PAPER

Dynamical effects of switching a super-critical well potential on pair creation from a vacuum*

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The dynamical effects on electron-positron pair creation from a vacuum caused by the switching processes of a super-critical well potential are investigated in detail. The results show that only when the switching on and switching off time both increase will the final pair yield converge to the integer of embedded bound states nearly exponentially. But a single adiabatic switching on or switching off cannot lead to an integer pair yield. If the potential is turned on abruptly, associated with the discrete and embedded bound states, there is multi-frequency oscillation around the pair number's saturation. The slowly switching on can suppress the amplitude of this oscillation and reduce the final pair yield. The switching off can also reduce the final pair number in the same order of magnitude. The evolution of a single-pair number shows a robust long range correlation between particle and antiparticle. For an adiabatic switching case, the single-pair dominates the early pair creation, their upper limit value is equal to the integer, and these single-pairs will totally disentangle during the switching off.

Keywords: electron–positron pair creation, dynamical effect

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1. Introduction

The possibility of electron–positron pair creation from a vacuum due to an extremely strong external field is an important prediction of quantum electrodynamics.[1] Pairs have been generated by the relativistic heavy-ion collisions[2] or the collision of an intense laser pulse and a 46-GeV electron beam.[3] However, a direct conversion from energy to matter, i.e., electron–positron pair creation from pure intense laser light, has not been observed yet. In light of the rapid advance of laser technology,[4] a good theoretical understanding of the pair creation in a strong field becomes highly desirable.

Theoretically, in a super-critical potential, electrons from the negative continuum will fill the empty embedded bound states (EBSs) once they are degenerated, and the Pauli principle will prevent further occupation. The holes left in the Dirac sea are identified as positrons. The number of pairs created should be the number of the EBSs. However, it was found that in an abruptly turned on super-critical well potential, the final pair number is not an integer, but is greater than the number of the EBSs. The small excess amount was attributed to the non-adiabatic turn on and partially occupied subcritical bound states.[5–8] The authors addressed that if the potential is turned on slowly enough the final pair number will be precisely equal to the number of the EBSs; however, the dynamical behaviors caused by the turn on or turn off of a super-critical potential have not been studied in detail yet.

Experimentally the external field is always time dependent, and the creation process must be treated with time effects being taken into account. Similar to the charge resonance enhanced ionization of molecular physics,[9–11] bound states (for example, supported by a super strong nuclear Coulomb field characteristic of two colliding high-Z ions) play an important role in the pair creation process.[12,13] Without losing generality, we choose a one dimensional well potential to study the dynamical effects with bound states existed. This model, defined by two Sauter potentials, can be realized in principle by two localized electric fields that have identical intensities and frequencies, but phases differ by a shift of π, though a possible experimental set-up would be more complicated.

In this paper, for a supercritical well potential with fixed holding on duration, we study how the turn on and turn off determine the pair creation process. We examine whether and how the pair number saturate to the integer when the potential well is turned on or turned off slowly enough. Close attention is paid to the relation between pair number evolution and the EBSs. The entanglement between a single electron and positron pair is also studied for a better understanding. We use the space–time resolved numerical method which has been introduced recently (for a review see Ref. [14]). This method works for arbitrary spacial and temporal field construction, and provides information inside the interaction zone. It has

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contributed to the resolution of various conceptual problems such as the Zitterbewegung,[15] the Klein paradox,[16] as well as the fermion pair creation.[5–8,12]

This paper is organized as follows. In Section 2, we introduce the model and numerical method. In Section 3, numerical results are shown and discussed. First, the well-known adiabatic limit is examined. Then the effects of turn on, turn off, and their combinations are studied sequentially. We also examine the single-pair number evolution. In the last section we give a brief summary and discussion.

