Berry curvature as a lower bound for multiparameter estimation

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Quantum Fisher information (QFI) is a key concept in quantum state estimation, and Berry curvature (BC) is another basic concept to describe geometric properties of quantum states. In this paper, we consider pure states undergoing unitary parametrization processes and show that the BC serves as a lower bound for the product of QFIs corresponding to two different parameters through the Heisenberg uncertainty relation. This relation between QFI and BC implies that the estimation precisions of two different parameters are mutually restrictive due to finite BC, and the notion of QFI squeezing is introduced. A scenario of general su(2) parametrization is considered in detail to verify the relation between the QFI and BC.

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I. INTRODUCTION

In information theory, estimation of parameters is a central task, among which the limit of estimation precision is of great interest [1–3]. In a classical scenario of estimation, given the probability distribution and an unbiased estimator, Fisher information sets an attainable lower bound for the variance of the estimator due to the Cramér-Rao inequality [4,5]. In a quantum estimation and detection scenario for a single parameter θ, the extension of classical Fisher information is not unique [6–11]. One prominent definition of the quantum Fisher information (QFI), $F(\rho_\theta) = \mathrm{tr}(\rho_\theta L_\theta^2)$, is based on the symmetric logarithmic derivative (SLD) operator L_θ [9,12], which is Hermitian and determined by the equation $\partial_\theta \rho_\theta = \frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta)$. The Braunstein-Caves theorem [13] states that QFI defined via SLD is maximal among all classical Fisher informations which are induced by positive-valued-operator measurement, indicating that $F(\rho_\theta)$ utilizes the information encoded in the state. The QFI is also closely related to the fidelity susceptibility, which is a measure of the distinguishability between the state $\rho(\theta)$ and its neighboring state $\rho(\theta + \delta\theta)$ [14–16].

For a multivariate parameter estimation, the counterpart of $F(\rho_\theta)$ is known as the QFI matrix $\mathcal{F}$. The definition of QFI matrix (QFIM) is also not unique [17] due to different versions of logarithmic derivative operators, and one possible choice is the SLD. Elements within the SLD-QFIM are $\mathcal{F}_{\alpha\beta} = \mathrm{tr}(\rho\{L_\alpha, L_\beta\})$, where $L_{\alpha(\beta)}$ are SLD operators for parameters $\alpha(\beta)$. The matrix's diagonal terms represent the QFIs for the corresponding single-parameter estimation. A matrix inequality for multivariate parameter estimation reads $\mathrm{Cov}(\hat{\boldsymbol{\theta}}) \geqslant \mathcal{F}^{-1}$, where on the left-hand side is the covariance matrix induced by the locally unbiased measurement and on the other side is the inverse of the SLD-QFIM [9,12]. However, the lower bound given by the matrix inequality in general cannot be achieved due to the noncommutativity ingrained in quantum mechanics [17]. Though the bound may not be achieved, it has been pointed out that [13,17–19] the matrix elements of $\mathcal{F}$ are identical to the Fubini-Study metric [20] for a pure state, which is a gauge-invariant metric in projective Hilbert space, $\mathcal{P}H = H/U(1)$, thus linking the problem of parameter estimation with state distinguishability.

Recently, there has been a growing interest in the study of quantum metrology for a general parameter estimation [21,22], where parameters are embedded in the Hamiltonian rather than as overall multiplicative factors [23–25]. Given a general unitary parametrization process $U(\theta)$, it has been shown that such a process can be characterized by a Hermitian operator $\mathcal{H}_\theta \equiv i(\partial_\theta U^\dagger)U$ [26,27], which can also be viewed as the generator of parameter θ [28]. It is to be noted that when θ does not depend on t, i.e., $U = \exp[-itH(\theta)]$, the explicit expression of $\mathcal{H}_\theta$ is given in the form of a series expansion [22].

The Berry phase, originally discovered as a gauge-invariant phase factor accompanying adiabatic changes [29], was later realized to be a holonomy effect in Hilbert space [30–32]. The loop integral expression of the Berry phase enables Berry to rewrite the geometric phase as an integral of an antisymmetric second-rank tensor field, i.e., the Berry curvature (BC) [33]. Defined on the parameter space (or projective Hilbert space), the BC not only gives the Berry phase but also appears in the equation of motion when the system's evolution involves slowly varying variables, e.g., anomalous velocity [34,35]. The Berry phase, along with the BC, not only helps to explain the phenomena in solid-state physics [36] such as the quantum Hall effect and the anomalous Hall effect [37,38], but also becomes a candidate for fault-tolerant quantum computation [39–41]. For quantum estimation theory, it was noted that BC serves as an indicator of noncommutativity in estimating different parameters, which means a large BC indicates that it is hard to estimate two different parameters simultaneously [42].

