Zitterbewegung for ultracold atoms in the merging of Dirac points

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(Received 4 January 2015; published 23 February 2015)

We investigate *Zitterbewegung* (ZB) of the ultracold atoms in a tunable honeycomb optical lattice. By tuning the parameters of the lattice one can realize the merging of two Dirac points. The process of two Dirac points merging into a hybrid point implies a topological phase transition from a semimetallic to a band insulator phase. After merging, it presents a linear-quadratic dispersion relation which is linear in one direction and quadric in the other orthogonal direction. We show that ZB occurs isotropically before merging but occurs only in the direction of the linear dispersion after merging. Furthermore, we obtain that ZB is exponentially damping for a negative energy gap but is a stable oscillation for a positive energy gap. The frequency and amplitude of ZB can be controlled in a detectable range, so the phenomena can be observable in the ultracold atomic experiments.

DOI: 10.1103/PhysRevA.91.023623 PACS number(s): 67.85.Fg, 37.10.Jk, 03.65.Pm, 81.05.ue

I. INTRODUCTION

Zitterbewegung (ZB), first predicted by Schrödinger in 1930, represents a spontaneously trembling motion of free relativistic particles with extremely high frequency and small amplitude [1]. While the early research of ZB was merely focusing on Dirac electrons, recent findings show that ZB oscillation may not be a unique character of Dirac electrons but may also exist in many spinor systems featuring linear dispersion relations such as graphene [2–4], trapped ions [5], photonic crystals [6], and ultracold atoms [7–11]. All of them can be described by an effective Dirac equation. According to very recent reports, ZB phenomenon has been observed in experiments of trapped ions [12], photons [13], and cold atoms [14–16].

Furthermore, it is noticeable that the merging of Dirac points (DPs), where two DPs move and merge into a hybrid point (HP) in the Brillouin zone, has attracted much interest of physicists in recent years. Such a merging leads the final dispersion relation to be quite exotic, which is linear in one direction but parabolic in the other orthogonal direction [17]. The merging signals a quantum phase transition between a semimetallic and a band insulator phase [17–25], i.e., a Lifshitz phase transition. Very recently, a well-designed experiment has been carried out by Tarruell *et al.*, in which the creating, moving, and merging of DPs have been realized with ultracold atoms loaded into a tunable honeycomb lattice [26]. Moreover, ultracold atoms in an optical lattice provide a versatile tool in which the properties of condensed-matter systems [27] and many-body physics [28] can be simulated in a highly controllable manner, such as the superfluid-Mott insulator transitions of Hubbard models [29–31]. A quantum-optical analog of graphene can be realized by ultracold relativistic atoms loaded in a hexagonal optical lattice [32,33]. The observation of many remarkable relativistic phenomena in table-top experiments become possible, such as Klein tunneling [34], the relativistic extension of Landau levels [35], and even ZB, which usually only occurs in high-energy physics [36]. Thus, it provides a good experimental platform to probe ZB and may bring intriguing phenomena of two-dimensional many-body dynamics. Much more intriguing phenomena can be expected when ultracold atoms are subject to the special dispersion relation of HP.

In this work the ZB phenomenon in the process of two DPs merging into a HP has been studied. We directly simulate the free evolution of Dirac particles at different times in the process of merging and provides the analytical expression for the center of wave packet as a function of time in such process. We show that ZB occurs isotropically before the merge but occurs only in the direction of the linear dispersion after merging. Furthermore, we obtain that ZB is exponentially damping for a negative energy gap but is a stable oscillation for a positive energy gap. Our numerical and analytical results have good agreement with each other and since the parameter of ultracold atoms in optical lattice are tunable in such a wide range the frequency and amplitude of ZB can be controlled in a detectable range [14–16]. Thus, the observation of ZB phenomenon is expectable in ultracold atoms.

