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LETTER TO THE EDITOR

The configuration of a topological current and its physical structure: an application and paradigmatic evidence

Li-Bin Fu[1], Jie Liu[1], Shi-Gang Chen[1] and Yi-Shi Duan[2]

[1] Institute of Applied Physics and Computational Mathematics, PO Box 8009(28), Beijing 100088, People's Republic of China
[2] Institute of Theoretical Physics, Department of Physics, Lanzhou University, 730000, People's Republic of China

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Abstract
In the ϕ-mapping theory, the topological current constructed from the order parameters can possess different inner structures. The difference in topology must correspond to the difference in physical structure. The transition between different structures happens at the bifurcation point of the topological current. In a self-interaction two-level system, the change of topological particles corresponds to a change of energy levels.

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In recent years, topology has established itself as an important part of the physicist's mathematical arsenal [1]. The concepts of the topological particle and its current have been widely used in particle physics [2, 3] and topological defect theory [4]. Here, the topological particles are regarded as abstract particles, such as monopoles and point defects.

In this letter, we achieve an understanding new to topology and physics. Many physical systems can be described by employing the order parameters. By making use of the ϕ-mapping theory, we find that the topological current constructed from the order parameters can possess different inner structures. The topological properties are basic properties for a physical system, so the difference in configuration of the topological current must correspond to a difference in physical structure.

Consider an $(n+1)$-dimensional system with an n-component-vector order parameter field $\vec{\phi}(\boldsymbol{x})$, where $\boldsymbol{x} = (x^0, x^1, x^2, \ldots, x^n)$ corresponds to local coordinates. The direction unit field of $\vec{\phi}$ is defined by

$$n^a = \frac{\phi^a}{\|\phi\|}, \qquad a = 1, 2, \ldots, n \tag{1}$$

where

$$\|\phi\| = (\phi^a \phi^a)^{1/2}.$$

The topological current of this system is defined by

$$j^\mu(x) = \frac{\varepsilon^{\mu\mu_1\ldots\mu_n}}{A(S^{n-1})(d-1)!}\varepsilon_{a_1\ldots a_n}\partial_{\mu_1}n^{a_1}\cdots\partial_{\mu_n}n^{a_n} \tag{2}$$

where $A(S^{n-1})$ is the surface area of the $(n-1)$-dimensional unit sphere S^{n-1}. Obviously, the current is identically conserved:

$$\partial_\mu j^\mu = 0.$$

If we define a Jacobian by

$$\varepsilon^{a_1\ldots a_n} D^\mu\left(\frac{\phi}{x}\right) = \varepsilon^{\mu\mu_1\ldots\mu_n}\partial_{\mu_1}\phi^{a_1}\cdots\partial_{\mu_n}\phi^{a_n}, \tag{3}$$

then, as has been proved before [5], this current takes the form

$$j^\mu = \delta(\vec{\phi}) D^\mu\left(\frac{\phi}{x}\right). \tag{4}$$

Then, we can obtain

$$j^\mu = \sum_{i=1}^{l}\beta_i\eta_i\delta(\vec{x}-\vec{z}_i(x^0))\frac{\mathrm{d}z_i^\mu}{\mathrm{d}x^0}, \tag{5}$$

in which $z_i(x^0)$ are the zero-lines where $\vec{\phi}(\boldsymbol{x}) = 0$, the positive integer β_i and $\eta_i = \operatorname{sgn} D(\frac{\phi}{x})$ are the Hopf index and Brouwer degree of ϕ-mapping [7] respectively, and l is the total number of zero-lines. This current is similar to a current of point particles (and the ith one has the charge $\beta_i\eta_i$) and the zero-lines $z_i(x)$ are just the trajectories of the particles; for convenience we define these point particles as topological particles. Then the total topological charge of the system is

$$Q = \int_M j^0\,\mathrm{d}^n x = \sum_{i=1}^{l}\beta_i\eta_i,$$

where M is a n-dimensional spatial space for a given x^0. This is a topological invariant and corresponds to some basic conditions of this physical system. However, it is important that the inner structure of the topological invariant can be constructed in different configurations, i.e., the number of topological particles and their charge can be changed. This change in configuration of the topological current must correspond to some change in physical structure.

All of the above discussion is based on the condition that

$$D\left(\frac{\phi}{x}\right) = D^0\left(\frac{\phi}{x}\right)\bigg|_{z_i} \neq 0$$

when $D(\phi/x)|_{z_i} = 0$ at some points $p_i^* = z^*(x_c^0)$ at $x^0 = x_c^0$ along the zero-line $z_i(x^0)$; it is shown that there exist several crucial cases of branch processes which correspond to the topological particle being generated or annihilating at limit points and splitting, encountering another particle, or merging at the bifurcation points. A vast amount of literature has been devoted to discussing these features of the evolution of the topological particles [8]. Here, we will not spend more time on describing these evolutions, but will focus our attention on the physical substance of these processes.