In this paper we consider the problem of a general multivariate parameter estimation of $\boldsymbol{\theta}$ inside the Hamiltonian, and the process is as follows. First, we prepare a pure initial state with no parameters to be estimated, then let the state

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undergo a unitary evolution which is determined by the Hamiltonian $H(\boldsymbol{\theta})$. We prove that, after the unitary evolution, the QFIM of $\boldsymbol{\theta}$ and the BC on the parameter space of $\boldsymbol{\theta}$ are connected through the Heisenberg uncertainty relation. To be explicit, there exists an inequality linking the BC to the lower bound of the product of different parameters' QFIs. Furthermore, by applying the Robertson-Schrödinger relation, a more stringent bound on the product of QFIs is derived involving the off-diagonal terms of the QFIM. A scenario of position and momentum measurement and a general su(2) parametrization process are discussed to demonstrate the bound, and we derive the Heisenberg uncertainty relations for position and momentum operators and the angular momentum operators from the perspective of parameter estimation.

The structure of this paper is as follows. In Sec. II, we introduce the concepts of the QFI and QFIM together with the BC, and the parameter generator $\mathcal{H}$ as well. In Sec. III, with the application of the Heisenberg and Robertson-Schrödinger uncertainty relations, two inequalities between the QFI and BC are established along with the concept of QFI squeezing. In Sec. IV, a scenario of general su(2) parametrization is considered and discussed in detail to verify the inequality. A brief discussion and summary are given in Sec. V.

II. QUANTUM FISHER INFORMATION AND BERRY CURVATURE

In this section we briefly review the notion of the QFI, QFIM, and BC. The most general case of estimation is to consider a parametrized quantum state ρ_θ, whose spectrum decomposition is

$$\rho_\theta = \sum_i p_i(\theta)\,|\psi_i(\theta)\rangle\,\langle\psi_i(\theta)|\,, \tag{1}$$

where θ is the parameter to be estimated. For any unbiased estimator $\hat{\theta}$, i.e., $E(\hat{\theta}) = \theta$, its variance is bounded from below by the inverse of the QFI [13]:

$$\mathrm{Var}(\hat{\theta}) \geqslant F^{-1}(\theta), \tag{2}$$

and it is to be noted that the dimension of the QFI is the negative square of the parameter to be estimated. The QFI in Eq. (2) is defined as [8,9]

$$F(\theta) \equiv \mathrm{Tr}(\rho_\theta L_\theta^2), \tag{3}$$

where L_θ is the symmetric logarithmic derivative operator given by

$$\partial_\theta \rho_\theta = \tfrac{1}{2}\{\rho_\theta, L_\theta\} \tag{4}$$

with $\{.,.\}$ denoting anticommutator. The explicit expression of the QFI is found to be [43]

$$F(\theta) = \sum_i \frac{1}{p_i}(\partial_\theta p_i)^2 + \sum_i 4p_i\,\langle\partial_\theta\psi_i|\partial_\theta\psi_i\rangle - \sum_{i,j}\frac{8p_ip_j}{p_i+p_j}|\,\langle\psi_i|\partial_\theta\psi_j\rangle\,|^2. \tag{5}$$

Now we assume that θ is brought in through a unitary evolution $U(\theta)$ whereas the initial state is

$$\rho_0 = \sum p_i\,|\psi_i(0)\rangle\,\langle\psi_i(0)|\,, \tag{6}$$

which contains no parameter to be estimated, and the state becomes

$$\rho_\theta = U^\dagger(\theta)\rho_0 U(\theta) \tag{7}$$

after the unitary evolution. Since p_i is conserved under unitary evolution, the QFI of ρ_θ now is

$$F(\theta) = \sum_i 4p_i(\langle\partial_\theta\psi_i|\partial_\theta\psi_i\rangle - |\,\langle\psi_i|\partial_\theta\psi_i\rangle\,|^2) - \sum_{i\neq j}\frac{8p_ip_j}{p_i+p_j}|\,\langle\psi_i|\partial_\theta\psi_j\rangle\,|^2. \tag{8}$$

The generator of parameter θ [21,22,28] is given by

$$\mathcal{H} \equiv i(\partial_\theta U^\dagger)U, \tag{9}$$

which is Hermitian since $U^\dagger U = I$. By introducing $\mathcal{H}$, Eq. (8) can be rewritten as

$$F(\theta) = \sum_i 4p_i\,\langle\Delta\mathcal{H}^2\rangle_i - \sum_{i\neq j}\frac{8p_ip_j}{p_i+p_j}|\,\langle\psi_i(0)|\mathcal{H}|\psi_j(0)\rangle\,|^2 \tag{10}$$

where $\langle\Delta\mathcal{H}^2\rangle_i$ is the variance of $\mathcal{H}$ on the ith eigenstate of ρ_0.