The paper is organized as follows. In Sec. II, we review the recent merging experiments and the corresponding theoretical model. In Sec. III, we investigate and discuss ZB in the process of merging by numerical and analytical methods, respectively. We give a brief summary in Sec. VI.

II. MODEL

We consider the case of Ref. [26] in which an ultracold relativistic gas of ultracold atoms in a two-dimensional (2D) tunable optical lattice is formed by three retro-reflected laser beams of wavelength $\lambda = 1064$ nm. The interference of these three beams result in a potential of the form

$$V(x,y) = -V_{\bar{X}}\cos^2(kx+\theta/2) - V_X\cos^2(kx) - V_Y\cos^2(ky) - 2\alpha\sqrt{V_X V_Y}\cos(kx)\cos(ky)\cos(\varphi), \quad (1)$$

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where $V_{\bar{X}}$ (V_X, V_Y) denotes the single-beam lattice depth which is proportional to the laser beam intensities, α is the visibility of the interference pattern, and $k = 2\pi/\lambda$. Here we choose $\theta = \pi$ and $\varphi = 0$. It is clearly that one can continuously adjust them to create square, triangular, dimer, and honeycomb structures by tuning the intensities of the beams. We start from a honeycomb lattice with $V_{\bar{X}}/E_R = 5.3(3)$, $V_X/E_R = 0.28(1)$, and $V_Y/E_R = 1.8(1)$, where $E_R = h^2/2m\lambda^2$ is the recoil energy, h denotes Planck's constant, and m is the mass of ultracold atoms. By decreasing the intensity of beam $\bar{X}$, the topological phase transition occurs. Corresponding dynamical properties of ultracold relativistic particles in the honeycomb lattice can be described by the following time-dependence free Dirac equation [24,26],

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi, \tag{2}$$

with

$$H = \sigma_x\left(\Delta + \frac{p_x^2}{2m^*}\right) + \sigma_y c p_y, \tag{3}$$

where m^* is the effective mass and Δ is a parameter for driving the transition which can be controlled by tuning the intensity of beam $\bar{X}$. The effective mass m^* and the transition parameter Δ are defined by Eqs. (5) and (7) of Ref. [24], respectively. σ_x (σ_y) is Pauli matrix. p_x (p_y) are momentum operators in $x(y)$ direction. The expression represents a type of energy-momentum relation of relativity,

$$E = \sqrt{\left(\Delta + \frac{p_x^2}{2m^*}\right)^2 + (cp_y)^2}. \tag{4}$$

Obviously, the merging process occurs along the direction of p_x. By tuning the parameter Δ, one can control the position of Dirac points (DPs, which have linear-linear dispersion relation) and make them merge into a hybrid point (HP, which has linear-quadratic dispersion relation). As shown in the top layer of Fig. 1, when $\Delta \ll 0$, the spectrum exhibits the two DPs and a saddle point in the middle of them. By increasing the parameter Δ, the two DPs get closer and finally the saddle point evolves into a HP at the critical point of transition ($\Delta = 0$). In the merging process, the slope of the linear dispersion relation decreases by increasing Δ along the p_x direction, whereas that along the p_y direction holds constant. They show a special linear-linear dispersion relation which has different slopes in different directions and results in different expansion speeds in two different orthogonal directions. After merging, when $\Delta > 0$, a 2Δ gap opens and a linear-quadratic dispersion relation appears which is linear in a direction and quadratic in the other.

III. *ZITTERBEWEGUNG* IN THE PROCESS OF DIRAC POINTS' MERGING

We directly solve the time-dependent Dirac equation with different value of transition driving parameter Δ. We consider

