQFI for pure states Eq. (C1) is

$$\begin{aligned}\Delta\mathcal{F}(|\psi\rangle\langle\psi|) &= \mathcal{F}(|\psi\rangle\langle\psi|) - \mathcal{F}(\rho_A) \\ &= \sum_{a,b}\frac{4d_a d_b}{d_a+d_b} - 2\sum_a d_a^2 \\ &= 2\sum_{a\neq b}\left(\frac{2d_a d_b}{d_a+d_b} + d_a d_b\right). \end{aligned} \tag{C11}$$

APPENDIX D: rQFI OF MAXIMALLY MIXED MARGINAL STATES

1. Calculation of $\Delta\mathcal{F}(\rho_{AB}^{\mathrm{MS}})$

We study the rQFI of maximally mixed marginal states in the form

$$\rho_{AB}^{\mathrm{MS}} = \frac{\boldsymbol{I}}{4} + \sum_i \beta_i\sigma_i\otimes\sigma_i. \tag{D1}$$

The symmetric logarithmic derivative corresponding to generator $\sigma_x^A/2$, $\sigma_y^A/2$, and $\sigma_z^A/2$ is

$$\begin{aligned} L_x &= \frac{4(4\beta_1\beta_2+\beta_3)}{16\beta_1^2-1}\sigma_y\otimes\sigma_z - \frac{4(4\beta_1\beta_3+\beta_2)}{16\beta_1^2-1}\sigma_z\otimes\sigma_y, \\ L_y &= \frac{4(4\beta_2\beta_3+\beta_1)}{16\beta_2^2-1}\sigma_z\otimes\sigma_x - \frac{4(4\beta_1\beta_2+\beta_3)}{16\beta_2^2-1}\sigma_x\otimes\sigma_z, \\ L_z &= \frac{4(4\beta_1\beta_3+\beta_2)}{16\beta_3^2-1}\sigma_x\otimes\sigma_y - \frac{4(4\beta_2\beta_3+\beta_1)}{16\beta_3^2-1}\sigma_y\otimes\sigma_x, \end{aligned} \tag{D2}$$

respectively. Thus the sum of quantum Fisher information is

$$\begin{aligned}\mathcal{F}(\rho_{AB}^{\mathrm{MS}}) &= \sum_i \mathcal{F}(\rho_{AB}^{\mathrm{MS}}, T_i^A) = \mathrm{tr}[(L_x^2+L_y^2+L_z^2)\rho_{AB}] \\ &= 3 - 4\sum_{i>j}\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}, \end{aligned} \tag{D3}$$

where λ_i are eigenvalues of ρ_{AB}^{MS} with

$$\begin{aligned}\lambda_1 &= 1/4 - \beta_1 + \beta_2 + \beta_3, \lambda_2 = 1/4 + \beta_1 - \beta_2 + \beta_3, \\ \lambda_3 &= 1/4 + \beta_1 + \beta_2 - \beta_3, \lambda_4 = 1/4 - \beta_1 - \beta_2 - \beta_3. \end{aligned} \tag{D4}$$

$\rho_A^{\mathrm{MS}} = \mathrm{tr}_B(\rho_{AB}^{\mathrm{MS}})$ is a maximally mixed state, hence the sum of QFI $\mathcal{F}(\rho_A^{\mathrm{MS}}) = \sum_i(\rho_A^{\mathrm{MS}}, T_i)$ is zero. Together with Eq. (D3), the rQFI of ρ_{AB}^{MS} is

$$\Delta\mathcal{F}(\rho_{AB}^{\mathrm{MS}}) = 3 - 4\sum_{i>j}\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}. \tag{D5}$$