2. Model and method

The Sauter-like well potential is defined as

$$V(z,t)=\frac{V_0(t)}{2}\left[\tanh\left(\frac{z-W/2}{D}\right)-\tanh\left(\frac{z+W/2}{D}\right)\right], \quad (1)$$

where D is the extension of each edge, W is the total width. The corresponding electric field is

$$E(z,t)=\frac{V_0(t)}{2D}\left[\operatorname{sech}^2(\frac{z-W/2}{D})-\operatorname{sech}^2(\frac{z+W/2}{D})\right].$$

We set $D=0.3\lambda_C$ and $W=4.55\lambda_C$. λ_C is the Compton wave length of electron, $\lambda_C=1/c$. The atomic units (short as a.u.) are used: $m=\hbar=e=1$, $c=1/\alpha$. α is the fine-structure constant. We choose the potential depth as $V_0(t)=2.53c^2 f(t)$, where $f(t)$ is a piecewise function of time t:

$$\begin{cases} f(t)=\sin^2\left(\dfrac{\pi}{2T_{\rm on}}t\right), & 0<t<T_{\rm on},\\ f(t)=1, & T_{\rm on}\le t\le T+T_{\rm on},\\ f(t)=\cos^2\left(\dfrac{\pi}{2T_{\rm off}}(t-T-T_{\rm on})\right), & T+T_{\rm on}<t<T_{\rm total}. \end{cases}$$

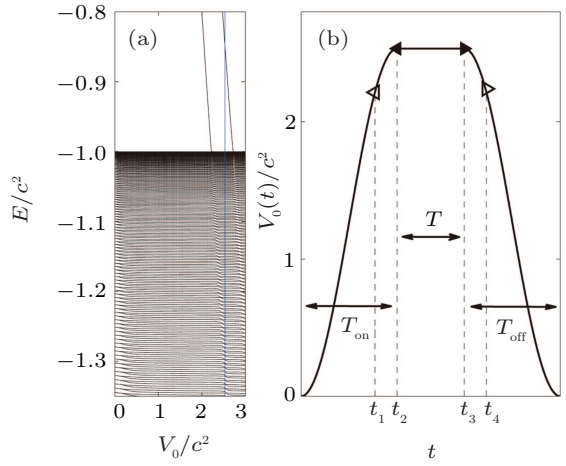

Fig. 1. (a) The energy spectrum of the total Hamiltonian as a function of the depth. The vertical blue line denotes $V_0=2.53c^2$. (b) $V_0(t)$ as a function of time. Left and right empty triangles (at t_1 and t_4) mark the times when the bound state dives and leaves the Dirac sea. Left and right filled triangles (at t_2 and t_3) separate the three time regions.

The energy spectrum of the total Hamiltonian $\hat{H}=c\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{p}}_z+c^2\boldsymbol{\sigma}_3+V(z,t)$ for each depth of the potential are presented in Fig. 1(a). It was shown that when $V_0(t_1)=2.22c^2$ the lowest bound state dives into the negative continuum and the potential becomes super-critical. As shown in Fig. 1(b), there are three time regions separated by the left and right filled triangles (at t_2 and t_3). The first and third regions ($T_{\rm on}$ and $T_{\rm off}$) are the turn on and off processes. During the second region T, between t_2 and t_3, $T_{\rm on}\le t\le T+T_{\rm on}$, $V_0=2.53c^2$, there exists one bound state embedded in the Dirac sea at $E=-1.31c^2$. In this paper we set T as a constant $T=0.012$. The total time is $T_{\rm total}=T+T_{\rm on}+T_{\rm off}$.

In the following we will briefly review the numerical method based on constructing a field operator by a single particle wave function of the Dirac equation. The field operator $\hat{\Psi}(z,t)$ can be expressed in terms of the electron annihilation and positron creation operators as[14]

$$\begin{aligned}\hat{\Psi}(z,t)&=\sum_p \hat{b}_p W_p(z,t)+\sum_n \hat{d}_n^{\dagger} W_n(z,t)\\&=\sum_p \hat{b}_p(t) W_p(z)+\sum_n \hat{d}_n^{\dagger}(t) W_n(z),\end{aligned} \quad (2)$$

in which $W_{p(n)}(z)=\langle z|p(n)\rangle$ is the solution of the field-free Dirac Hamiltonian $\hat{H}_0=\left[c\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{p}}_z+c^2\boldsymbol{\sigma}_3\right]$, and the term $\sum_{p(n)}$ denotes the summation over all states with positive (negative) energy. In order to cut down computation costs, the Dirac matrices are reduced to Pauli matrices since the spin is invariant here. The time dependent single particle wave function $W_{p(n)}(z,t)$ can be obtained by introducing the time-evolution operator