We further assume that the initial state is a pure state; then its QFI after unitary evolution $U(\theta)$ has a simple form as

$$F(\theta) = 4\,\langle\Delta\mathcal{H}^2\rangle\,. \tag{11}$$

When the unitary parametrization contains more than one parameter, $\boldsymbol{\theta} = (\ldots,\theta_i,\theta_j,\ldots)$, to be estimated, following a similar procedure we can obtain a QFIM whose elements are

$$\begin{aligned}\mathcal{F}_{ij} &= 4\left(\left\langle\frac{\{\mathcal{H}_i,\mathcal{H}_j\}}{2}\right\rangle - \langle\mathcal{H}_i\rangle\,\langle\mathcal{H}_j\rangle\right)\\ &= 4\mathrm{Cov}(\mathcal{H}_i,\mathcal{H}_j),\end{aligned} \tag{12}$$

where $\mathcal{H}_i = i(\partial_i U^\dagger)U$ is the generator of parameter θ_i, and $\langle\cdots\rangle$ denotes taking the average on the initial state. From now on, we set Roman letters in subscript to represent the corresponding parameter to be estimated, i.e., subscript i represents θ_i.

On the other side, for a cyclic evolution of the pure state, the geometric phase is defined as [29,31]

$$\gamma_g = \oint_{\mathcal{C}} i\,\langle\psi|d\psi\rangle\,, \tag{13}$$

where the integrand is a 1-form Berry connection and $\mathcal{C}$ is a closed loop in projective Hilbert space. After applying the Stokes theorem to Eq. (13), the integral can be written as

$$\gamma_g = \iint_S i\,\langle d\psi|\wedge|d\psi\rangle\,, \tag{14}$$

where the integrand is the phase 2-form with $\wedge$ as the exterior product, and S is a surface enclosed by $\mathcal{C}$. The explicit expression of the phase 2-form is

$$\frac{i}{2}(\langle\partial_i\psi|\partial_j\psi\rangle - \langle\partial_j\psi|\partial_i\psi\rangle)d\theta_i\wedge d\theta_j. \tag{15}$$

Here and hereafter we set $\partial_i \equiv \partial/\partial\theta_i$ and Einstein's summation convention is assumed. The BC [33,36], which is an antisymmetric tensor in projective Hilbert space, can be defined from Eq. (15) as

$$\Omega_{ij} = i(\langle \partial_i \psi | \partial_j \psi \rangle - \langle \partial_j \psi | \partial_i \psi \rangle). \tag{16}$$

Since the integral of BC over θ and ϕ gives the dimensionless geometric phase, the dimension of BC is the inverse of the dimension of θ times the dimension of ϕ. Suppose the parameters $\boldsymbol{\theta} = (\ldots,\theta_i,\theta_j,\ldots)$ are encoded in quantum states through a unitary parametrization process $U(\boldsymbol{\theta})$; then according to Eq. (9) the generators of θ_i and θ_j are

$$\mathcal{H}_i = i(\partial_i U^\dagger)U, \tag{17a}$$

$$\mathcal{H}_j = i(\partial_j U^\dagger)U. \tag{17b}$$

After substituting Eqs. (17a) and (17b) back into Eq. (16), we have

$$\Omega_{ij} = i\,\langle [\mathcal{H}_i,\mathcal{H}_j] \rangle\,. \tag{18}$$

The BC has a clear geometric meaning due to the presence of the Berry phase, and likewise for QFIM, $\mathcal{F}$ can be interpreted geometrically [6,13,17]. Since the accuracy of estimation of parameters is equivalent to the ability of distinguishing $\rho(\boldsymbol{\theta})$ from its neighboring state $\rho(\boldsymbol{\theta} + d\boldsymbol{\theta})$, $\mathcal{F}$ can be identified as a quantum distinguishability metric [13]. For a pure state, $\mathcal{F}$ in Eq. (12) is identical, up to a constant, to the Fubini-Study metric [17,20], which measures the distance of the rays in $\mathcal{P}H$. Furthermore, since the metric and curvature are both defined in $\mathcal{P}H$, for a pure state there exists a unifying description of the QFIM and BC known as the quantum geometric tensor (QGT) [33,44],

$$Q_{ij} = \langle \partial_i \psi | (1 - |\psi\rangle\langle\psi|) | \partial_j \psi \rangle\,, \tag{19}$$

with the following properties:

$$\mathrm{Re}Q_{ij} = \frac{\mathcal{F}_{ij}}{4}, \tag{20a}$$

$$\mathrm{Im}Q_{ij} = -\frac{\Omega_{ij}}{2}. \tag{20b}$$

Thus for a unitary parametrization process, the QFIM and BC are connected through the QGT, and by using parameter generators, the QGT can be expressed concisely as

$$Q_{ij} = \langle \mathcal{H}_i \mathcal{H}_j \rangle - \langle \mathcal{H}_i \rangle \langle \mathcal{H}_j \rangle\,. \tag{21}$$