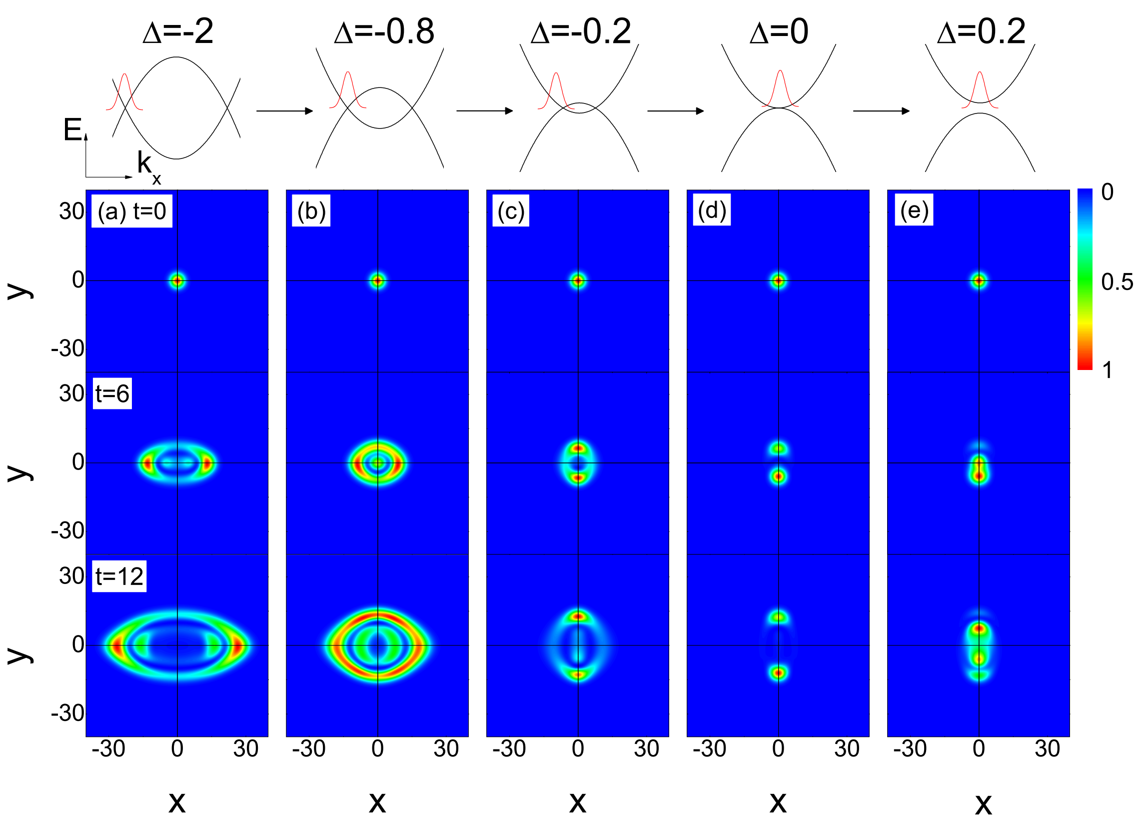

FIG. 1. (Color online) Numerically calculated probability distributions, $|\Psi(x,y)|^2$, at times $t = 0$ (top), 6 (middle), 12 (bottom) with $\Delta = -2, -0.8, -0.2, 0, 0.2$ [from panels (a) to (e)]. The corresponding Δ are marked. The width of the initial wave packet $d = 2$. Throughout t is in units of $m^*/(\hbar k_y^2)$, the gap parameter Δ is in units of $(\hbar k_y^2)/m^*$, and x and y are in units of $(p_y t)/m^*$.

a general Gaussian initial wave packet

$$\Psi = (1/\sqrt{2\pi}d)e^{ik_1x}e^{ik_2y}e^{-(x^2+y^2)/4d^2}\Phi, \tag{5}$$

where $\Phi = (c_1,c_2)^T$ is the unit vector, T stands for matrix transposition, d is the width of the wave packet, and k_1 (k_2) is the wave vector in the x (y) direction. These two-component wave functions are called Dirac "spinors," but they actually reflect the appearance of positive and negative energies in low-dimensional space. The corresponding wave function in momentum space is