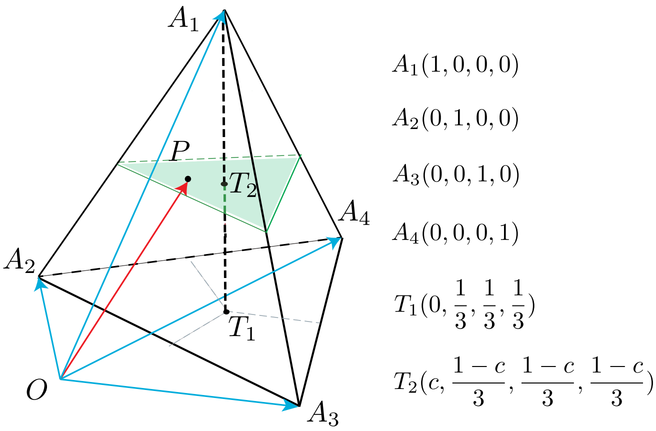

FIG. 3. Barycentric coordinate system.

2. Barycentric coordinate system and the bound of maximally mixed marginal states

The barycentric coordinate system (BCS) is usually defined on a simplex. We exemplify it with the tetrahedra in Fig. 3, where the four vertices are denoted by A_i, where $i = 1, 2, 3, 4$, respectively. Put particles on each of the vertices, and set the mass of particle on A_i as λ_i with $\lambda_i \geqslant 0$ and $\sum_i \lambda_i = 1$. The coordinate $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ locates the mass center of the four particles. Alternatively, one can define the BCS with vectors. For the point P with coordinate $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and an arbitrary point O, we have $\overrightarrow{OP} = \sum_i \lambda_i \overrightarrow{OA_i}$.

Furthermore, the surface with $\lambda_1 = c$, $0 \leqslant c \leqslant 1$ is a plane parallel to the plane $A_2A_3A_4$, and the distance satisfies $T_2T_1/A_1T_1 = c$.

APPENDIX E: rQFI OF SEPARABLE STATES

In this section, we study the rQFI for states

$$\rho_{AB}^{\mathrm{SS}} = \frac{\boldsymbol{I}}{4} + n\sigma_z^A + m\sigma_z^B + \beta_3\sigma_z \otimes \sigma_z. \tag{E1}$$

The symmetric logarithmic derivatives of ρ_{AB}^{SS} corresponding to generators $\sigma_x^A/2$, $\sigma_y^A/2$, and $\sigma_z^A/2$ are

$$\begin{aligned} L_x &= \frac{4(4\beta_3 m - n)}{1-16m^2}\sigma_y \otimes I_2 + \frac{4(4mn-\beta_3)}{1-16m^2}\sigma_y \otimes \sigma_z, \\ L_y &= \frac{4(n-4\beta_3 m)}{1-16m^2}\sigma_x \otimes I_2 + \frac{4(\beta_3-4mn)}{1-16m^2}\sigma_x \otimes \sigma_z, \\ L_z &= 0, \end{aligned} \tag{E2}$$

respectively. Hence the sum of QFI for ρ_{AB}^{SS} is

$$\begin{aligned} \mathcal{F}(\rho_{AB}^{\mathrm{SS}}) &= \sum_i \mathcal{F}(\rho_{AB}^{\mathrm{SS}}, T_i^A) = \mathrm{tr}\big[\rho_{AB}^{\mathrm{SS}}(L_x^2 + L_y^2 + L_z^2)\big] \\ &= \frac{32(\beta_3^2 - 8\beta_3 mn + n^2)}{1-16m^2}. \end{aligned} \tag{E3}$$

The eigenvalues of the reduced matrix

$$\rho_A^{\mathrm{SS}} = \frac{\boldsymbol{I}_2}{2} + 2n\sigma_z \tag{E4}$$

are $1/2 - 2n$, $1/2 + 2n$. Substituting it into Eq. (C10), the sum of QFI for reduced matrix ρ_A^{SS} is

$$\mathcal{F}(\rho_A^{\mathrm{SS}}) = \sum_i \mathcal{F}(\rho_A^{\mathrm{SS}}, T_i) = 32n^2. \tag{E5}$$

