Alternative fidelity measure between two states of an *N*-state quantum system

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An alternative fidelity measure between two states of a qunit, an *N*-state quantum system, is proposed. It has a hyperbolic geometric interpretation, and it reduces to the Bures fidelity in the special case when $N=2$.

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I. INTRODUCTION AND MOTIVATION

The concept of fidelity is important in communication theory. In particular, the Bures fidelity is a most important distance measure for quantum computation and quantum information [1–7]. For any pair of density operators ρ_1 and ρ_2, the Bures fidelity

$$F(\rho_1,\rho_2)=[\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}]^2 \tag{1}$$

quantifies the extent to which ρ_1 and ρ_2 can be distinguished from one another. The Bures fidelity has useful properties. Thus, for instance, $0\leqslant F(\rho_1,\rho_2)\leqslant 1$, and $F(\rho_1,\rho_2)=1$ if and only if $\rho_1=\rho_2$, and for any unitary transformation $U, F(U\rho_1U^\dagger,U\rho_2U^\dagger)=F(\rho_1,\rho_2)$.

A qubit is a two-state quantum system represented by the 2×2 density matrix

$$\rho(\mathbf{n})=\tfrac{1}{2}(\mathbf{1}+\vec{\sigma}\cdot\mathbf{n}),|\mathbf{n}|\leqslant 1, \tag{2}$$

where $\mathbf{1}$ is the unit matrix, $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices in vector notation, and $\mathbf{n}$ is the three-dimensional Bloch vector. Equality, $|\mathbf{n}|=1$, in Eq. (2) corresponds to a pure state, otherwise a mixed state. Let

$$\rho_1=\tfrac{1}{2}(\mathbf{1}+\vec{\sigma}\cdot\mathbf{u}),$$

$$\rho_2=\tfrac{1}{2}(\mathbf{1}+\vec{\sigma}\cdot\mathbf{v}) \tag{3}$$

be two states of a qubit. Then

$$F(\rho_1,\rho_2)=\tfrac{1}{2}[1+\mathbf{u}\cdot\mathbf{v}+\sqrt{1-|\mathbf{u}|^2}\sqrt{1-|\mathbf{v}|^2}]. \tag{4}$$

Following [8], we introduce the hyperbolic parameter $\phi_{\mathbf{u}}$, called *rapidity*, representing the Bloch vector by the equation

$$\mathbf{u}=\hat{\mathbf{u}}\tanh\phi_{\mathbf{u}}, \tag{5}$$

where $\hat{\mathbf{u}}=\mathbf{u}/|\mathbf{u}|$ is a unit vector. Clearly, $\phi_{\mathbf{u}}=0$ corresponds to $|\mathbf{u}|=0$, and $\phi_{\mathbf{u}}\to\infty$ corresponds to $|\mathbf{u}|=1$. As shown in [8], the density matrix $\rho(\mathbf{u})$ is related to the Lorentz boost matrix $L(\mathbf{u})$,

$$L(\mathbf{u})=\exp\left(\frac{\varphi_{\mathbf{u}}}{2}\vec{\sigma}\cdot\hat{\mathbf{u}}\right)=\mathbf{1}\cosh\left(\frac{\varphi_{\mathbf{u}}}{2}\right)+\vec{\sigma}\cdot\hat{\mathbf{u}}\sinh\left(\frac{\varphi_{\mathbf{u}}}{2}\right), \tag{6}$$

by the equation

$$\rho(\mathbf{u})=\frac{L(\mathbf{u})}{2\cosh\phi_{\mathbf{u}}},\quad \phi_{\mathbf{u}}=\varphi_{\mathbf{u}}/2. \tag{7}$$

Clearly, $\rho(\mathbf{u})$ and $L(\mathbf{u})$ are in one-to-one correspondence. Interestingly, the vector $\mathbf{u}$ in the former is the Bloch vector of quantum mechanics, while the vector $\mathbf{u}$ in the latter is the generic relativistically admissible velocity. Relativistically admissible velocities, in turn, give rise to the Thomas precession [9], and are regulated by the hyperbolic geometry of Bolyai and Lobachevski as explained in [10] and [11].

Viewing the Bloch vector $\mathbf{u}$ in Eq. (4) as a relativistically admissible velocity, the identity

$$F(\rho_1,\rho_2)=\frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{u}}}\,\frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{v}}} \tag{8}$$

was established in Ref. [8]. Here $\mathbf{w}$ is the Einstein sum $\mathbf{w}=\mathbf{u}\oplus\mathbf{v}$, $\oplus$ being the Einstein addition operation between relativistically admissible velocities. It is given by the equation

$$\mathbf{w}=\mathbf{u}\oplus\mathbf{v}=\frac{1}{1+\dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left[\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right], \tag{9}$$

where $\gamma_{\mathbf{u}}=1/\sqrt{1-|\mathbf{u}|^2/c^2}=\cosh\phi_{\mathbf{u}}$ is the Lorentz factor, and where c is the vacuum speed of light. The positive constant c is normalized to $c=1$, when $\mathbf{u}$ is viewed as a Bloch vector. The rapidity $\phi_{\mathbf{w}}$ satisfies the cosine law of hyperbolic geometry,

$$\cosh\phi_{\mathbf{w}}=\cosh\phi_{\mathbf{u}}\cosh\phi_{\mathbf{v}}(1+\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}\tanh\phi_{\mathbf{u}}\tanh\phi_{\mathbf{v}}), \tag{10}$$

a result already known to Silberstein in 1914 [12].