III. INEQUALITY BETWEEN QUANTUM FISHER INFORMATION AND BERRY CURVATURE

In this section we consider a situation of a pure state undergoing a unitary parametrization process; then we establish two inequalities between the QFI and BC, and the inequality is verified under a simple parametrization scenario.

Derivation of the inequalities is based on two familiar uncertainty relations in quantum mechanics: one is the Heisenberg uncertainty relation [45]

$$\langle (\Delta \hat{A})^2 \rangle \langle (\Delta \hat{B})^2 \rangle \geqslant \tfrac{1}{4} |\langle [\hat{A},\hat{B}] \rangle|^2, \tag{22}$$

where $\hat{A}$ and $\hat{B}$ are Hermitian operators and $\Delta\hat{A} = \hat{A} - \langle \hat{A} \rangle$; the other one is the Robertson-Schrödinger uncertainty relation (R-S inequality) [46], which is the stronger version of Eq. (22),

$$\begin{aligned}\langle (\Delta \hat{A})^2 \rangle \langle (\Delta \hat{B})^2 \rangle &\geqslant \tfrac{1}{4} |\langle [\hat{A},\hat{B}] \rangle|^2 + \tfrac{1}{4}\langle \{\Delta\hat{A},\Delta\hat{B}\} \rangle^2 \\ &= \tfrac{1}{4} |\langle [\hat{A},\hat{B}] \rangle|^2 + \mathrm{Cov}(\hat{A},\hat{B})^2,\end{aligned} \tag{23}$$

where in the second line the anticommutator is replaced with covariance of operators, which is defined in Eq. (12).

A. Application of the Heisenberg uncertainty relation

First we consider the application of the Heisenberg uncertainty relation. We substitute the parameter generators $\mathcal{H}_i$ and $\mathcal{H}_j$ into Eq. (22); then we get

$$\begin{aligned}\langle \Delta\mathcal{H}_i^2 \rangle \langle \Delta\mathcal{H}_j^2 \rangle &\geqslant \left| \frac{1}{2i} \langle [\mathcal{H}_i,\mathcal{H}_j] \rangle \right|^2 \\ &= \frac{1}{4}\Omega_{ij}^2,\end{aligned} \tag{24}$$

where the commutator is expressed as the BC due to Eq. (18). According to Eq. (11), the variance of parameter generators can be substituted with QFI, and Eq. (24) becomes

$$F_i F_j \geqslant 4\Omega_{ij}^2. \tag{25}$$

Thus for a pure state under a unitary parametrization process, its QFI and BC are connected. Now we consider a scenario where the BC of parameters (θ_i,θ_j) and QFI of one parameter θ_i, F_i, are given; then F_j is bounded from below. Since Eq. (2) states that it is the inverse of F_j that gives the lowest attainable bound to any unbiased estimator $\hat{\theta}_j$, unlike the Heisenberg uncertainty relation that limits the precision of estimation from below, the inequality of Eq. (25) indicates that there exists an upper bound to the precision of estimation.

Analogous to bosonic squeezing [47] and inspired by Eq. (25), we may introduce a similar concept called QFI squeezing. The definition is as follows: if θ_j's QFI satisfies that $F_j < 2|\Omega_{ij}|$ then F_i is said to be squeezed ($F_i > 2|\Omega_{ij}|$), and we can define a QFI squeezing parameter as

$$\xi_j = \frac{F_j}{2|\Omega_{ij}|}, \tag{26}$$

and the dimension of the squeezing parameter is determined by QFI and BC. As seen from Eq. (2), the larger the QFI is the more accurate the estimation is. Thus for a pair of parameters with QFI squeezing, one parameter's estimation limit must be more precise than the other one's limit or vice versa, which is consistent with bosonic squeezing. A detailed discussion of QFI squeezing is presented in Sec. IV.