Hence, the rQFI for ρ_{AB}^{SS} is

$$\begin{aligned} \Delta\mathcal{F}(\rho_{AB}^{\mathrm{SS}}) &= \mathcal{F}(\rho_{AB}^{\mathrm{SS}}) - \mathcal{F}(\rho_A^{\mathrm{SS}}) \\ &= \frac{32(\beta_3 - 4mn)^2}{1-16m^2} = \frac{8(\lambda_1\lambda_4 - \lambda_2\lambda_3)^2}{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)}, \end{aligned} \tag{E6}$$

where $\lambda_1 = 1/4 + n + m + \beta_3$, $\lambda_2 = 1/4 - n + m - \beta_3$, $\lambda_3 = 1/4 + n - m - \beta_3$, and $\lambda_4 = 1/4 - n - m + \beta_3$ are eigenvalues of ρ_{AB}^{SS}.

APPENDIX F: PROPERTIES OF $\delta\mathcal{F}(\rho_{AB})$

In the main text, we build a measure $\delta\mathcal{F}(\rho_{AB})$ to characterize the contribution of discord to the enhancement of sensitivity:

$$\delta\mathcal{F}(\rho_{AB}) = \min_{\{|a\rangle\}} \sum_c \big[\mathcal{F}\big(\rho_{AB}, T_{cc}^{(3)A}\big) - \mathcal{F}\big(\rho_A, T_{cc}^{(3)}\big)\big], \tag{F1}$$

where $\{|a\rangle\}$ is an unknown orthogonal complete basis of subspace $\mathcal{H}_A$. $T_{cc}^{(3)}$ is the diagonal generator defined with Eq. (B10) in basis $\{|a\rangle\}$. The summation is done over the set of these diagonal generators. The minimization is done over all bases of $\mathcal{H}_A$.

Next, we will show $\delta\mathcal{F}(\rho_{AB})$ satisfies the established necessary properties to be defined as a measure of generalized discord.

(b1) *$\delta\mathcal{F}(\rho_{AB})$ is invariant under local unitary operations in state space.* For arbitrary bipartite state ρ_{AB} and local unitary operation $U_A \otimes U_B$, we have equality $\mathcal{F}(\rho'_{AB}, T_{cc}^{\prime(3)A}) = \mathcal{F}(\rho_{AB}, T_{cc}^{(3)A})$ with $O' = (U_A \otimes U_B)O(U_A^\dagger \otimes U_B^\dagger)$, $O = \rho_{AB}, T_{cc}^{(3)}$. Based on that, if $\delta\mathcal{F}(\rho_{AB}) = F$ is reached with generator $\{T_{cc}^{(3)}\}$ diagonal in basis $\{|a\rangle\}$, we have $\delta\mathcal{F}(\rho'_{AB}) = F$ reached with generator $\{T_{cc}^{\prime(3)}\}$ diagonal in the basis $\{U_A|a\rangle\}$.

(b2) *$\delta\mathcal{F}(\rho_{AB})$ is non-negative.* It is straightforward with Eq. (B3), which says the difference $\mathcal{F}(\rho_{AB}, T_{cc}^{(3)A}) - \mathcal{F}(\rho_A, T_{cc}^{(3)})$ is non-negative for bipartite state ρ_{AB} and local rotation generator $T_{cc}^{(3)A}$.

(b3) *$\delta\mathcal{F}(\rho_{AB})$ vanishes if ρ_{AB} is a classical state.* The classical state $\chi_{aB} = \sum_a p_a |a\rangle\langle a| \otimes \rho_{B|a}$ is invariant under the local rotations generated by $\{T_{cc}^{(3)}\}$, the generators which are diagonal in basis $\{|a\rangle\}$. Hence the corresponding QFI is zero, and $\delta\mathcal{F}(\rho_{AB})$ vanishes.