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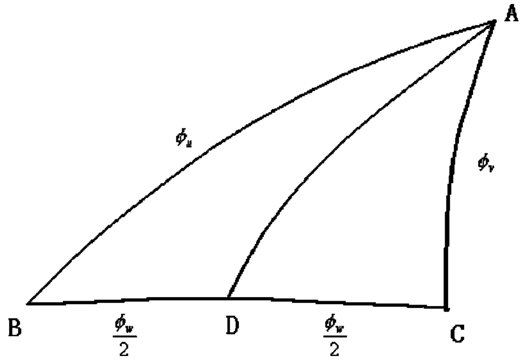

FIG. 1. The hyperbolic triangle ABC. Its three sides are $|AB| = \phi_{\mathbf{u}} = \tanh^{-1}|\mathbf{u}|$, $|AC| = \phi_{\mathbf{v}} = \tanh^{-1}|\mathbf{v}|$, and $|BC| = \phi_{\mathbf{w}} = \tanh^{-1}|\mathbf{w}|$. D is the midpoint of the side BC. The angle between AB and AC is equal to $\pi - \arccos(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}})$.

The hyperbolic angles $\{\phi_{\mathbf{u}}, \phi_{\mathbf{v}}, \phi_{\mathbf{w}}\}$ form a hyperbolic triangle (see Fig. 1, where D is the midpoint of the side BC). Interestingly, the Bures fidelity for a qubit appears in Eq. (8) as the product of two similar factors. Furthermore, it follows from Eq. (8) that $F(\rho_1,\rho_2)$ is symmetric in its arguments, and is invariant under unitary transformations on the state space.

A remarkable property that Eq. (4) exhibits is that $F(\rho_1,\rho_2)$ is solely dependent on the magnitudes of $\mathbf{u}$ and $\mathbf{v}$, and the angle between them (that is, $\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}$). However, this remarkable property is lost when one calculates the Bures fidelity for an N-state quantum system in the case of $N \geqslant 3$; as a result, the simple geometric interpretation for the quantum fidelity as shown in Eq. (8) and in Fig. 1 is no longer valid. For instance, we take two states of a qutrit as

$$\rho_1 = \frac{1}{3}(\mathbf{1} + \sqrt{3}\vec{\lambda}\cdot\mathbf{u}) = \frac{1}{3}\begin{pmatrix} 1+\sqrt{3}u_3+u_8 & 0 & 0 \\ 0 & 1-\sqrt{3}u_3+u_8 & 0 \\ 0 & 0 & 1-2u_8 \end{pmatrix},$$
$$\rho_2 = \frac{1}{3}(\mathbf{1} + \sqrt{3}\vec{\lambda}\cdot\mathbf{v}) = \frac{1}{3}\begin{pmatrix} 1+\sqrt{3}v_3+v_8 & 0 & 0 \\ 0 & 1-\sqrt{3}v_3+v_8 & 0 \\ 0 & 0 & 1-2v_8 \end{pmatrix}, \tag{11}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_8)$ are the eight Hermitian generators of SU(3), $\mathbf{u} = (0,0,u_3,0,\ldots,0,u_8)$ and $\mathbf{v} = (0,0,v_3,0,\ldots,0,v_8)$. Then, the Bures fidelity for a qutrit is given by the equation

$$F(\rho_1,\rho_2) = \tfrac{1}{9}\Big[\sqrt{(1+\sqrt{3}u_3+u_8)(1+\sqrt{3}v_3+v_8)} + \sqrt{(1-\sqrt{3}u_3+u_8)(1-\sqrt{3}v_3+v_8)} + \sqrt{(1-2u_8)(1-2v_8)}\Big]^2, \tag{12}$$

which is not solely dependent on $|\mathbf{u}|, |\mathbf{v}|$, and $\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}$.

We therefore propose in the following theorem an alternative definition for quantum fidelity, $\mathcal{F}(\rho_1,\rho_2)$, following which the fidelity measure for any two states of a qunit [13] has the geometric interpretation suggested by Eq. (8).