$$\hat{U}(t_2,t_1)=\hat{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2}\mathrm{d}t'\hat{H}(t')\right],$$

where $\hat{T}$ denotes the Dyson time ordering operator. We use the numerical split operator technique[17,18] to compute $W_{p(n)}(z,t)=\hat{U}(t,0)W_{p(n)}(z)$. The number of electrons created from the vacuum (defined as $\hat{b}_p\|\mathrm{vac}\rangle=0$, $\hat{d}_n\|\mathrm{vac}\rangle=0$) is obtained from the positive part of the field operator,

$$\begin{aligned}N^{\rm el.}(t)&=\int \mathrm{d}z\,\langle\mathrm{vac}\|\hat{\Psi}^{(+)\dagger}(z,t)\hat{\Psi}^{(+)}(z,t)\|\mathrm{vac}\rangle\\&=\sum_p\langle\mathrm{vac}\|\hat{b}_p^{\dagger}(t)\hat{b}_p(t)\|\mathrm{vac}\rangle\\&=\sum_{pn}\left|U_{pn}(t)\right|^2,\end{aligned} \quad (3)$$

where

$$\hat{\Psi}^{(+)}(z,t)=\sum_p\hat{b}_p(t)W_p(z),$$

$$U_{pn}(t)=\left\langle W_p(z)\middle|W_n(z,t)\right\rangle=\int \mathrm{d}z W_p^*(z)W_n(z,t).$$

Because electron and positron are always generated in pairs, the pair number $N(t)$ and positron number $N^{\rm po.}(t)$ are equal to the electron number $N^{\rm el.}(t)$. We should point out that each $N(t)$ is obtained by projecting the time dependent single particle wave functions $|W_n(t)\rangle$ onto the field-free positive eigenstates $|W_p\rangle$, as a result $N(t)$ is actually the pair number if the field is turned off abruptly at time t.

3. Dynamical effects of switching

3.1. Pair yield in the adiabatic limit

In this subsection, we study the combined effects of slowly turn on and turn off, with $T_{on} = T_{off}$ fixed. During the second time region T, this super-critical field can trigger the spontaneous creation of electron-positron pairs from a vacuum. If the potential is turned on abruptly ($T_{on} = 0$, blue curve in Fig. 2), as discussed in Ref. [5], the growth of the pair number can be characterized by four distinct regimes in time. In the fourth regime, $t \gg 1/c^2 + W/c$, the pair number undergoes an asymptotic behavior and saturates to a number larger than the number of EBSs.[5–8] For potential parameters in this work and the potential duration $T = 0.012$, this number is $N \approx 1.259$ when $t = t_3$, point A on the blue curve in Fig. 2. We also compute $N(t)$ for different $T_{on} = T_{off}$: in case B ($T_{on} = T_{off} = 10/c^2$), C ($T_{on} = T_{off} = 50/c^2$), and D ($T_{on} = T_{off} = 100/c^2$). Pair number $N(t)$ at time points as marked in Fig. 2 are listed in Table 1. From points A, B_3, C_3, and D_3, we can see that the turn on can reduce the pair number. After B_3, C_3, and D_3, the pair number $N(t)$ decreases in the turning off process. As shown in the last column of the table, more slowly turning on and turning off, more closely the final pair number reaches 1 (number of EBSs, no spin considered).

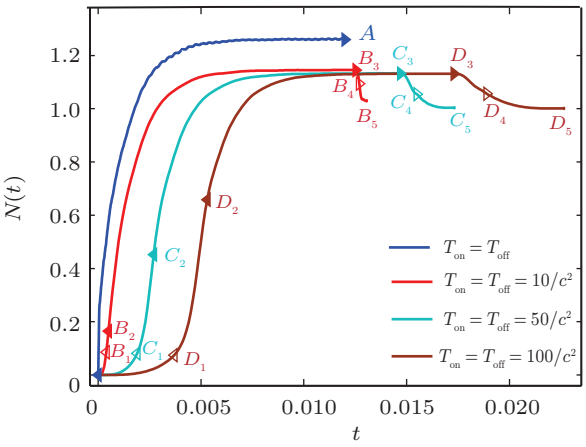

Fig. 2. (color online) Pair number $N(t)$ as a function of time for different $T_{on} = T_{off}$. $T = 0.012$ is fixed. The empty and filled triangles mark the moments $t = t_1, t_2, t_3, t_4$ in Fig. 1.

