Degree of entanglement for two qubits

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In this paper, we present a measure to quantify the degree of entanglement for two qubits in a pure state.

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I. INTRODUCTION

Quantum entanglement is the most surprising nonclassical property of composite quantum systems [1]. As it is well known, a qubit (or a spin-(1/2) particle) is described by the 2×2 density matrix $\rho(\mathbf{n})=(\mathbf{1}+\vec{\sigma}\cdot\mathbf{n})/2$, $|\mathbf{n}|\leqslant 1$, where $\mathbf{1}$ is the unit matrix, $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ the Pauli matrices vector, and $\mathbf{n}$ the Bloch vector. $|\mathbf{n}|=1$ corresponds to a pure state, otherwise a mixed state. Whereas, an entangled pairs of two qubits is completely described by the following 4×4 density matrix:

$$\rho_{AB}=\frac{1}{4}\left(\mathbf{1}\otimes\mathbf{1}+\vec{\sigma}^A\cdot\mathbf{u}\otimes\mathbf{1}+\mathbf{1}\otimes\vec{\sigma}^B\cdot\mathbf{v}+\sum_{i,j=1}^{3}\beta_{ij}\sigma_i^A\otimes\sigma_j^B\right) \tag{1}$$

from which one could obtain two reduced density matrices

$$\rho_A=\mathrm{tr}_B(\rho_{AB})=\tfrac{1}{2}(\mathbf{1}+\vec{\sigma}^A\cdot\mathbf{u}),$$

$$\rho_B=\mathrm{tr}_A(\rho_{AB})=\tfrac{1}{2}(\mathbf{1}+\vec{\sigma}^B\cdot\mathbf{v}), \tag{2}$$

for the two qubits A and B, where $\mathbf{u}$ and $\mathbf{v}$ are Bloch vectors for particles A and B, respectively; β_{ij} are some real numbers.

It has been shown that entangled pairs are a more powerful resource than separable, i.e., disentangled, pairs in a number of applications, such as quantum cryptography [2], dense coding [3], teleportation [4] and investigations of quantum channels [5], communication protocols, and computation [6,7]. The superior potentiality of entangled states has raised a natural question: "How much are two particles entangled?," since pairs with a high degree of entanglement should be a better resource than less entangled ones. Many measures of entanglement proposed in the past have relied on either the Schmidt decomposition [8] or decomposition in a *magic basis* [9]. In an interesting paper, Abouraddy *et al.* devised a new measure of entanglement for pure bipartite states of two qubits, based on a decomposition of the state vector as a superposition of a maximally entangled state vector and an orthogonal factorizable one [10]. Although there are many such decompositions, the weights of the two superposed states are remarkably unique. The square of the weight of the maximally entangled state vector (i.e., $P_E=p^2$) is then defined as the degree of entanglement for two qubits, such a measure is consistent with three measures of entanglement previously set forth: maximal violation of Bell's inequality [11], concurrence [9], and two-particle visibility [12].

The purpose of this paper is to propose an approach to the problem of defining the degree of entanglement for two qubits in a pure state. In Sec. II, a new measure is formulated to quantify the degree of entanglement. Some examples are given in Sec. III. Conclusion and discussion are made in the last section.

II. FORMALISM

Theorem. If ρ_{AB} is a pure state, then its degree of entanglement P_E is equal to

$$P_E=(-\det\hat{\alpha})^{1/4}, \tag{3}$$

where the matrix $\hat{\alpha}$ is

$$\hat{\alpha}=\begin{pmatrix} 1 & v_1 & v_2 & v_3 \\ u_1 & \beta_{11} & \beta_{12} & \beta_{13} \\ u_2 & \beta_{21} & \beta_{22} & \beta_{23} \\ u_3 & \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}. \tag{4}$$

Proof. ρ_{AB} is a pure state implies that $\rho_{AB}^2=\rho_{AB}$, from which one obtains the following constraints among u_i, v_i and β_{ij} $(i,j=1,2,3)$:

$$u_i=\beta_{i1}v_1+\beta_{i2}v_2+\beta_{i3}v_3, \tag{5}$$

$$v_i=\beta_{1i}u_1+\beta_{2i}u_2+\beta_{3i}u_3, \tag{6}$$

$$\sum_{i,j}\beta_{ij}^2=3-|\mathbf{u}|^2-|\mathbf{v}|^2, \tag{7}$$

$$\beta_{ij}=u_iv_j-(-1)^{i+j}M_{ij}, \tag{8}$$

where M_{ij} is the algebraic complement of the matrix element β_{ij} for the following $\hat{\beta}$ matrix:

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$$\hat{\beta}=\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}. \tag{9}$$