II. FORMALISM

Theorem. Let

$$\rho(\mathbf{n}) = \frac{1}{N}\left(\mathbf{1} + \sqrt{\frac{N(N-1)}{2}}\vec{\lambda}\cdot\mathbf{n}\right) \tag{13}$$

be the density matrix of a qunit, where $\mathbf{1}$ is the $N\times N$ unit matrix, $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{N^2-1})$ are the generators of SU(N), and $\mathbf{n}$ is the (N^2-1)-dimensional Bloch vector. Furthermore, let $\rho_1 = \rho(\mathbf{u})$ and $\rho_2 = \rho(\mathbf{v})$ be two states of a qunit. Then the fidelity measure

$$\mathcal{F}(\rho_1,\rho_2) = \frac{1-r}{2} + \frac{1+r}{2}\Big[\mathrm{tr}(\rho_1\rho_2) + \sqrt{1-\mathrm{tr}(\rho_1^2)}\sqrt{1-\mathrm{tr}(\rho_2^2)}\Big], \tag{14}$$

where $r = 1/(N-1)$, can be written as

$$\mathcal{F}(\rho_1,\rho_2) = \frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{u}}}\,\frac{\cosh(\phi_{\mathbf{w}}/2)}{\cosh\phi_{\mathbf{v}}}, \tag{15}$$

where $\mathbf{w} = \mathbf{u}\oplus\mathbf{v}$.

Proof. From the well-known trace relations

$$\mathrm{tr}(\lambda_i) = 0,\ \mathrm{tr}(\lambda_i\lambda_j) = 2\delta_{ij} \tag{16}$$

for the generators of SU(N) we have

$$\mathrm{tr}(\rho_1\rho_2) = \frac{1+(N-1)\mathbf{u}\cdot\mathbf{v}}{N},$$
$$\mathrm{tr}(\rho_1^2) = \frac{1+(N-1)|\mathbf{u}|^2}{N},$$
$$\mathrm{tr}(\rho_2^2) = \frac{1+(N-1)|\mathbf{v}|^2}{N}. \tag{17}$$

Substituting these equations into Eq. (14), noting $r = 1/(N-1)$, we obtain Eq. (15), and the proof is complete.

III. CONCLUSION AND DISCUSSION

We have proposed an alternative fidelity measure between two states of a qunit. For any N-state quantum system, the fidelity measure possesses the geometric interpretation that Eq. (15) uncovers. The following observations may be noted.

(a) Geometric interpretation of the parameter r in Eq. (14): It is well known that a density matrix must satisfy three conditions. (i) Trace unity $\mathrm{tr}\rho = 1$; (ii) Hermiticity; and (iii) positivity, i.e., all eigenvalues of ρ are non-negative. Indeed, the operator $\rho(\mathbf{n})$ in Eq. (13) satisfies the first two conditions. However, not every vector $\mathbf{n}, |\mathbf{n}| \leqslant 1$, allows $\rho(\mathbf{n})$ to satisfy the positivity condition. Assuming that $\rho(\mathbf{u})$ is a density matrix satisfying the above three conditions, if $\rho(\mathbf{v})$ is a density matrix, one must have the constraint $\mathrm{tr}(\rho_1\rho_2) \geqslant 0$, that is $-\mathbf{u}\cdot\mathbf{v} \leqslant r$. For instance, let $\rho(\hat{\mathbf{u}}) = (\mathbf{1} + \sqrt{3}\vec{\lambda}\cdot\hat{\mathbf{u}})/3$ be a pure state of a qutrit, where $\hat{\mathbf{u}}$ is a unit vector (e.g., $\hat{\mathbf{u}} = (0,0,\sqrt{3}/2,0,\ldots,0,1/2)$. Then, $\rho(-\hat{\mathbf{u}})$ is not a density matrix since it violates the positivity condition. The operator $\rho(-\mathbf{u}) = (\mathbf{1} - \sqrt{3}|\mathbf{u}|\vec{\lambda}\cdot\hat{\mathbf{u}})/3$ is a density operator if $|\mathbf{u}| \leqslant 1/2$ (Note that $r=1/2$ for $N=3$). Thus, if $|\mathbf{n}| \leqslant r, \rho(\mathbf{n})$ is always a density matrix regardless of the direction of $\mathbf{n}$. Geometrically, r is the radius of a *characteristic ball* inside the Bloch sphere such that any point located either on the surface of the ball or inside the ball corresponds to a physical state of a qunit.

(b) For $N=2$, we have $r=1$, which implies that the *characteristic ball* is identical to the usual Bloch sphere of a qubit. Therefore one obtains $\mathcal{F}(\rho_1,\rho_2) = F(\rho_1,\rho_2)$, that is, in the case of a qubit the alternative fidelity measure is identical to the Bures fidelity.

(c) The radius r decreases when N increases, $r \to 0$ when $N \to \infty$. For Bloch vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying $0 \leqslant |\mathbf{u}|, |\mathbf{v}| \leqslant r$, one can obtain the usual *trace distance* $d = |\mathbf{u} - \mathbf{v}|/2$ from the distance measure [1] $d^2(\rho_1,\rho_2) = 2[1 - \sqrt{\mathcal{F}(\rho_1,\rho_2)}]$ as a first approximation.

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