As we already know, all of these branch processes keep the total topological charge conserved, but it is very important that these branch processes change the number and the charge of the topological particles. i.e. change the inner structure of the topological current. From our point of view, a different configuration of a topological current corresponds to a different physical structure.

We consider x^0 as a parameter λ of a physical system. Let us define

$$f_i(\lambda) = D^0\left(\frac{\phi}{x}\right)\bigg|_{z_i}. \tag{6}$$

As λ changes, the value of $f_i(\lambda)$ changes along the zero-lines $z_i(\lambda)$. At a critical point $\lambda = \lambda_c$, when $f_i(\lambda_c) = 0$, we know that the inner structure of the topological current will be changed in some way, and at the same time the physical structure will also be changed, i.e., the physical structure when $\lambda < \lambda_c$ will be different from the one when $\lambda > \lambda_c$. The transition between these structures occurs at the bifurcation points where $f_i(\lambda) = 0$.

As an application and example, let us consider a self-interacting two-level model introduced in [9]. The nonlinear two-level system is described by the dimensionless Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = H(\gamma)\begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

with the Hamiltonian given by

$$H(\gamma) = \begin{pmatrix} \gamma/2 + (C/2)(|b|^2 - |a|^2) & V/2 \\ V/2 & -\gamma/2 - (C/2)(|b|^2 - |a|^2) \end{pmatrix}, \tag{8}$$

in which γ is the level separation, V is the coupling constant of the two levels, and C is the nonlinear parameter describing the interaction. The total probability $|a|^2 + |b|^2$ is conserved and is set to 1.

We assume $a = |a|\mathrm{e}^{\mathrm{i}\varphi_1(t)}$, $b = |b|\mathrm{e}^{\mathrm{i}\varphi_2(t)}$; the fractional population imbalance and relative phase can be defined by

$$z(t) = |b|^2 - |a|^2, \qquad \varphi(t) = \varphi_2(t) - \varphi_1(t). \tag{9}$$

From equations (7) and (8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = -V\sqrt{1 - z^2(t)}\sin[\varphi(t)] \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \gamma + Cz(t) + \frac{Vz(t)}{\sqrt{1 - z^2(t)}}\cos[\varphi(t)]. \tag{11}$$

If we chose $x = 2|a||b|\cos(\varphi)$, $y = 2|a||b|\sin(\varphi)$, it is easy to see that $x^2+y^2+z^2 = 1$ by considering $|a|^2+|b|^2 = 1$, which describes a unit sphere S^2 with z and φ a pair of coordinates. We define a vector field on this unit sphere:

$$\phi^1 = -V\sqrt{1 - z^2}\sin(\varphi), \tag{12}$$

$$\phi^2 = \gamma + Cz + \frac{Vz}{\sqrt{1 - z^2}}\cos(\varphi). \tag{13}$$

Apparently, there are singularities at the two pole points $z = \pm 1$, which make the vector $\vec{\phi}$ discontinuous at these points. However, the direction unit vector $\vec{n}$ is continuous. In the ϕ-mapping theory, we only need the unit vector $\vec{n}$ to be continuous and differentiable on the whole sphere S^2 (at the zero-points of $\vec{\phi}$, the differential of $\vec{n}$ is a general function), and the vector $\vec{\phi}$ to be continuous and differentiable in the neighbourhoods of its zero-points. Then from ϕ-mapping theory, we can obtain a topological current:

$$\vec{j} = \sum_{i=1}^{l} \beta_i\eta_i\delta(\varphi - \varphi_i(\gamma))\delta(z - z_i(\gamma))\frac{\mathrm{d}z_i}{\mathrm{d}\gamma}\bigg|_{p_i}, \tag{14}$$

in which $p_i = p_i(z_i, \varphi_i)$ is the trajectory of the ith topological particle P_i and

$$\eta_i = \mathrm{sgn}(D(\gamma)) = \mathrm{sgn}\left(\det\begin{pmatrix} \partial\phi^1/\partial\varphi & \partial\phi^2/\partial\varphi \\ \partial\phi^1/\partial z & \partial\phi^2/\partial z \end{pmatrix}\Bigg|_{p_i}\right). \tag{15}$$

From equations (12) and (13), it is easy to see that $\vec{\phi}$ is single valued on S^2, which indicates that the Hopf index $\beta_i = 1$ $(i = 1, 2, \ldots, l)$ here. It can be proved that the total charge of this system is just the Euler number of S^2 which is 2 [6]. This is a topological invariant of S^2 which corresponds to the basic condition of this system: $|a|^2 + |b|^2 = 1$. The following discussion can show that this topological invariant can possess different configurations when γ changes. This difference in topology corresponds to the change in adiabatic levels of this nonlinear system.