$$\Psi_k = (d/\sqrt{\pi})e^{-d^2[(k_x-k_1)^2+(k_y-k_2)^2]}\Phi. \tag{6}$$

For convenience, we consider a monocomponent initial packet in which there is only the positive-energy part, i.e., $\Phi = (1,0)^T$. The packet is centered at ($k_1 = \sqrt{2m^*|\Delta|}, k_2 = 0$) for $\Delta < 0$ and ($k_1 = 0, k_2 = 0$) for $\Delta > 0$, which guarantees the center of wave packet always at the center of DP in the whole process of merging, as shown in top layer of Fig. 1. The numerical results for different Δ are shown in Fig. 1. Here after we use units $\hbar = m^* = c = 1$.

As shown in Figs. 1(a)–1(c), when $\Delta < 0$ (before transition), the space-time evolution of the wave packets resembles no-gap DPs with different speeds of expansion in two orthogonal directions. The slopes of dispersion relation in the y direction are constant, whereas the slopes along the merging direction, which plays the key role in controlling the speed of expansion of wave packet, increases by decreasing Δ. When $\Delta = 0$ (critical point), two Dirac points merge into a HP, which corresponds to a linear-quadratic dispersion relation. The dynamical properties are nonrelativistic in the merging direction and relativistic in the other. The wave packet represents two asymmetric parts and drifts away along the y axis as shown in Fig. 1(d). When $\Delta > 0$ (after transition), a 2Δ gap opens and the density distribution oscillates with time along the y direction and the drift disappears as shown in Fig. 1(e). Obviously, the gap is good for the localization of the wave packet, which means the greater the gap the more localized the wave packet. Furthermore, the oscillation merely appears in a single direction because the dynamical properties of the HP system are relativistic only in one direction.

We now calculate the expectation value of the position operator, which is crucial in studying ZB. In Heisenberg picture, the time-dependent position operator is as follows:

$$\vec{r}(t) = e^{iHt/\hbar}\vec{r}(0)e^{-iHt/\hbar}. \tag{7}$$

After inserting the Hamiltonian H into the above expression, we obtain the following two orthogonal coordinate components of $\vec{r}(t)$, respectively:

$$\begin{aligned} x(t) &= x_0 + \alpha_x t + \beta_x\left[\cos\left(\frac{2Et}{\hbar}\right) - 1\right] + \gamma_x \sin\left(\frac{2Et}{\hbar}\right), \\ y(t) &= y_0 + \alpha_y t + \beta_y\left[\cos\left(\frac{2Et}{\hbar}\right) - 1\right] + \gamma_y \sin\left(\frac{2Et}{\hbar}\right), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \alpha_x &= \frac{\left[\sigma_x\left(\Delta + \frac{p_x^2}{2m^*}\right)^2 + \sigma_y c p_y\left(\Delta + \frac{p_x^2}{2m^*}\right)\right]p_x}{E^2 m^*} \\ \beta_x &= \frac{\hbar(-\sigma_z)c p_y p_x}{2E^2 m^*} \\ \gamma_x &= \frac{\hbar\left[\sigma_x E^2 - \sigma_x\left(\Delta + \frac{p_x^2}{2m^*}\right)^2 - \sigma_y c p_y\left(\Delta + \frac{p_x^2}{2m^*}\right)\right]p_x}{2E^3 m^*} \end{aligned} \tag{9}$$

for x direction and

$$\begin{aligned} \alpha_y &= \frac{c^2\sigma_x\left(\Delta + \frac{p_x^2}{2m^*}\right)p_y + c^3\sigma_y p_y^2}{E^2} \\ \beta_y &= \frac{\hbar c\sigma_z\left(\Delta + \frac{p_x^2}{2m^*}\right)}{2E^2} \\ \gamma_y &= \frac{\hbar\left[c\sigma_y E^2 - c^2\sigma_x\left(\Delta + \frac{p_x^2}{2m^*}\right)p_y - c^3\sigma_y p_y^2\right]}{2E^3} \end{aligned} \tag{10}$$