Table 1. Pair number $N(t)$ at time points (use 'P' for short in this table) in Fig. 2.

P	$N(t)$	P	$N(t)$	P	$N(t)$	P	$N(t)$	P	$N(t)$
–	–	–	–	A	1.259	–	–	–	–
B_1	0.098	B_2	0.165	B_3	1.145	B_4	1.094	B_5	1.030
C_1	0.115	C_2	0.452	C_3	1.133	C_4	1.054	C_5	1.004
D_1	0.124	D_2	0.658	D_3	1.131	D_4	1.056	D_5	1.001

The final pair number N_F behavior as a function of $T_{on} = T_{off}$ is shown in Fig. 3. The pairs surviving the turn off can converge to 1 nearly exponentially as $T_{on} = T_{off}$ increases. So, one can get a final pair number precisely equal to the integer of EBSs by turn on and turn off both adiabatically. Another numerical evidence is shown in Ref. [19]. There, more EBSs are involved. A single quasi-adiabatic sinusoidal pulse can produce integer pairs. The abrupt changes of the final pair number locate at the critical depths or widths where each discrete bound state dives into the Dirac sea.

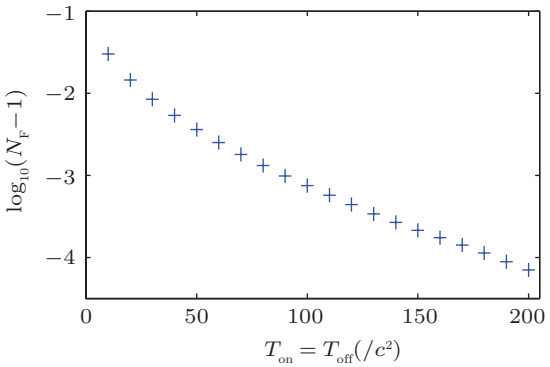

Fig. 3. (color online) $\log_{10}(N_F - 1)$ as a function of $T_{on} = T_{off}$. N_F is the value of $N(t)$ at the end of T_{off}. $T = 0.012$ is fixed. When $T_{on} = T_{off} = 200/c^2$, $(N_F - 1)$ is 7.00×10^{-5}.

3.2. Effects of slow turn on

In this subsection we fix $T_{off} = 0$ and analyze the effect of the turn on process. A bound state dives into the Dirac sea and the potential becomes critical at $t = t_1$, indicated by the left empty triangles in Fig. 2. Although the potential is subcritical before $t = t_1$, $N(t)$ are non-zero due to the time effect.[20] Pair numbers are 0.098, 0.115, 0.124 (points B_1, C_1, D_1) respectively. The reason for this monotonic increasing behavior is that for larger T_{on} the creation time is longer. We can expect that $N(t)$ can come to its halt within T_{on} if T_{on} is extremely large. When the potential is completely turned on, at points B_2, C_2, D_2, there are more pairs for larger T_{on}. The four characterized time regimes for curve $\boldsymbol{A}$ are destroyed if $T_{on} > 0$. In the very early time regime, a quartic growth can be observed, $N = \sum_{n,p} |\langle n|V|p\rangle|^2 t^2 \sim t^4$. The following process within T_{on} depends on the function of turning on, which is $\sin^2\left((\pi/2T_{on})t\right)$ here.

The potential hold on static during T, and pair numbers approach to its final values N_F at $t = t_3$. N_F decreases as T_{on} increases. Now, we come to a question: when T_{on} is large enough, can N_F equal 1, the number of EBSs? The simulation results are shown in Fig. 4. Strangely, as T_{on} increases, N_F converges to 1.131 quickly. Here T is fixed to a constant $T = 0.012$. If we extend T, N_F will get a larger value, as we will discuss later for Fig. 5. The number 1.131 also depends on the concrete parameters of the potential well. So the slow turn on can reduce the pair number, but cannot get a pair number precisely equal to the integer of EBSs. In Refs. [5]–[8] the excess amount is attributed to the partially populated discrete bound states located in the gap. In our opinion, the excess

amount is associated with the virtual particles which have no time to annihilate, as discussed at the end.