(b4) *$\delta\mathcal{F}(\rho_{AB})$ is not increased under local CPTP maps on subsystem B.* Equation (B31) shows that, for each diagonal generator $T_{cc}^{(3)}$, $\mathcal{F}(\rho_{AB}, T_{cc}^{(3)A}) - \mathcal{F}(\rho_A, T_{cc}^{(3)})$ is not increased under CPTP maps on subsystem B, nor is $\delta\mathcal{F}(\rho_{AB})$.

Since $\delta\mathcal{F}(\rho_{AB})$ satisfies those criteria, we can take it as a valid generalized discord measure.

APPENDIX G: rQFI AND QUANTUM ERROR CORRECTION

The mechanism of enhancing the precision of parameter estimation through correlations is similar to quantum error correction. We will show it below with a toy model of a two-qubit system.

Suppose we aim to prepare state $|1\rangle \in \mathcal{H}_A$, but the spin-flip error happens with probability p. The state we prepared is $\rho_1 = (1-p)|1\rangle\langle 1| + p|0\rangle\langle 0|$. The parameter to be estimated is imprinted into ρ_1 with rotation $U_i(\theta_i) = e^{-i\sigma_i/2\theta_i}$, where $i = x, y, z$, respectively, which gives us the parametrized state $\rho_1(\theta_i) = U_i(\theta_i)\rho_1 U_i^\dagger(\theta_i)$. As proposed in the main text, the sensitivity of ρ_1 to the local rotation is measured by $\sum_i \mathcal{F}(\rho_1, \sigma_i/2)$. We can see the error decrease the sensitivity from 2 to $2(1-2p)^2$.

Now suppose there is an ancillary qubit initialized to state $|1\rangle$, we aim to prepare state $|11\rangle$. The spin flip error happens with probability p. It can be recorded by the ancillary qubit through a controlled-NOT gate $|0\rangle\langle 0| \otimes \sigma_x + |1\rangle\langle 1| \otimes I_2$. The state we prepared is $\rho_2 = (1-p)|11\rangle\langle 11| + p|00\rangle\langle 00|$. Parametrize this state with local rotation $U_i^A(\theta_i) \equiv U_i(\theta_i) \otimes I_2$ and state $\rho_2(\theta_i) = U_i^A(\theta_i)\rho_2 U_i^{A\dagger}(\theta_i)$ is obtained, with $i = x, y, z$, respectively. Now, perform projective measurement $\{|1\rangle\langle 1|, |0\rangle\langle 0|\}$ on qubit B. If result $|1\rangle$ is acquired, the conditional state of qubit A is pure state $U_i(\theta_i)|1\rangle$. If result $|0\rangle$ is acquired, we know that the error happened, and the state of system A is $U_i(\theta_i)|0\rangle$. With this knowledge, a different schedule to estimate the parameter θ_i can be applied accordingly. With the overall scheme, the "sensitivity," measured by $\sum_i \mathcal{F}(\rho_2, \sigma_i^A/2)$, is 2, which equals to the sensitivity of a pure state of qubit A, e.g., state $|1\rangle$. From this point of view, the error is corrected. And the enhancement of "sensitivity" contributed by this "error correction scheme" is measured by rQFI as

$$\begin{aligned}\Delta\mathcal{F}(\rho_2) &= \sum_i \mathcal{F}\big(\rho_2, \sigma_i^A/2\big) - \sum_i \mathcal{F}(\rho_1, \sigma_i/2)\\ &= 8p(1-p).\end{aligned} \tag{G1}$$

To summarize, the error creates new components in a state of sensor system A. If there are correlations between system A and ancillary system B, one may recognize the error components through measurement on ancillary system B. Then a different estimation schedule can be applied conditionally. The overall "precision" may be enhanced by this protocol. In this manner, the error is corrected with the help of the ancillary system and correlation. And rQFI is a measure of the enhancement of SLR contributed by this "error correction scheme."

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