Now we apply the inequality in Eq. (25) to a simple parametrization estimation scenario. Consider the case of a simple two-parameter estimation case (with $t = 1$ and $\hbar = 1$):

$$|\psi(x,p)\rangle = e^{i(x\hat{p}+p\hat{x})}|\psi(0)\rangle\,, \tag{27}$$

where (x,p) are parameters to be estimated and $|\psi(0)\rangle$ is an arbitrary pure initial state. According to Eq. (9) and with the help of Glauber's formula, the parameter generators of (x,p)

can be calculated as

$$\mathcal{H}_x = i(\partial_x U^\dagger)U = \hat{p} + \frac{3p}{2}, \tag{28a}$$

$$\mathcal{H}_p = i(\partial_p U^\dagger)U = \hat{x} - \frac{3x}{2}. \tag{28b}$$

The BC on parameter space is

$$\Omega_{xp} = i\,\langle[\mathcal{H}_x,\mathcal{H}_p]\rangle = 1, \tag{29}$$

which is independent of initial state. The QFIs of (x,p) are

$$F_x = 4\,\langle \Delta\mathcal{H}_x^2\rangle = 4\,\langle \Delta\hat{p}^2\rangle\,, \tag{30a}$$

$$F_p = 4\,\langle \Delta\mathcal{H}_p^2\rangle = 4\,\langle \Delta\hat{x}^2\rangle\,. \tag{30b}$$

Substituting the QFI and BC into Eq. (25), we obtain

$$\sigma_{\hat{x}}\sigma_{\hat{p}} \geqslant \tfrac{1}{2}, \tag{31}$$

where $\sigma_{\hat{A}}$ denotes the standard deviation of an operator $\hat{A}$ on the initial state. Equation (31) recovers the famous Heisenberg uncertainty relation for the position and momentum operators, but viewed from the multiparameter estimation theory. The calculation above also shows that the bound in Eq. (25) is attainable with a coherent state.

B. Application of the Robertson-Schrödinger uncertainty relation

Now we consider the more stringent Robertson-Schrödinger inequality of Eq. (23). We denote (θ_i,θ_j) as (θ,ϕ) for the sake of simplicity. Following a similar procedure, we have

$$\begin{aligned}\frac{F_\theta F_\phi}{16} &\geqslant \left|\frac{1}{2i}\,\langle|[\mathcal{H}_\theta,\mathcal{H}_\phi]|\rangle\right|^2 + \mathrm{Cov}^2(\mathcal{H}_\theta,\mathcal{H}_\phi)\\ &= \frac{1}{4}\Omega_{\theta\phi}^2 + \mathrm{Cov}^2(\mathcal{H}_\theta,\mathcal{H}_\phi)\\ &= \frac{1}{4}\Omega_{\theta\phi}^2 + \frac{1}{16}F_{\theta\phi}^2,\end{aligned} \tag{32}$$

where in the first line Eq. (11) is used and in the last line Eq. (12) is used. Note that the off-diagonal element of QFIM, $F_{\theta\phi}$, now appears in the inequality.

To give a better understanding of Eq. (32), let us consider the case of a two-parameter estimation of (θ,ϕ); then we have

$$\begin{aligned}&\frac{F_\theta F_\phi}{16} - \frac{F_{\theta\phi}^2}{16} \geqslant \frac{\Omega_{\theta\phi}^2}{4}\\ &\Rightarrow \det\mathcal{F} \geqslant 4\Omega_{\theta\phi}^2\end{aligned} \tag{33}$$

where $\mathcal{F}$ is the QFIM of the two-parameter estimation process. Equation (33) shows that the determinant of QFIM is bounded from below by four times the square of BC in the parameter space.

As mentioned in Sec. II, the QFI is identical to the Fubini-Study metric up to a constant for a pure state; therefore, Eqs. (25) and (32) can also be viewed as an inequality connecting the Fubini-Study metric and the BC on the parameter space of pure states, and such a connection may imply a geometric explanation. It is worth mentioning that in Ref. [48], following as the consequence of the Hermitian Schwartz inequality in Hilbert space, Brody *et al.* proved a similar inequality involving the Fubini-Study metric and BC after mapping the whole quantum system to a real manifold.

IV. THE INEQUALITY IN A GENERAL su(2) PARAMETRIZATION PROCESS

With the inequalities connecting the QFI and BC at hand, now we consider a general su(2) parametrization process. We assume that there are two parameters to be estimated, (θ,ϕ), and the unitary evolution is

$$U = \exp[-itH(\theta,\phi)] \tag{34}$$

(with $\hbar = 1$) whose Hamiltonian takes the form

$$H(\theta,\phi) = \mathbf{r}\cdot\mathbf{J}. \tag{35}$$

Inside Eq. (35), $\mathbf{r}$ is a unit vector with (θ,ϕ) as its spherical coordinates:

$$\mathbf{r} = (\sin\theta\cos\phi,\,\sin\theta\sin\phi,\,\cos\theta), \tag{36}$$

and $\mathbf{J} = (J_x,J_y,J_z)$ are the generators of su(2) algebra. Note that $\mathbf{r}$ is a periodic function of θ or ϕ; thus the QFI corresponds to θ or ϕ is also periodic.