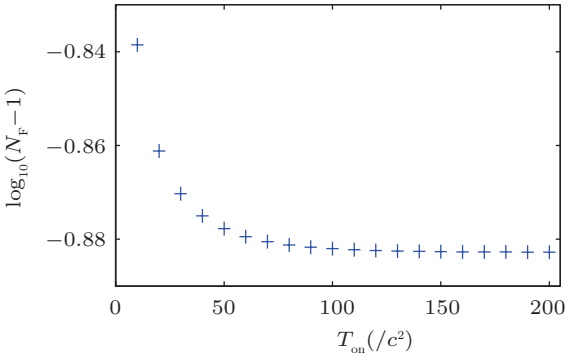

Fig. 4. (color online) $\log_{10}(N_F-1)$ as a function of T_{on}. N_F is the value of $N(t)$ at the end of T. $T=0.012$ and $T_{off}=0$ are fixed. When $T_{on}=200/c^2$, (N_F-1) is 0.131.

To understand the way in which $N(t)$ approaches to its final value N_F at $t=t_3$ and how it is affected by the slow turn on, we refer to the energy spectrum. During T, the system has one bound state embedded in the Dirac sea at $E=-1.31c^2$, and four discrete bound states in the gap $-c^2<E<c^2$. The lowest discrete bound state is at $E=-0.84c^2$, see Fig. 1. Then oscillation of $N(t)$ with two frequencies can be expected: (i) from each edge of the well, the electrons created at $E=-1.31c^2$ with speed $v=0.85c$, can reach another edge after a time of order of $W/v=2.9\times10^{-4}$, suppress the creation process there, and then be reflected; and (ii) employing an effective two level model,[21] the gap between the lowest bound state and the Dirac sea upper limit $(\Delta E=(-0.84c^2)-(-c^2)=0.16c^2)$ determine another oscillation with period $2\pi/\Delta E=2.1\times10^{-3}$.

In Fig. 5(a), to clearly show how $N(t)$ approach to its N_F in Fig. 2, we move curves B, C, D to make all right full triangles located at the same point. Obviously, for $T_{on}=0$, there are two frequency oscillations with periods $T_1=3.2\times10^{-4}$ and $T_2=2.1\times10^{-3}$. The oscillation T_1 matches the estimation $W/v=2.9\times10^{-4}$ with error about 10%. The error can be attributed to the previous process when the EBS going down to $E=-1.31c^2$ from the Dirac sea upper limit $E=-c^2$. The oscillation T_2 matches the estimation $2\pi/\Delta E=2.1\times10^{-3}$ precisely. As shown in the figure, the oscillation T_1 can be suppressed by slow turn on, because the electrons created earlier have a significant proportion and their speeds cover from zero to $v=0.85c$. For larger T_{on}, the oscillation T_2 is suppressed too.

The long time behavior of the pair creation can be described by an exponential saturation, which is determined by the widths of the EBSs.[6] In all the simulations here, the exponential saturation has no difference because the systems are the same during the time region T. In Fig. 5(b), we plot $N_F-N(t)$ as a function of time. The exponential decay of $N_F-N(t)$ can be fitted by $\exp(-\Gamma t)$, where Γ can be obtained from the complex coordinate scaling technique.[6] Thus in a word briefly, the pair number evolution at time region T can be characterized by an exponential saturation accompanied by a two frequency oscillation, and the oscillation can be suppressed by the slow turn on while the exponential saturation remains robust.