Equations (5) and (6) can be recast as $\hat{\beta}\mathbf{v}=\mathbf{u}$, $\hat{\beta}^T\mathbf{u}=\mathbf{v}$, where T represents transpose and $\mathbf{u}=(u_1,u_2,u_3)^T$. An interesting result, i.e., $|\mathbf{u}|=|\mathbf{v}|$, will be obtained immediately from Eqs. (5) and (6) for the pure state ρ_{AB} [13]. From Eq. (8) we have

$$\begin{aligned}\beta_{11}^2+\beta_{12}^2+\beta_{13}^2=&\beta_{11}u_1v_1+\beta_{12}u_1v_2+\beta_{13}u_1v_3\\ &-[(-1)^{1+1}\beta_{11}M_{11}+(-1)^{1+2}\\ &\times\beta_{12}M_{12}+(-1)^{1+3}\beta_{13}M_{13}].\end{aligned} \tag{10}$$

Due to $\det\hat{\beta}=\beta_{11}M_{11}-\beta_{12}M_{12}+\beta_{13}M_{13}$ and Eq. (5), one obtains

$$\beta_{11}^2+\beta_{12}^2+\beta_{13}^2-u_1^2=-\det\hat{\beta}. \tag{11}$$

Similarly

$$\beta_{21}^2+\beta_{22}^2+\beta_{23}^2-u_2^2=-\det\hat{\beta},$$

$$\beta_{31}^2+\beta_{32}^2+\beta_{33}^2-u_3^2=-\det\hat{\beta}. \tag{12}$$

After combining Eqs. (7), (11), and (12), and taking $|\mathbf{u}|=|\mathbf{v}|$ into account, one easily obtains $-\det\hat{\beta}=1-|\mathbf{u}|^2$ [13]. Consequently, we have

$$(-\det\hat{\alpha})^{1/4}=[(-\det\hat{\beta})(1-|\mathbf{u}|^2)]^{1/4}=\sqrt{1-|\mathbf{u}|^2}. \tag{13}$$

One can know from Ref. [10] that $P_E=2\kappa_1\kappa_2$, where κ_1 and κ_2 are the two coefficients in the Schmidt decomposition $|\Psi\rangle=\kappa_1|x_1,y_1\rangle+\kappa_2|x_2,y_2\rangle$, $\rho_{AB}=|\Psi\rangle\langle\Psi|$, where $\{|x_1\rangle,|x_2\rangle\}$ and $\{|y_1\rangle,|y_2\rangle\}$ are orthogonal bases for the Hilbert spaces of particles A and B, respectively. It is easy to prove that $\kappa_1=\sqrt{(1+|\mathbf{u}|)/2}$, $\kappa_2=\sqrt{(1-|\mathbf{u}|)/2}$, which are square roots of the two eigenvalues of the reduced matrix ρ_A or ρ_B. Therefore, we have $P_E=(-\det\hat{\alpha})^{1/4}$. This ends the proof.

III. EXAMPLES

Example 1. For the state $|\Psi\rangle=(|00\rangle+|01\rangle+|11\rangle)/\sqrt{3}$, one obtains the density matrix

$$\rho_{AB}=|\Psi\rangle\langle\Psi|=\frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

with the Bloch vectors $\mathbf{u}=(2/3,0,1/3)^T$, $\mathbf{v}=(2/3,0,-1/3)^T$, and the alpha matrix

$$\hat{\alpha}=\frac{1}{3}\begin{pmatrix} 3 & 2 & 0 & -1 \\ 2 & 2 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}. \tag{14}$$

One can have $P_E=2/3$, which is consistent with the result in Ref. [10].

Example 2. For the state $|\Psi\rangle=[|00\rangle+2(|01\rangle+|11\rangle)]/3$, the density matrix is

$$\rho_{AB}=|\Psi\rangle\langle\Psi|=\frac{1}{9}\begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 4 \end{pmatrix}$$

with $\mathbf{u}=(8/9,0,1/9)^T$, $\mathbf{v}=(4/9,0,-7/9)^T$, and the alpha matrix

$$\hat{\alpha}=\frac{1}{9}\begin{pmatrix} 9 & 4 & 0 & -7 \\ 8 & 4 & 0 & -8 \\ 0 & 0 & -4 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}. \tag{15}$$

Hence the degree of entanglement is $P_E=4/9$.

Example 3. For the maximally entangled state $|\Psi\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$, one obtains the density matrix

$$\rho_{AB}=|\Psi\rangle\langle\Psi|=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

with the Bloch vectors $\mathbf{u}=\mathbf{v}=(0,0,0)^T$, and the alpha matrix

$$\hat{\alpha}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Thus $P_E=1$ reaches the highest value.