For the specific unitary parametrization process in Eq. (34), the parameter generator can be expressed in a form of series expansion [22],

$$\mathcal{H}_\theta = i\sum_{n=0}^{\infty}\frac{(it)^{n+1}}{(n+1)!}(H^\times)^n(\partial_\theta H), \tag{37}$$

where $H^\times$ is a superoperator defined as $H^\times(\cdot) \equiv [H,\cdot]$. With the help of Eq. (37), the explicit expression of generators of θ is [49]

$$\mathcal{H}_\theta = -\sin t\,\mathbf{v}_\theta\cdot\mathbf{J} + (1-\cos t)(\mathbf{r}\times\mathbf{v}_\theta)\cdot\mathbf{J}, \tag{38}$$

where $\mathbf{v}_\theta = \partial_\theta\mathbf{r}$, and the expression for $\mathcal{H}_\phi$ is similar with the substitution of ϕ for θ. The explicit expressions of $\mathbf{v}_\theta$ and $\mathbf{v}_\phi$ are

$$\mathbf{v}_\theta = (\cos\theta\cos\phi,\,\cos\theta\sin\phi,\,-\sin\theta), \tag{39a}$$

$$\mathbf{v}_\phi = (-\sin\theta\sin\phi,\,\sin\theta\cos\phi,0). \tag{39b}$$

Note that $\mathbf{v}_\theta$ is unit vector while $\mathbf{v}_\phi$ is not ($|\mathbf{v}_\phi| = |\sin\theta|$).

The calculations of QFI and BC are straightforward with the explicit expression of parameter generators. For the sake of convenience we rewrite the parameter generators by introducing two new vectors,

$$\mathcal{H}_\theta = \mathbf{A}_\theta\cdot\mathbf{J}, \tag{40a}$$

$$\mathcal{H}_\phi = \mathbf{A}_\phi\cdot\mathbf{J}, \tag{40b}$$

where the explicit expressions of $\mathbf{A}_\theta$ or $\mathbf{A}_\theta$ are given in Eq. (38). These two vectors can be decomposed into the product of its norm and a unit vector, $\mathbf{A}_\theta = |\mathbf{A}_\theta|\mathbf{a}_\theta$ and $\mathbf{A}_\phi = |\mathbf{A}_\phi|\mathbf{a}_\phi$, where the norms are $|\mathbf{A}_\theta| = 2|\sin\frac{t}{2}|$ and $|\mathbf{A}_\phi| = 2|\sin\theta\sin\frac{t}{2}|$. It is easy to verify that $\mathbf{a}_\theta$, $\mathbf{a}_\phi$, and $\mathbf{r}$ are perpendicular to each other. The QFI can be expressed as

$$F_\theta = 16\sin^2\frac{t}{2}(\Delta J_{\mathbf{a}_\theta})^2, \tag{41a}$$

$$F_\phi = 16\sin^2\theta\sin^2\frac{t}{2}(\Delta J_{\mathbf{a}_\phi})^2, \tag{41b}$$

where

$$\Delta J_{\mathbf{a}_\theta} = \sqrt{\langle(\mathbf{a}_\theta\cdot\mathbf{J})^2\rangle - \langle\mathbf{a}_\theta\cdot\mathbf{J}\rangle^2}, \tag{42a}$$

$$\Delta J_{\mathbf{a}_\phi} = \sqrt{\langle(\mathbf{a}_\phi\cdot\mathbf{J})^2\rangle - \langle\mathbf{a}_\phi\cdot\mathbf{J}\rangle^2} \tag{42b}$$

are the standard deviations of $\mathbf{J}$ on the initial state along the direction of $\mathbf{a}_\theta$ or $\mathbf{a}_\phi$, and the coefficients before the variance $\Delta J_{\mathbf{a}_\theta}$ or $\Delta J_{\mathbf{a}_\phi}$ are proportional to the maximal QFI of θ or ϕ [49].