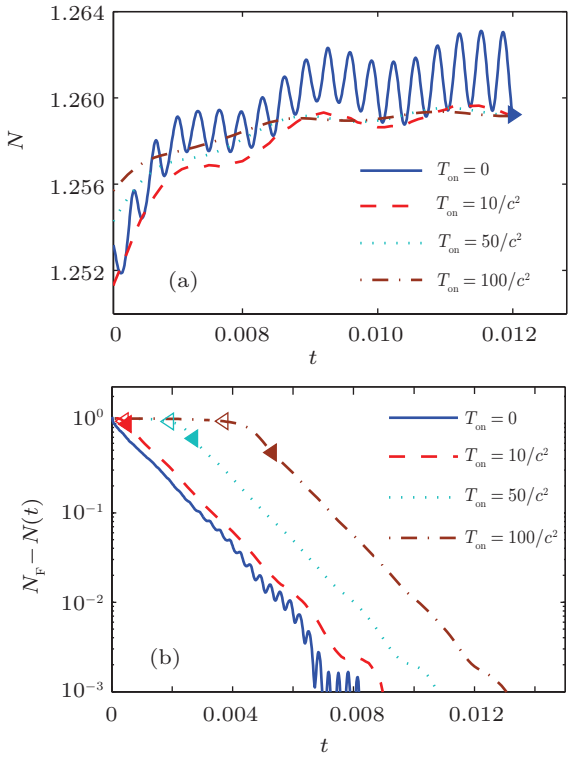

Fig. 5. (color online) (a) Curves B, C, D in Fig. 2 are moved to make their right full triangles locate at the right full triangle of curve A for comparison. (b) For every case the final pair number N_F minus its $N(t)$ is shown to illustrate the exponential saturation. The empty and filled triangles mark the moments as described in Fig. 1.

3.3. Effects of slow turn off

In this subsection we fix $T_{on}=0$ and analyze the effect of the turn off process. Pairs annihilate during T_{off}. It seems that (N_F-1) will vanish continuously as T_{off} increases. Actually, as figure 6 shows, the final pair number N_F will converge to 1.136 quickly in the same way as N_F converges to 1.131 in Fig. 4. Here 1.136 is approximately equal to the value 1.131. No matter how long T_{off} is, an abruptly turned on potential well can only provide a larger pair yield than the number of EBSs which depends on the duration T and potential configuration parameters.

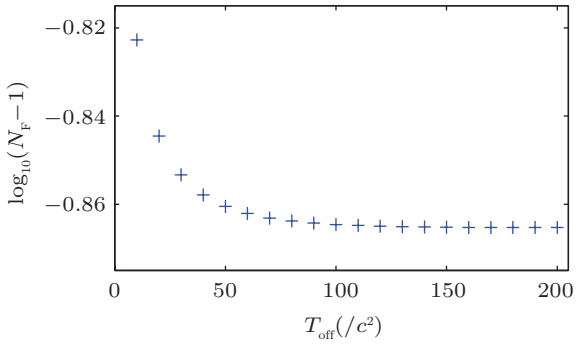

Fig. 6. (color online) $\log_{10}(N_F-1)$ as a function of T_{off}. N_F is the value of $N(t)$ at the end of T_{off}. $T_{on}=0$ and $T=0.012$ are fixed. When $T_{off}=200/c^2$, (N_F-1) is 0.136.

3.4. Combined effects of slow turn on and turn off

For different sets of $(T_{\rm on}, T_{\rm off})$, final pair numbers $N_{\rm F}$ are listed in Table 2. The decreased amounts caused by $T_{\rm on}$ or $T_{\rm off}$ are in the same order of magnitude, and the table is on diagonal symmetry. Figure 3 shows asymptotic behavior along the diagonal of the table. Up to the numerical precision, the final pair number for $(T_{\rm on}, T_{\rm off}) = (10,50)$, $(10,100)$, $(50,100)$ are equal to that for $(50, 10)$, $(100,10)$, $(100,50)$ respectively. On the other hand, if the total time is fixed, we should set $T_{\rm on} = T_{\rm off}$ to get a reasonable pair yield. For example, when $T_{\rm on} + T_{\rm off} = 100$, if $(T_{\rm on}, T_{\rm off}) = (50,50)$, the final total number is 1.003603, the deviation from 1 is no more than 0.4%, far less than the 13% when $(T_{\rm on}, T_{\rm off}) = (100,0)$ or $(0,100)$.

Table 2. The final pair number $N_{\rm F}$ for different sets of $(T_{\rm on}, T_{\rm off})$. $T = 0.012$ is fixed.