Example 4. For the disentangled pure state $\rho_{AB}=\frac{1}{2}(\mathbf{1}+\vec{\sigma}^A\cdot\mathbf{u})\otimes\frac{1}{2}(\mathbf{1}+\vec{\sigma}^B\cdot\mathbf{v})$, where $|\mathbf{u}|=|\mathbf{v}|=1$, we have the alpha matrix as

$$\hat{\alpha}=\begin{pmatrix} 1 & v_1 & v_2 & v_3 \\ u_1 & u_1v_1 & u_1v_2 & u_1v_3 \\ u_2 & u_2v_1 & u_2v_2 & u_2v_3 \\ u_3 & u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}. \tag{17}$$

Obviously $P_E=0$ indicates that ρ_{AB} is disentangled.

IV. CONCLUSION AND DISCUSSION

In conclusion, we have presented a measure to quantify the degree of entanglement for two qubits in a pure state. We would like to make some discussion in the following.

(i) The similar idea developed in this paper could be generalized to quantify the degree of entanglement for two qu Nits (i.e., N-state quantum systems, $N=2$ and $N=3$ correspond to a qubit and a qutrit, respectively) [14,15] in a pure state. For instance, the density matrix for two entangled qutrits could be written as

$$\rho_{AB}=\frac{1}{9}\Bigg(\mathbf{1}\otimes\mathbf{1}+\sqrt{3}\vec{\lambda}^{A}\cdot\mathbf{u}\otimes\mathbf{1}+\sqrt{3}\mathbf{1}\otimes\vec{\lambda}^{B}\cdot\mathbf{v}+\frac{3}{2}\times\sum_{i,j=1}^{8}\beta_{ij}\lambda_i^{A}\otimes\lambda_j^{B}\Bigg), \tag{18}$$

where λ_i ($i=1,2,\ldots,8$) are the eight Hermitian generators of SU(3) (namely, the usual Gell-mann matrices). For the state of two entangled qutrits

$$|\Psi\rangle=\tfrac{1}{3}(|00\rangle+|11\rangle+|22\rangle), \tag{19}$$

its corresponding density matrix is [15]

$$\rho_{AB}=\frac{1}{9}\left(\mathbf{1}\otimes\mathbf{1}+\frac{3}{2}\sum_{i,j=1}^{8}\beta_{ij}\lambda_i^{A}\otimes\lambda_j^{B}\right), \tag{20}$$

with the nonzero coefficients $\beta_{11}=\beta_{33}=\beta_{44}=\beta_{66}=\beta_{88}=1$, $\beta_{22}=\beta_{55}=\beta_{77}=-1$. The elements β_{ij}, 1, $\mathbf{u}$ and $\mathbf{v}$ form a 9×9 matrix $\hat{\alpha}$, it is easy to show that $P_E=(-\det\hat{\alpha})^{1/4}=1$, which indicates that the state $|\Psi\rangle$ in Eq. (19) is just a maximally entangled state.

(ii) After making the parametrization $\mathbf{u}=\hat{\mathbf{u}}\tanh\phi_{\mathbf{u}}$, where $\hat{\mathbf{u}}=\mathbf{u}/|\mathbf{u}|$, the density matrix of a qubit $\rho(\mathbf{u})=(\mathbf{1}+\vec{\sigma}\cdot\mathbf{u})/2$ can be connected to the Lorentz boost matrix $L(\mathbf{u})=\exp(\varphi_{\mathbf{u}}\vec{\sigma}\cdot\hat{\mathbf{u}}/2)=\mathbf{1}\cosh(\varphi_{\mathbf{u}}/2)+\vec{\sigma}\cdot\hat{\mathbf{u}}\sinh(\varphi_{\mathbf{u}}/2)$ as [16]

$$\rho(\mathbf{u})=\frac{L(\mathbf{u})}{2\cosh\phi_{\mathbf{u}}}, \quad \phi_{\mathbf{u}}=\varphi_{\mathbf{u}}/2. \tag{21}$$

Obviously, $\rho(\mathbf{u})$ and $L(\mathbf{u})$ are in one-to-one correspondence. For the former, the physical meaning of the vector $\mathbf{u}$ is the Bloch vector in quantum mechanics, while for the latter it is the relativistic velocity. Due to the rapidity φ, i.e., the hyperbolic angle, special relativity can be formulated in terms of hyperbolic geometry. As a result, some physical quantities have been found to have geometric significance, such as the Thomas rotation angle corresponds to the defect of a hyperbolic triangle [17,18]. After viewing the Bloch vector $\mathbf{u}$ as an analogous relativistic velocity, the Bures fidelity $F(\rho_1,\rho_2)=(\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}})^2$ was found to have a geometric interpretation in the framework of hyperbolic geometry [16]. Similarly, with the aid of the parametrization $\mathbf{u}=\hat{\mathbf{u}}\tanh\phi_{\mathbf{u}}$, it is not difficult to find that the entanglement degree $P_E=\sqrt{1-|\mathbf{u}|^2}=1/\cosh\phi_{\mathbf{u}}$ for two qubits in a pure state is the reciprocal of the Lorentz factor [18] in the hyperbolic geometry. The extension of our approach to the mixed states of two entangled qubits will be discussed elsewhere.

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