The explicit expression of the BC is

$$\Omega_{\theta\phi} = i\,\langle[\mathcal{H}_\theta,\mathcal{H}_\phi]\rangle = -4\sin\theta\sin^2\frac{t}{2}\,\langle\mathbf{r}\cdot\mathbf{J}\rangle\,, \tag{43}$$

where the commutation relation for su(2) algebra $[\mathbf{a}\cdot\mathbf{J},\mathbf{b}\cdot\mathbf{J}] = i(\mathbf{a}\times\mathbf{b})\cdot\mathbf{J}$ is utilized, and we set

$$\langle\mathbf{J}\rangle = (\langle J_x\rangle\,,\langle J_y\rangle\,,\langle J_z\rangle). \tag{44}$$

Now we select a unique initial state to test the effectiveness of the inequality and the coherent spin state (CSS) serves as a benchmark in the field of quantum measurement [50]. Without loss of generality, we assume a specific form of CSS as the initial state:

$$|\eta\rangle = e^{i\frac{\pi}{2}J_y}\,|j,j\rangle\,, \tag{45}$$

where $|j,j\rangle$ is the eigenstate of J_z with eigenvalue j. The QFIs of θ and ϕ with respect to $|\eta\rangle$ are

$$\begin{aligned}F_\theta &= 8j\sin^2\frac{t}{2}(1-\mathbf{a}_{\theta x}^2)\\ &= 2j\sin^2\frac{t}{2}\left[3+\cos^2\phi\left(1-4\cos^2\theta\cos^2\frac{t}{2}\right)\right.\\ &\qquad\left.-2\cos\theta\sin t\sin 2\phi + (2\cos t-1)\sin^2\phi\right]\end{aligned} \tag{46a}$$

$$\begin{aligned}F_\phi &= 8j\sin^2\theta\sin^2\frac{t}{2}(1-\mathbf{a}_{\phi x}^2)\\ &= 8j\sin^2\theta\sin^2\frac{t}{2}\left[1-\cos^2\theta\sin^2\frac{t}{2}\cos^2\phi\right.\\ &\qquad\left.+\frac{1}{2}\cos\theta\sin t\sin 2\phi - \cos^2\frac{t}{2}\sin^2\phi\right],\end{aligned} \tag{46b}$$

where the subscript x denotes a vector's component along the x axis. As for the BC, its explicit expression is

$$\begin{aligned}\Omega_{\theta\phi} &= 4j\sin\theta\sin^2\frac{t}{2}\mathbf{r}_x\\ &= 4j\sin^2\theta\sin^2\frac{t}{2}\cos\phi.\end{aligned} \tag{47}$$

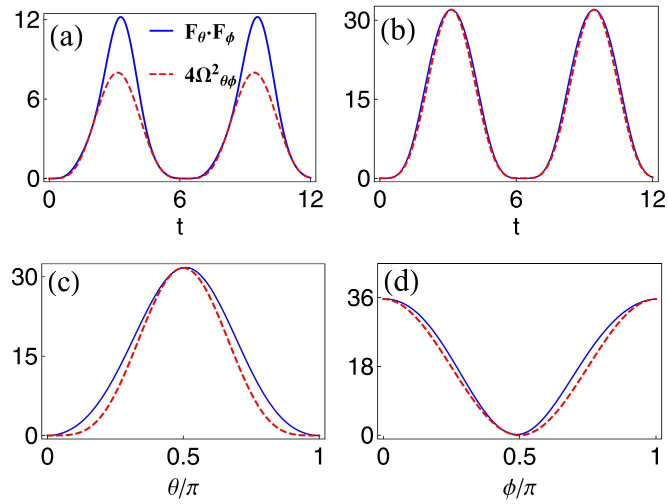

FIG. 1. Plot of four times the square of BC (dashed red line) together with the product of QFIs (solid blue line) of θ and ϕ, with $j = 1$. It is to be noted that both QFI and BC are dimensionless due to the dimensionless θ and ϕ. The parameters are (a) $\theta = \phi = \pi/4$ and the square of curvature oscillates along with the product of the QFIs; (b) $\theta = \pi/2$ and $\phi = \pi/4$ and two lines almost coincide while actually there exists a tiny difference between them; (c) for $t = 3$ and $\phi = \pi/4$ the lower bound becomes tight near the point $\theta = \pi/2$; and (d) for $t = 3$ and $\theta = \pi/3$ the lower bound becomes tight around the point $\phi = \pi/2$.

The product of F_θ and F_ϕ in Eqs. (46a) and (46b) together with $4\Omega_{\theta\phi}^2$ in Eq. (47) as a function of θ, ϕ, and t are shown in Fig. 1 (with $j = 1$). In Figs. 1(a) and 1(b), θ and ϕ are fixed with t as a free parameter. The product of QFIs shows an oscillating behavior and so does $4\Omega_{\theta\phi}^2$; especially for the parameters set in Fig. 1(b) the bound is almost tight regardless of t yet a detailed investigation shows that the difference between $F_\theta F_\phi$ and $4\Omega_{\theta\phi}^2$ does exist. In Figs. 1(c) and 1(d), t is fixed and $4\Omega_{\theta\phi}^2$ follows the rise and fall of the product of QFI; in addition, the bound becomes tight for certain parameter setting, e.g., the point near $\theta = \pi/2$ in Fig. 1(c). All plots show that the inequality works satisfactorily with the CSS.