$T_{\rm off}/c^2$ \ $T_{\rm on}/c^2$	0	10	50	100
0	1.259156	1.150411	1.137880	1.136583
10	1.145043	1.030028	1.016886	1.015523
50	1.132511	1.016886	1.003603	1.002187
100	1.131215	1.015523	1.002187	1.000746

3.5. The single electron-positron pairs

To get a deeper insight of the pair number evolution, we compute the single electron-positron pair number,[5,15,22]

$$P_1(t) \equiv \iint \mathrm{d}x\mathrm{d}y\,|\Phi(x,y,t)|^2,$$

where $\Phi(x,y,t) \equiv \langle \mathrm{vac}\|\hat{\Psi}^{(+)}(x,t)\hat{\Psi}_c^{(+)}(y,t)\|\mathrm{vac}\rangle$ is the single electron-positron pair wave function and the subscript c denotes the charge conjugation operation. In the early process, $\Phi(x,y,t)$ contains all the information about creation. The single-pair number $P_1(t)$ is equal to the total pair number $N(t)$. As time increases, multiple pairs are created, and single-pair number $P_1(t)$ becomes less than total pair number $N(t)$.[22] If the potential is quickly turned on, for example, see curves $T_{\rm on} = 0$ (as discussed in Ref. [5]) and $T_{\rm on} = 10/c^2$ in the above panel of Fig. 7, $P_1(t)$ even suffers a shrink because some electrons become disentangled with corresponding positrons.

For larger $T_{\rm on}$, $P_1(t)$ comes to a halt in time region T. As $T_{\rm on}$ increases, the single-pair will dominate the pair creation on an even longer time scale within $T_{\rm on}$, and multiple pairs are created much later. Since the electrons are trapped in the well and positrons are ejected, this also illustrates a robust long range correlation between particle and antiparticle. The halt number of $P_1(t)$ approaches to 1 (number of EBSs) for extremely large $T_{\rm on}$, as $T_{\rm on} = 2000/c^2$, shown in Fig. 7.

In time region $T_{\rm off}$, single-pairs cannot survive. The disentanglement makes P_1 decrease, and finally shrink to zero. Then the accompanied pair annihilation lead to that $N(t)$ approach to 1. Therefore, the decreasing of total pair number $N(t)$ in time region $T_{\rm off}$ is associated with the disentangling process of the single-pair and the consequent annihilation of particle-antiparticle pairs.

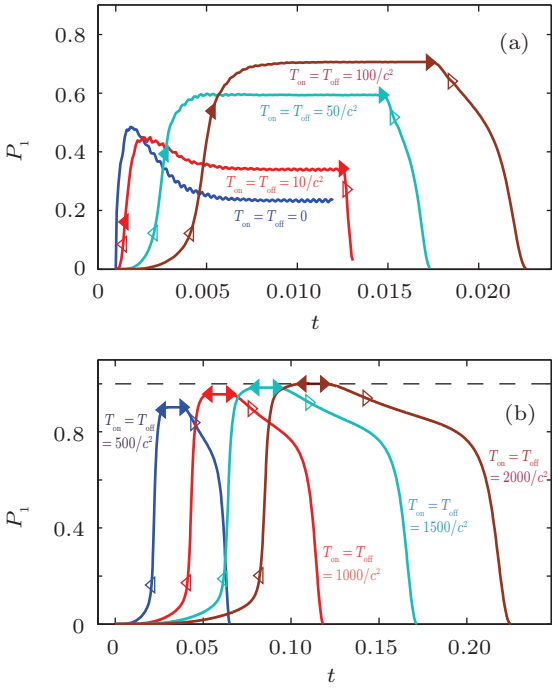

Fig. 7. (color online) P_1 as a function of time for different $T_{\rm on} = T_{\rm off}$. (a) $T_{\rm on} = T_{\rm off} = 0$, $10/c^2$, $50/c^2$, and $100/c^2$. (b) $T_{\rm on} = T_{\rm off} = 500/c^2$, $1000/c^2$, $1500/c^2$, and $2000/c^2$. The field duration is always $T = 0.012$. The empty and filled triangles mark the moments as described in Fig. 1.

4. Summary and discussion

In summary, using a space–time resolved numerical method, we have studied the dynamical effects of electron-positron pair creation caused by switching processes of supercritical well potential with bound states embedded in the Dirac sea. Due to the single-pair disentanglement and electron–positron annihilation process, the final pair number can converge to the integer of embedded bound states nearly exponentially as $T_{\rm on}$ and $T_{\rm off}$ both increase. In the adiabatic limit, all early pairs are created in the form of single electron–positron pairs, and the single-pair number will approach to the number of embedded bound states during the holding on of the potential. We also have found that the decreased amounts of final pair number caused by $T_{\rm on}$ and $T_{\rm off}$ are in the same order of magnitude, and the oscillations which were superposed on the pair number's exponential saturation can be suppressed by $T_{\rm on}$.