We can prove that every topological particle corresponds to an eigenstate of the nonlinear two-level system. By solving for $\vec{\phi} = 0$ from (12) and (13), we find that there are two different cases to discuss.

Case 1. For $|C/V| \leqslant 1$, there only two topological particles P_1 and P_2, which are located on the lines $\varphi = 0$ and π respectively as shown in the upper panel of figure 1. All of them have topological charge +1, and $D(\gamma)|_{P_{1,2}} > 0$ for any γ. Correspondingly, in this case, there are only two adiabatic energy levels in this nonlinear two-level mode for various γ [9], as shown in the lower panel of figure 1; P_1 corresponds to the upper level and P_2 corresponds to the lower level.

Case 2. For $C/V > 1$, two more topological particles can appear when γ lies in a window $-\gamma_c < \gamma < \gamma_c$. The boundary of the window can be obtained by assuming $D(\gamma)|_{P_i} = 0$, yielding

$$\gamma_c = (C^{2/3} - V^{2/3})^{3/2}. \tag{16}$$

There is a striking feature at $\gamma = -\gamma_c$: there exists a critical point $T_1^*(z_c, \pi)$ with $D(\gamma_c)|_{T_1^*} = 0$; as we have shown in [8], we can prove that this point is a limit point. A pair of topological particles P_3 and P_4 are generated with opposite charges -1 and +1 respectively; both of the new topological particles lie on the line $\varphi = \pi$. One of the original topological particle, P_2 with charge +1 on the line $\varphi = \pi$, moves smoothly up to $\gamma = \gamma_c$, where it collides with P_3 and annihilates with it at another limit point $T_2^*(-z_c, \pi)$. The other, P_1, which lies on the line $\varphi = 0$, still moves appropriately with γ.

As pointed out by Wu and Niu [9], when the interaction is strong enough $(C/V > 1)$, a loop appears at the tip of the lower adiabatic level when $C/V > 1$ while $-\gamma_c \leqslant \gamma \leqslant \gamma_c$. We show the interesting structure in figure 2, in which $C/V = 2$. For $\gamma < -\gamma_c$, there are two adiabatic levels; the upper level corresponds to the topological particle P_1, while the lower one corresponds to the topological particle P_2; for $\gamma > \gamma_c$, there are also only two adiabatic levels, but this time the lower one corresponds to P_4 while the upper one still corresponds to P_1. The arc part of the loop on the tip of the lower level when $-\gamma_c < \gamma < \gamma_c$ just corresponds to P_3, which merges with the level corresponding to P_4 at the point M on the left and with the one corresponding to P_1 at the point T on the right.

From the above discussion, one sees that when the structure of the topological current is changed by generating or annihilating a pair of topological particles (the upper panel of figure 2), at the same time the physical structure is changed by adding two energy levels or subtracting two energy levels respectively (the lower panel of figure 2). The critical behaviours happen at the limit points where $D(\gamma_c)|_{T^*_{1,2}} = 0$.

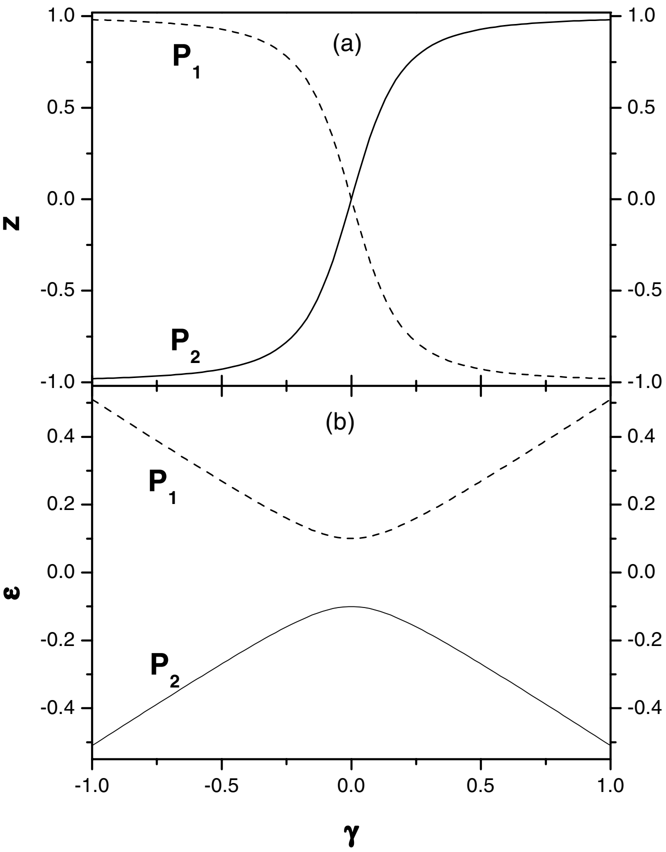

Figure 1. (*a*) The projection of the trajectory of topological particles on the (z–γ) plane for $C/V = 0$. P_i denotes the ith topological particle. (*b*) The energy levels for $C/V = 0$. Each level is labelled by the topological particle which corresponds to it.