for y direction. It is clearly that both $x(t)$ and $y(t)$ are 2×2 matrix. The first two terms in $x(t)[y(t)]$ account for the classical kinematics, as expected for a free Dirac particle, whereas the third and fourth, oscillating, terms induce ZB. The frequency of ZB, which can be estimated as $\omega_z \sim 2|E|/\hbar = 2\sqrt{[\Delta + p_x^2/(2m^*)]^2 + (cp_y)^2}/\hbar \sim 2|\Delta|$, grows linearly with increasing $|\Delta|$. The same monospinor initial packet of Eq. (6) is taken here in following theoretical analysis. For the case of $\Phi = (1,0)^T$, the [11] component of the matrix $\left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right]$ ($\left[\begin{smallmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{smallmatrix}\right]$) is selected and the calculation is direct; $\bar{x}(t)$ and $\bar{y}(t)$ can be obtained as

$$\begin{aligned} \bar{x}(t) &= \langle\Psi_k|x_{11}(t)|\Psi_k\rangle \\ &= \frac{d^2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-\hbar c p_y p_x}{2E^2}\left[\cos\left(\frac{2Et}{\hbar}\right) - 1\right] \\ &\quad \times e^{-2d^2[(k_x-k_1)^2+(k_y-k_2)^2]}dk_x dk_y, \end{aligned} \tag{11}$$

$$\begin{aligned} \bar{y}(t) &= \langle\Psi_k|y_{11}(t)|\Psi_k\rangle \\ &= \frac{d^2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\hbar c\left(\Delta + \frac{p_x^2}{2m^*}\right)}{2E^2}\left[\cos\left(\frac{2Et}{\hbar}\right) - 1\right] \\ &\quad \times e^{-2d^2[(k_x-k_1)^2+(k_y-k_2)^2]}dk_x dk_y. \end{aligned} \tag{12}$$

A trivial result $\bar{x}(t) = 0$ is obtained by considering the parity of integrand. From the expression of $\bar{y}(t)$ we can see there exists an oscillation term with frequency of $\sim 2E/\hbar$ and when $\Delta < 0$ we have ($k_1 = \sqrt{2m^*|\Delta|}, k_2 = 0$) so that the exponential term results in $e^{-2d^2[(k_x-\sqrt{2m^*|\Delta|})^2+k_y^2]} = e^{-4d^2m^*|\Delta|}e^{-2d^2(k_x^2-4k_xm^*|\Delta|+k_y^2)}$ in which an exponential damping term appears. This means that as the value of $|\Delta|$ increases, the oscillation damping is faster. This means the expression has good agreement with former numerical results: $\bar{y}(t)$ in varies with time throughout the evolution when $|\Delta| \sim \infty$. When $\Delta = 0$, the damping term is gone and the frequency of the oscillating term tends to 0, which means there is no turning-back point under this circumstance, so the wave center shows a directional drift. When $\Delta > 0$, after merging, the system satisfies the linear-quadratic dispersion relation. From expression (12), we can see that a periodic oscillation along

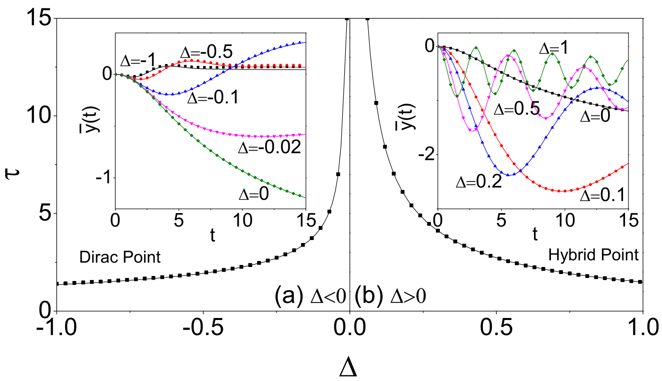

FIG. 2. (Color online) Analytical (lines) and numerically (symbols) calculated τ vs Δ in the process of merging. The left inset shows the average values $\bar{y}(t)$ change over time with different Δ before merging and the right one shows that after merging. The values of Δ are marked. The width of the initial wave packet $d = 2$. Throughout t and τ are in units of $m^*/(\hbar k_y^2)$, gap parameter Δ is in units of $(\hbar k_y^2)/m^*$, and $y(t)$ is in units of $(p_y t)/m^*$.