Finally, let us review the definition of the pair number in Eq. (3). The time-dependent operator and the initial field free operator are connected through the Bogoliubov transformation, similar to the quantum kinetic theory (QKT) based on the quantum Vlasov equation.[23,24] The total pair number $N(t)$

here, as in QKT, is actually a mixture of real and virtual excitation at finite time. It is presently still an unresolved conceptual problem to distinguish real and virtual excitation as long as the field is non-vanishing in various theoretical approaches. Here in this model, in light of the well-known knowledge of the adiabatic limit, i.e., the convergence of the pair yield to the integer of embedded bound states, an estimation of virtual excitation is feasible. It is the excess amounts more than the integer, one order of magnitude smaller than the real excitation in this case, and it can be suppressed by slow switching.

References

[1] Greiner W, Müller B and Rafelski J 1985 *Quantum Electrodynamics of Strong Fields* (Berlin: Springer-Verlag)

[2] Belkacem A, Gould H, Feinberg B, Bossingham R and Meyerhof W E 1993 *Phys. Rev. Lett.* **71** 1514

[3] Burke D L, et al. 1997 *Phys. Rev. Lett.* **79** 1626

[4] Di Piazza A, Müller C, Hatsagortsyan K Z and Keitel C H 2012 *Rev. Mod. Phys.* **84** 1177

[5] Krekora P, Cooley K, Su Q and Grobe R 2005 *Phys. Rev. Lett.* **95** 070403

[6] Lv Q Z, Liu Y, Li Y J, Grobe R and Su Q 2013 *Phys. Rev. Lett.* **111** 183204

[7] Liu Y, Jiang M, Lv Q Z, Li Y T, Grobe R and Su Q 2014 *Phys. Rev. A* **89** 012127

[8] Lv Q Z, Liu Y, Li Y J, Grobe R and Su Q 2014 *Phys. Rev. A* **90** 013405

[9] Zuo T and Bandrauk A D 1995 *Phys. Rev. A* **52** R2511

[10] Wu J, Meckel M, Schmidt L Ph H, Kunitski M, Voss S, Sann H, Kim H, Jahnke T, Czasch A and Dörner R 2013 *Nat Commun.* **3** 1113

[11] Xu H, He F, Kielpinski D, Sang R T and Litvinyuk I V 2015 *Sci. Rep.* **5** 13527

[12] Tang S, Xie B S, Lu D, Wang H Y, Fu L B and Liu J 2013 *Phys. Rev. A* **88** 012106

[13] Fillion-Gourdeau F, Lorin E and Bandrauk A D 2013 *Phys. Rev. Lett.* **110** 013002

[14] Cheng T, Su Q and Grobe R 2010 *Contemp. Phys.* **51** 315

[15] Krekora P, Su Q and Grobe R 2004 *Phys. Rev. Lett.* **93** 043004

[16] Krekora P, Su Q and Grobe R 2004 *Phys. Rev. Lett.* **92** 040406

[17] Mocken G R and Keitel C H 2004 *J. Comput. Phys.* **199** 558

[18] Mocken G R and Keitel C H 2008 *Comput. Phys. Commun.* **178** 868

[19] Wang Q, Liu J and Fu L B 2016 *Sci. Rep.* **6** 25292

[20] Gerry C C, Su Q and Grobe R 2006 *Phys. Rev. A* **74** 044103

[21] Liu Y, Lv Q Z, Li Y T, Grobe R and Su Q 2015 *Phys. Rev. A* **91** 052123

[22] Krekora P, Cooley K, Su Q and Grobe R 2005 *Laser Phys.* **15** 282

[23] Smolyansky S A, Röpke G, Schmidt S, Blaschke D, Toneev V D and Prozorkevich A V 1997 arXiv: hep-ph/9712377

[24] Schmidt S, Blaschke D, Röpke G, Smolyansky S, Prozorkevich A and Toneev V 1998 *Int. J. Mod. Phys. E* **07** 709