In fact, this nonlinear two-level model is of widespread interest, for it is associated with a wide range of concrete physical systems, e.g., the Bose–Einstein condensate (BEC) in an optical lattice [10] or in a double-well potential [11], and the motion of small polarons [12]. So, the relation between topological particles and physical inner structure can be observed by experimental methods. Here we propose a suitable system in which to observe this striking phenomenon: a BEC in a double-well potential [11, 13]. The amplitudes of general occupations $N_{1,2}(t)$ and phases $\varphi_{1,2}$ obey the nonlinear two-mode Schrödinger equations, approximately [13]:

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial\phi_1}{\partial t} &= (E_1^0 + U_1N_1)\phi_1 - K\phi_2 \\ \mathrm{i}\hbar\frac{\partial\phi_2}{\partial t} &= (E_2^0 + U_2N_2)\phi_2 - K\phi_1 \end{aligned} \tag{17}$$

with $\phi_{1,2} = \sqrt{N_{1,2}}\exp(\mathrm{i}\varphi_{1,2})$, and the total number of atoms, $N_1 + N_2 = N_T$, is conserved. Here $E_{1,2}^0$ are the zero-point energies in the two wells, $U_{1,2}N_{1,2}$ are proportional to the atomic self-interaction energy, and K describes the amplitude of the tunnelling between the condensates. After introducing the new variables $z(t) = (N_2(t) - N_1(t))/N_T$ and $\varphi = \varphi_2 - \varphi_1$, one also obtains equations having the same form as equations (7) and (8) except for the parameters

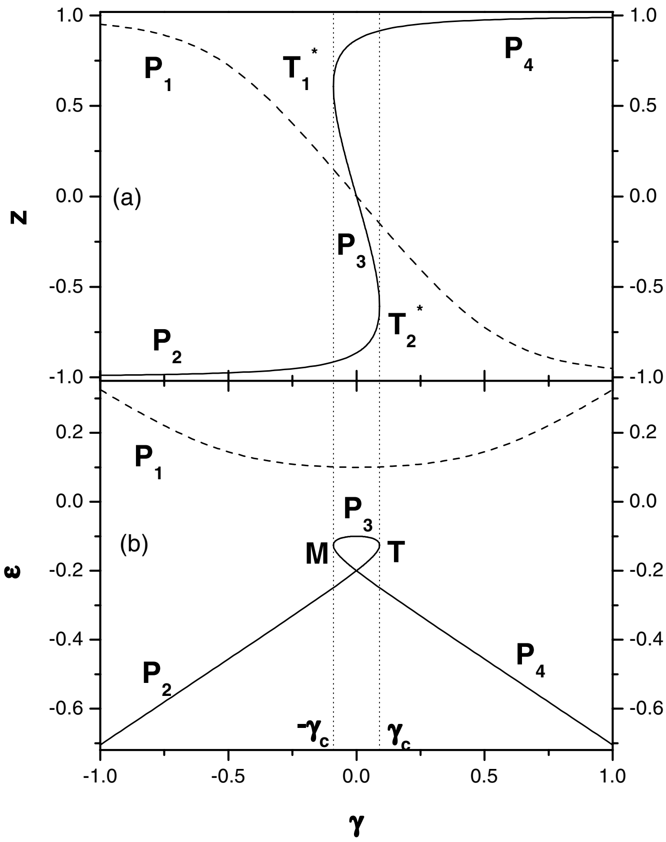

Figure 2. (*a*) The projection of the trajectory of the topological particles on the (z–γ) plane for $C/V = 2$. P_i denotes the ith topological particle. (*b*) The energy levels for $C/V = 2$. Each level is labelled by the topological particle which corresponds to it.

being replaced by

$$\gamma = -[(E_1^0 - E_2^0) - (U_1 - U_2)N_T/2]/\hbar, \tag{18}$$

$$V = 2K/\hbar, \qquad C = (U_1 + U_2)N_T/2\hbar. \tag{19}$$

With these explicit expressions, our theory and results can be directly applied to this system without intrinsic difficulty. In this system the topological particles can be located by the stable occupation and relative phase $\varphi = 0$ or π for a given parameter γ. Also, one can draw the zero-line for each topological particle by giving different values of γ. We hope that our discussions will stimulate experimental works in this direction.

We note that for a system, when the global property (topology) is given, the interesting feature is that under the same topology the topological configurations can be different; this difference must correspond to different physical structure. This relation between the topological configuration and physical structure gives an important property for classifying some physical systems which contain many different structures.

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