Actually, there exists a deeper explanation for the performance of the inequality in the case of su(2) parametrization, which is regardless of the specific form of initial state. We substitute the expression of QFI in Eqs. (41a) and (41b) and that of BC in Eq. (43) into the inequality in Eq. (25); we then have

$$(\Delta J_{\mathbf{a}_\theta})^2(\Delta J_{\mathbf{a}_\phi})^2 \geqslant \tfrac{1}{4}\,\langle\mathbf{r}\cdot\mathbf{J}\rangle^2\,, \tag{48}$$

which depends not on time t but only on $(\mathbf{r},\mathbf{a}_\theta,\mathbf{a}_\phi)$ and the initial state. Equation (48) shows that the product of the variances along $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are bounded from below by the expected value of angular momentum along the direction of $\mathbf{r}$. Since $(\mathbf{r},\mathbf{a}_\theta,\mathbf{a}_\phi)$ forms an orthonormal coordinate system, Eq. (48) is equivalent to the uncertainty relation for angular momentum operators with reference to a Cartesian coordinate system,

$$(\Delta J_\alpha)^2(\Delta J_\beta)^2 \geqslant \tfrac{1}{4}|\,\langle J_\gamma\rangle\,|^2, \tag{49}$$

where $[J_\alpha,J_\beta] = i\epsilon_{\alpha\beta\gamma}J_\gamma$. Once again the Heisenberg uncertainty relation is recovered, and this is the origin that

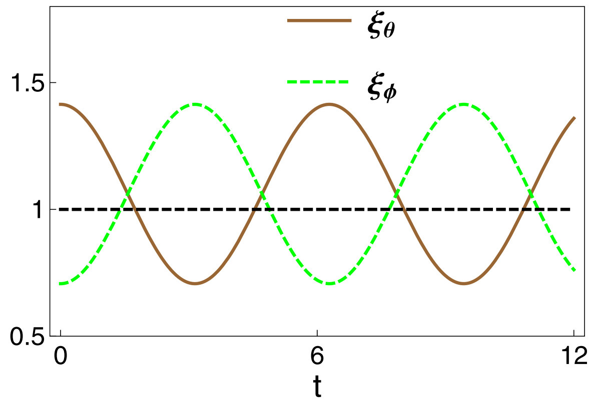

FIG. 2. Plot of the QFI squeezing parameters, ξ_θ (brown line) and ξ_ϕ (dashed green line), during the time evolution with parameter setting $\theta = \pi/2$ and $\phi = \pi/4$ (with $j = 1$); the squeezing parameters are dimensionless since QFI and BC are dimensionless. As shown in Eq. (26) that a less-than-one squeezing parameter guarantees its dual squeezing parameter to be larger than one, it is clear to see that when one squeezing parameter is below 1 (a relative smaller QFI) another squeezing parameter rises above 1 (a relative larger QFI), but not vice versa.

guarantees the effectiveness of the inequality in the case of su(2) parametrization.

The definition of QFI squeezing, defined in Eq. (26), has also been tested with the CSS given in Eq. (45) and the result is plotted in Fig. 2 under a parameter setting of $\theta = \pi/2$, $\phi = \pi/4$. When one QFI squeezing parameter, say ξ_θ, falls below 1, another squeezing parameter ξ_ϕ must be greater than 1. Note that ξ_θ and ξ_ϕ are proportional to the QFI of θ and ϕ; thus the behavior of squeezing parameters ξ_θ and ξ_ϕ shows that the QFIs for different parameters show mutually restrictive behavior in multiparameter estimations.

V. CONCLUSIONS

In summary, we investigated the relation between the QFI and BC for pure states under a unitary parametrization process. We find that for the multiparameter estimation process, the QFI and BC in parameter space can be described uniformly by exploiting the generators. Furthermore, we deduced two inequalities relating the QFI and BC through the Heisenberg uncertainty relation and the Robertson-Schrödinger uncertainty relation, and the notion of QFI squeezing was proposed. Two parametrization scenarios including the general su(2) parametrization were discussed to test the inequality and the squeezing parameter. Our results show that estimations of the precision of two parameters are mutually restrictive and controlled by the BC.

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