the y direction occurs, which is a direct evidence for ZB in this dispersion relation. The frequency of ZB increases with increasing Δ, whereas the amplitude decreases with that.

We define the time when the first extreme value of $\bar{y}(t)$ appears as the turning time and mark it as τ. As the value of τ is greater, the the critical point of the transition is closer. The analytical (lines) and numerical (symbols) results [by directly solving Eq. (2)] for τ vs Δ are plotted in Fig. 2. The inserts show details for $\bar{y}(t)$ as a function of time before and after the merging. As shown in the left inset of Fig. 2, when $\Delta < 0$ there is short-lived oscillation in the beginning of the evolution and rapidly damped to zero. This is because the expansion speed is fast, so that the wave packet will diffuse away in a short time. Therefore, as the value of Δ decreases, the time for interference of positive- and negative-energy states shortens so that the oscillation falls off rapidly. When $\Delta = 0$, the damping effect disappears and a directional drift of the center of wave packet occurs. When $\Delta > 0$, it can be seen clearly that all of them are periodic oscillations, which is a direct manifestation of ZB. The frequency increases with increasing Δ, whereas the amplitude decreases as shown in the right inset of Fig. 2. The analytical results show good agreement with numerical results.

IV. CONCLUSION

In summary, we have investigated the *Zitterbewegung* of ultracold atoms loaded in a honeycomb optical lattice in the process of merging of Dirac points, which has been realized in a recent experiment by Tarruell *et al.* [26]. The merging corresponds to a topological quantum phase transition from a semimetallic to a band insulator phase. We numerically simulate the density of distribution versus time for free relativistic particles during the merging process. From the numerical results we can see a well-visualized picture for the evolution of relativistic particles in the process of merging. Furthermore, we provide the analytical expression for the expectation value of wave packet as a function of time. We show that ZB occurs isotropically before merging but occurs only in the direction of the linear dispersion after merging and obtain that ZB is exponentially damping for a negative energy gap but is a stable oscillation for a positive energy gap.

Since the appearance of ZB oscillation in a specific direction is determined by the relativistic dispersion relation in that direction, our results should also be observable in other 2D systems with dispersion relations that can be switched from relativistic to nonrelativistic, this is to say, with Dirac points that can be moved and merged. We affirmed this point with the model in Ref. [25]. As for Dirac points merging in a 1D system, the dispersion relation is formed from a linear to quadruple one and therefore all ZB phenomenon will disappear after merging. In the case of a 3D system, we predict that ZB oscillation can be observed in the two directions that are orthogonal to the merging direction. In the followup work, we further study this field in more detail.

Due to the high tunability of the parameters in ultracold atoms experiments, the frequency and amplitude of ZB can be well controlled in a detectable range, and thus the observation of ZB phenomenon is expectable in ultracold atom experiments. We hope that experimental efforts in observing cold-atom *Zitterbewegung* and exploring physical phenomena based on complex lattice topologies [37] will benefit from our work.

ACKNOWLEDGMENTS

We thank X. Zhang, J. Liu, and S. L. Zhu for useful discussions. This work is supported by the SKPBR of China (Grant No. 2011CB922104) and the NSFC (Contracts No. 11125417 and No. 11474153). H. Cao and L. B. Fu are supported by the National Fundamental Research Program of China (Contracts No. 2013CBA01502, No. 2011CB921503, and No. 2013CB834100) and the NSFC (Contracts No. 11374040 and No. 11274051).

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